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A cross-validation-based statistical theory for point processes

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Summary

Motivated by the general ability of cross-validation to reduce overfitting and mean square error, we develop a cross-validation-based statistical theory for general point processes. It is based on the combination of two novel concepts for general point processes: cross-validation and prediction errors. Our cross-validation approach uses thinning to split a point process/pattern into pairs of training and validation sets, while our prediction errors measure discrepancy between two point processes. The new statistical approach, which may be used to model different distributional characteristics, exploits the prediction errors to measure how well a given model predicts validation sets using associated training sets. Having indicated that our new framework generalizes many existing statistical approaches, we then establish different theoretical properties for it, including large sample properties. We further recognize that nonparametric intensity estimation is an instance of Papangelou conditional intensity estimation, which we exploit to apply our new statistical theory to kernel intensity estimation. Using independent thinning-based cross-validation, we numerically show that the new approach substantially outperforms the state-of-the-art in bandwidth selection. Finally, we carry out intensity estimation for a dataset in forestry and a dataset in neurology.

Some key words: Kernel intensity estimation; Papangelou conditional intensity; Prediction; Spatial statistics; Thinning.

1. Introduction

A point process in a general space (Daley & Vere-Jones, 2003, 2008; Kallenberg, 2017) can be viewed as a generalized random sample, where we allow the sample size to be random

and/or the sample points to be dependent random variables; an independent and identically distributed sample is called a binomial point process (van Lieshout, 2000). Consequently, they have been extensively applied to analyse and model various event data sources, e.g., in forestry and epidemiology (Diggle, 2014). As one typically observes only one point pattern, i.e., point process realization, classical independent and identically distributed sample statistics is infeasible and, in addition, likelihood estimation is generally intractable (van Lieshout, 2000). This has led to the development of a range of innovative statistical approaches (Coeurjolly & Lavancier, 2019), where the associated estimation criteria to be optimized, which take the full observed pattern as input, are not explicitly based on predictive estimation ideas. Generally speaking, models with small cross-validation errors yield good out-of-sample prediction performances and tend to result in little overfitting and small mean square errors (Hastie et al., 2009; Arlot & Celisse, 2010). Hence, a general cross-validation-based statistical theory for point processes could reduce mean integrated square errors when fitting different distributional characteristics, but such a theory does not currently exist.

This paper addresses the development of a general cross-validation approach as well as a cross-validation-based predictive statistical theory for general point processes. Our cross-validation approach, which is inspired by our previous work on point process subsampling (Moradi et al., 2019), is defined through thinning and allows us to consider a form of conditional independent and identically distributed sampling of a point process. Besides cross-validation, our statistical approach is further based on a new notion of point process prediction errors, which is inspired by our previous work on nonparametric intensity estimation (Cronie & van Lieshout, 2018). Our prediction errors, which allow us to measure the quality of a proposed estimate, can be thought of as measures of discrepancy between two point processes. More specifically, any prediction error is given by the difference between two parameterized terms, a random one and a deterministic one. The random term is a sum over the first point process, where each summand depends on (i) the second point process, (ii) a point of the first point process and (iii) a candidate parameter. The deterministic term is equal to the expectation of the random term if and only if the candidate parameter is set to the true one for a well-specified model. When the two point processes coincide, i.e., under autoprediction, in a certain setting our prediction errors reduce to so-called innovations, originally introduced by Baddeley et al. (2005, 2008) to define residuals for Papangelou conditional intensity models. Our prediction errors further reduce to various loss functions for existing estimation approaches, e.g., (i) the approach of Fiksel (1984) and Takacs (1986) for conditional intensity modelling (Coeurjolly et al., 2016), with pseudolikelihood estimation as a special case, (ii) the quasilikelihood approach of Guan et al. (2015) for parametric intensity estimation, which has composite-/Poisson- and Palm-likelihood estimation (Coeurjolly & Lavancier, 2019) as special cases, and (iii) the nonparametric intensity estimation approach of Cronie & van Lieshout (2018). By combining our two new concepts, we arrive at the definition of our new statistical theory, referred to as point process learning due to its similarities with risk minimization in statistical learning (Vapnik, 1999).

We establish different properties and variations of point process learning; in particular, how it can be applied to parametric product density/intensity estimation and conditional intensity estimation. We then focus on nonparametric intensity estimation, which we indicate is an instance of conditional intensity estimation. In particular, we apply our new approach to bandwidth selection in kernel intensity estimation, when the cross-validation is achieved through independent thinning. We find that point process learning numerically outperforms the state-of-the-art method, i.e., the approach of Cronie & van Lieshout (2018),

in terms of the mean integrated square error, regardless of the degree of spatial interaction in the underlying point process.

For readability, in the main text we state all theory for first-order statistics and defer higher-order statements and proofs to the Supplementary Material.

2. Preliminaries

2.1. *Point processes and distributional characteristics*

Consider a general space S, by which we mean a complete separable metric space, which is endowed with a notion of size in the form of a locally finite and σ-finite Borel reference measure $A \mapsto |A| = \int_A \mathrm{d}u$, $A \subseteq S$; here we reserve the notation $\subseteq$ for Borel sets of S. For convenience, one may think of $S = \mathbb{R}^d$, $d \geqslant 1$, equipped with the d-dimensional Euclidean metric $d(u, v) = \|u - v\|$ and Lebesgue measure $|\cdot|$, or a linear network $S = L = \bigcup_{i=1}^{k} l_i$, i.e., a union of connected line segment $l_i \subseteq \mathbb{R}^2$, where $d(\cdot, \cdot)$ is the shortest-path metric and $|\cdot|$ represents arc length integration on L (Baddeley et al., 2015; Cronie et al., 2020); see Fig. 1 in § 3.2 for illustrations. Throughout, we implicitly assume that functions are sufficiently measurable/integrable.

Given a suitable probability space $(\Omega, \mathcal{F}, \mathrm{pr})$, a point process $X = \{x_i\}_{i=1}^{N}$, $0 \leqslant N \leqslant \infty$, in S may be defined as a random element in the measurable space $(\mathcal{X}, \mathcal{N}) = (\mathcal{X}_S, \mathcal{N})$ of point patterns $\mathbb{x} = \{x_1, \ldots, x_n\} \subseteq S$, $0 \leqslant n \leqslant \infty$, which are locally finite, i.e., where the cardinality $\#(\mathbb{x} \cap A) = \sum_{i=1}^{n} \mathbb{1}(x_i \in A)$ is finite for bounded $A \subseteq S$ (Daley & Vere-Jones, 2003, 2008; Møller & Waagepetersen, 2004). A member x_i of a point pattern $\mathbb{x}$ or a point process X is commonly called an event. We identify X with the random measure $X(A) = \#(X \cap A)$, $A \subseteq S$, which is simple, meaning that, almost surely, $X(\{u\}) \in \{0, 1\}$, $u \in S$, i.e., X has at most one event at any location.

The distribution of a point process X is most conveniently described by its Papangelou conditional intensity, λ. It satisfies the Georgii–Nguyen–Zessin theorem, which states that (Daley & Vere-Jones, 2008)

$$E\left\{ \sum_{x \in X} h(x, X \setminus \{x\}) \right\} = \int_S E\{h(u, X)\lambda(u; X)\} \, \mathrm{d}u$$

for nonnegative, possibly infinite, and integrable $h\colon S \times \mathcal{X} \to \mathbb{R}$. It has the interpretation that the conditional probability of finding a point of X in an infinitesimal neighbourhood du of $u \in S$, given that X agrees with $\mathbb{x}$ outside du, satisfies $\mathrm{pr}\{X(du) = 1 \mid X \cap S \setminus du = \mathbb{x} \cap S \setminus du\} = \lambda(u; \mathbb{x}) \, \mathrm{d}u$ (Coeurjolly et al., 2017). This interpretation is motivated by the fact that, for a finite point process, i.e., if $N = X(S) < \infty$ almost surely, we can express λ as a ratio of Janossy densities, which thus implies that $\lambda(\cdot)$ can be readily derived when the Janossy densities are known in closed form (Daley & Vere-Jones, 2008). Unfortunately, the Janossy densities, which yield the likelihood function, are generally not tractable (van Lieshout, 2000), but luckily there exist many models with explicit forms for λ, e.g., Poisson, Cox (Møller & Waagepetersen, 2004), Hawkes (Yang et al., 2019), Markov (van Lieshout, 2000) and hybrid Gibbs (Baddeley et al., 2015) point processes. Regarding dependencies in X, when $\mathbb{y} \subseteq \mathbb{x}$, if $\lambda(\cdot; \mathbb{y})$ is smaller/larger than or equal to $\lambda(\cdot; \mathbb{x})$, we call X attractive/repulsive.

By letting h in the Georgii–Nguyen–Zessin formula be constant over its second argument, we obtain the Campbell formula, which yields $E\{\lambda(u; X)\} = \rho(u)$, the intensity function of X, where $E\{X(A)\} = \int_A \rho(u) \, \mathrm{d}u$, $A \subseteq S$. Heuristically, $\mathrm{pr}\{X(du) = 1\} = E\{X(du)\} = \rho(u) \, \mathrm{d}u$ and whenever $\rho(\cdot) \equiv \rho > 0$ is (non)constant, we say that X is (in)homogeneous.

By replacing X by the point process consisting of all distinct n-tuples of elements of X, $X^n_{\neq} = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\} \subseteq S^n$, we obtain nth-order conditional intensities $\lambda^{(n)}$ and product densities/intensities $\rho^{(n)}$ (Coeurjolly et al., 2017); see the Supplementary Material for details.

2.2. *Point process statistics*

Assume that we observe a point pattern $\mathbb{x} = \{x_1, \ldots, x_n\}$ within some, potentially bounded, study region $W \subseteq S$, $|W| > 0$, which has been generated by $X \cap W$, for some unknown point process X. Here, in contrast to the classical independent and identically distributed setting, we have only one realization of the random element of interest.

Statistical settings typically deal with estimation of some particular characteristic of X and it turns out that the associated estimators can be characterized by what we refer to as a general parameterized estimator family $\Xi_\Theta = \{\xi_\theta : \theta \in \Theta\}$, where

$$\xi_\theta(u; \mathbb{x}), \qquad u \in S, \ \mathbb{x} \in \mathcal{X}, \ \theta \in \Theta, \tag{1}$$

are real valued and $\xi_\theta(\cdot; \mathbb{x})$ is either nonnegative or integrable for any $\mathbb{x}$. Typically, $\Theta \subseteq \mathbb{R}^l$, $l \geqslant 1$, but one could imagine other forms of parametrization; cf. Vapnik (1999). When each ξ_θ is constant over $\mathbb{x} \in \mathcal{X}$, i.e., it does not depend on $\mathbb{x}$, we set

$$\xi_\theta(u; \mathbb{x}) \equiv \xi_\theta(u), \qquad u \in S, \ \mathbb{x} \in \mathcal{X}, \ \theta \in \Theta. \tag{2}$$

This definition naturally extends to the nth-order setting; see the Supplementary Material.

To carry out estimation, one typically finds a minimizer, an estimate $\hat{\theta} = \hat{\theta}_W(\mathbb{x}) \in \Theta$, through some loss function $\mathcal{L}(\theta) = \mathcal{L}(\xi_\theta, \mathbb{x}, W)$, $\theta \in \Theta$. Ideally, $\mathcal{L}(\theta)$ is constructed such that the estimator, $\hat{\theta}_W(X)$, properly describes the characteristic of interest, in some suitable distributional sense, e.g., a mean-square-error sense. When we do not work under model misspecification, we assume that the true characteristic of interest is parameterized by some $\theta_0 \in \Theta$. Many common statistical frameworks (Møller & Waagepetersen, 2017; Coeurjolly & Lavancier, 2019) can be expressed through general parameterized estimator families with accompanying loss functions. Examples include parametric product density/intensity estimation, $\rho^{(n)}_\theta$, $\theta \in \Theta$, $n \geqslant 1$, also encompassing Palm-likelihood estimation and K-function-based minimum contrast estimation, taking the form (2), as well as parametric conditional intensity estimation, λ_θ, $\theta \in \Theta$, and nonparametric product density/intensity estimation, which have the form (1). Mathematically speaking, a nonparametric intensity estimator $\hat{\rho}_\theta$ (van Lieshout, 2012) has the form of a parameterized conditional intensity; it is an attractive model since the addition of a point to $\mathbb{x}$ close to u increases the value of $\hat{\rho}_\theta(u, \mathbb{x})$.

We illustrate our new theory by focusing on nonparametric intensity estimation, in particular, kernel intensity estimation (van Lieshout, 2012): for a point pattern $\mathbb{x} \subseteq W \subseteq S = \mathbb{R}^d$,

$$\hat{\rho}_\theta(u, \mathbb{x}) = \sum_{x \in \mathbb{x}} \frac{\kappa_\theta(u - x)}{e_\theta(u, x)} = \sum_{x \in \mathbb{x}} \frac{\theta^{-d} \kappa\{(u - x)/\theta\}}{e_\theta(u, x)}, \qquad u \in W, \tag{3}$$

where the kernel κ is a symmetric density function and e_θ is an edge correction term compensating for potential interactions with points outside W; examples include $e_\theta(u, x) = \int_W \kappa_\theta(v - x)\, \mathrm{d}v$, which ensures that $\int_W \hat{\rho}_\theta(u, \mathbb{x})\, \mathrm{d}u = \#\mathbb{x}$, and $e_\theta(u, x) \equiv 1$,

which represents no edge correction. The main challenge here is optimal selection of the bandwidth, i.e., the smoothing parameter $\theta \in \Theta = (0, \infty)$, as, generally speaking, the kernel choice has a much less pronounced role than the chosen θ (Silverman, 1986). For other, possibly non-Euclidean, domains, κ and thereby the kernel estimator may look somewhat different and also be quite abstract (Di Marzio et al., 2014; McSwiggan et al., 2017; Rakshit et al., 2019; Mateu et al., 2020). In certain cases, however, there are straightforward extensions of (3); see the Supplementary Material for the case of linear networks.

Our main focus in this paper will be bandwidth selection, and we here follow Cronie & van Lieshout (2018), who, in the context of Takacs–Fiksel estimation, implicitly suggested the following for nonparametric intensity estimation: fit (3) to $\mathbb{x}$, using a suitable conditional intensity estimation method, to obtain $\hat{\theta}$ and the final intensity estimate $\hat{\rho}_{\hat{\theta}}(\cdot; \mathbb{x})$. We see that if $\hat{\rho}_{\hat{\theta}} = \lambda_{\hat{\theta}}$ is close to the conditional intensity λ of X, we have $\hat{\rho}_{\hat{\theta}}(\cdot; \mathbb{x}) \approx \lambda(\cdot; \mathbb{x}) \approx E[\lambda(\cdot; X)] = \rho(\cdot)$; the last approximation follows since, on average, $\lambda(\cdot; \mathbb{x})$ is close to the expectation of $\lambda(\cdot; X)$. Since we apply the same model $\hat{\rho}_\theta$, $\theta \in \Theta$, regardless of the unknown underlying distribution, this is in general an instance of model misspecification.

3. Thinning-based cross-validation

3.1. *Thinning*

Heuristically, a thinning $Z \subseteq X$ is generated by applying some rule to X that either retains or deletes each $x \in X$ (Chiu et al., 2013). At the same time, marked point processes are used when each event carries additional information, not directly connected to S, e.g., a label, a quantitative measurement, a function or a set (Chiu et al., 2013; Cronie & van Lieshout, 2016; Ghorbani et al., 2021). Next we formalize thinning through bivariate markings of point processes.

Definition 1. *Given a point process $X = \{x_i\}_{i=1}^N \subseteq S$, a thinning Z of X with retention probability $p\colon S \times \mathcal{X} \to [0,1]$ may be defined as the marginal point process $Z = \{x\colon (x,m) \in \breve{X}, m = 1\}$ of a bivariate marking $\breve{X} = \{(x_i, m_i)\}_{i=1}^N \subseteq S \times \mathcal{M}$, $\mathcal{M} = \{0,1\}$, of X. Here, $m_i = m(x_i) \in \mathcal{M}$, $i = 1, \ldots, N$, for some possibly random marking function $m(\cdot)$, governing the retention probability. We let the reference measure on $\mathcal{M}$ be the counting measure.*

When $\breve{X}$ is independently marked, i.e., the marks are independent conditional on X, whereby the retention probability $p(u)$, $u \in S$, does not depend on X, we say that Z is an independent thinning. If, in addition, $p(\cdot) \equiv p \in [0,1]$, we say that Z is a p-thinning.

Independent thinnings are particularly tractable and in Theorem 1 below we provide important results on such thinnings, which will be used to establish certain properties of our statistical theory. Theorem 1 is stated and proved in the general nth-order setting in the Supplementary Material.

Theorem 1. *Let Z be an independent thinning of a point process X on S, with retention probability $p(u) \in (0,1)$, $u \in S$. Given $Y = X \setminus Z$, for any nonnegative or integrable $h\colon S \times \mathcal{X} \to \mathbb{R}$,*

$$E\left\{\sum_{x \in Z} h(x, Y)\right\} = E\left\{\sum_{x \in Y} h(x, Y \setminus \{x\}) \frac{p(x)}{1 - p(x)}\right\}.$$

Moreover, provided that they exist, the conditional intensity and the intensity of Z almost everywhere satisfy

$$\lambda_Z(u, Z) = p(u)E\{\lambda_X(u; X) \mid Z\} \quad \textit{almost surely},$$
$$\rho_Z(u) = p(u)\rho_X(u),$$

where λ_X and ρ_X are the conditional intensity and the intensity of X. Moreover, the associated Palm intensities satisfy $\rho^!_Z(u \mid v) = p(u)\rho^!_X(u \mid v)$; see the Supplementary Material. Given the associated marked point process representation $\breve{X}$ in Definition 1, when the conditional intensities of $\breve{X}$ and Y exist, they satisfy $E[\breve{\lambda}\{(u, 1); \breve{X}\} \mid Y] = \lambda_Y(u; Y)p(u)/\{1 - p(u)\}$ for almost all $u \in S$. In particular, for a p-thinning with retention probability $p \in (0, 1)$, we set $p(\cdot) \equiv p$ above.

3.2. *Cross-validation for point processes*

Broadly speaking, cross-validation refers to a family of techniques that, in different ways, repeatedly split the dataset $\mathbb{x}$ into a training set $\mathbb{x}_i^T \subseteq \mathbb{x}$ and a validation set $\mathbb{x}_i^V = \mathbb{x} \setminus \mathbb{x}_i^T$, $i = 1, \ldots, k$, with the aim of assessing a model's generalizability (Arlot & Celisse, 2010); we here use the terms training, validation and test sets in a rather loose fashion. In the classical random sample setting, generalizability is often described in terms of out-of-sample prediction performance with respect to new data, i.e., additional draws from the underlying distribution. The notion of new data makes little sense in the context of point processes, where one is typically dealing with just one realization of a sample that potentially possesses both dependence and a random sample size. However, the equally common terms *unseen* and *hold out* make more sense here since they have a clear meaning for point processes, namely, splitting through thinning.

Definition 2 (Point process cross-validation). *Given $k \geqslant 1$ thinnings $Z_1, \ldots, Z_k$ of a point process $X \subseteq S$, we refer to the collection of pairs $(X_i^T, X_i^V) = (Y_i, Z_i)$, $Y_i = X \setminus Z_i$, $i = 1, \ldots, k$, as a cross-validation splitting. For a point pattern $\mathbb{x}$, we write $(\mathbb{x}_i^T, \mathbb{x}_i^V)$.*

If we have access to $k' \geqslant 1$ independent copies $X_1, \ldots, X_{k'}$ of X, we consider k splits for each X_i, giving $k \times k'$ training-validation pairs. Clearly, we may carry out the partitioning in an infinite number of ways by using different thinning strategies. However, dependent thinnings are hard to deal with since, for arbitrary point processes, it is generally hard to derive distributional properties for them; we essentially have no control over the dependence structures between the training and validation sets. In classical k-fold cross-validation, $\mathbb{x}$ is split into k folds, having approximately the same fixed cardinality, and in each round, $i = 1, \ldots, k$, the ith fold plays the role of $\mathbb{x}_i^V$, while the union of the remaining $k-1$ folds plays the role of $\mathbb{x}_i^T$. By setting $k = \#\mathbb{x}$, so that $\mathbb{x}_i^V = \{x_i\}$ and $\mathbb{x}_i^T = \mathbb{x} \setminus \{x_i\}$, $i = 1, \ldots, \#\mathbb{x}$, we obtain leave-one-out cross-validation. These sequential algorithms do not result in independent thinning, since the assignment of a point of $\mathbb{x}$ to a given fold depends on the assignments of other points; a fold runs full once it contains a given number of points of $\mathbb{x}$. Thus, due to the related intractability, we will not look closer at these. Instead, we argue that cross-validation procedures for point processes should be based on independent thinning. The main argument is that then, as we saw in Theorem 1, we have control over distributional properties of most characteristics of interest, e.g., intensity functions. There are, however, infinitely many ways to carry out independent thinning.

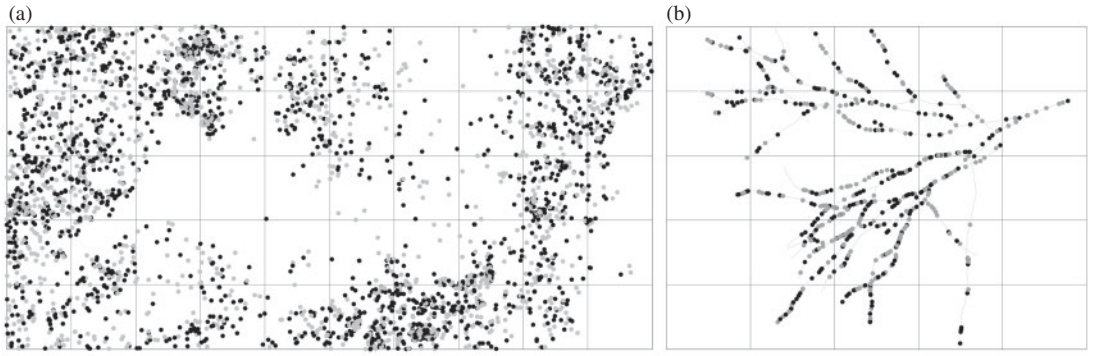

Fig. 1. A point pattern $\mathbb{x} \subseteq W \subseteq S$ together with a partition $\{W_i\}_{i=1}^k$, yielding validation sets $\mathbb{x}_i^V = \mathbb{x} \cap W_i$, $i = 1, \ldots, k \geqslant 2$, as well as a p-thinning-based training-validation pair (black dots indicate validation points). (a): Euclidean domain, $S = \mathbb{R}^2$. (b): Linear network, $S = L$.

For simplicity reasons, we argue that cross-validation procedures for point processes should be based on p-thinning; see Fig. 1 for illustrations in two different spatial domains. Next, we provide two p-thinning-based procedures: Monte Carlo cross-validation, which in the literature is also referred to as repeated random subsampling validation, and multinomial cross-validation, which is our variant of classical k-fold cross-validation. The main difference between the two is that the latter, which is computationally more efficient than the former, involves only one parameter, k, and does not allow for $\mathbb{x}_i^V$ and $\mathbb{x}_j^V$, $i \neq j$, to overlap.

DEFINITION 3. *Given $k \geqslant 1$ p-thinnings $\mathbb{z}_1, \ldots, \mathbb{z}_k$, $p \in (0, 1)$, of a point pattern $\mathbb{x}$, we define Monte Carlo cross-validation as setting $\mathbb{x}_i^V = \mathbb{z}_i$ and $\mathbb{x}_i^T = \mathbb{x} \setminus \mathbb{z}_i$, $i = 1, \ldots, k$.*

Given some $k \geqslant 2$, randomly label the point pattern $\mathbb{x}$ with independent and identically distributed marks $m(x) \in \{1, \ldots, k\}$, $x \in \mathbb{x}$, from a multinomial distribution with parameters k and $p_1 = \cdots = p_k = 1/k$. We define k-fold multinomial cross-validation by $\mathbb{x}_i^V = \{x \in \mathbb{x} : m(x) = i\}$ and $\mathbb{x}_i^T = \mathbb{x} \setminus \mathbb{x}_i^V$, $i = 1, \ldots, k$.

In Monte Carlo cross-validation, for any point pattern $\mathbb{x}$ and a suitable function f on $\mathcal{X}$, by the law of large numbers and the central limit theorem, conditionally on $X = \mathbb{x}$, the mean $k^{-1} \sum_{i=1}^k f(X_i^V)$ converges almost surely to $E[f(X_1^V)]$ and weakly to a Gaussian random variable; in practice, associated statistical procedures may be stopped when we see indications of convergence. Furthermore, when $p \approx 0$, we obtain something similar to the classical leave-one-out approach, where the advantage of the former over the latter is that by Theorem 1 we have theoretical control over distributional properties of X_i^T and X_i^V. In multinomial cross-validation, each validation set is a p-thinning with retention probability $1/k$ and each training set is a p-thinning with retention probability $1 - 1/k = (k-1)/k$, whereby various distributional properties are known, e.g., the intensity of X_i^V is $\rho_p(\cdot) = \rho(\cdot)/k$; recall Theorem 1. Moreover, for large datasets, multinomial and classical k-fold cross-validation should yield very similar results since, proportionally, the point counts of the folds become approximately the same in the two. Additionally, common rules of thumb for k-fold cross-validation in the independent and identically distributed setting (Arlot & Celisse, 2010), e.g., $k = 5$ or $k = 10$, do not necessarily apply in the current context. Sometimes, there are optimal choices for p and k such that our new statistical approach performs better with Monte Carlo cross-validation than with multinomial cross-validation, but such choices seem related to, e.g., the sample size or the dependence structure of X; see § 5.2.

Another natural and appealing approach here is block cross-validation (Roberts et al., 2017): choose $k \geqslant 1$ subsets $W_i \subseteq W$, $i = 1, \ldots, k$, potentially a partition of W, and let $\mathbb{x}_i^V = \mathbb{x} \cap W_i$; see Fig. 1 for an illustration. This is a form of independent thinning-based k-fold cross-validation, where the ith fold is obtained through the retention probability $p_i(u) = \mathbb{1}(u \in W_i)$, $u \in W$, $i = 1, \ldots, k$. Note the philosophical difference between this, where we sample from left to right and top to bottom, and the p-thinning-based approach, where we sample from above. As our general statistical theory in § 5 accommodates any thinning-based cross-validation approach, this extrapolation/interpolation kind of approach can also be combined with our new theory. However, in contrast to Definition 3, where the procedures are governed by the hyperparameters k and p, we here see a few challenges. For example, it is not evident how to best choose the sets W_i for an unknown model (cf. Mattfeldt et al., 2013), the counts $\#\mathbb{x}_i^V$ may be highly varying and we might introduce edge effects to be corrected for (Cronie & Särkkä, 2011; Chiu et al., 2013; Baddeley et al., 2015).

In the Supplementary Material we further discuss alternatives to and merits of Definition 3 and illustrate that, in addition to training and validation sets, we may generate test sets by treating each training set as an original point pattern to which cross-validation is applied.

4. Point process prediction

4.1. *Prediction errors*

Our statistical approach heavily relies on a new notion of prediction errors for point processes, which, e.g., may be used to predict properties of one point process from another point process.

Definition 4 (Point process prediction errors). *Let* $\Xi_\Theta = \{\xi_\theta : \theta \in \Theta\}$ *and* $\mathcal{H}_\Theta = \{h_\theta : \theta \in \Theta\}$ *be two general parameterized estimator families, both either of the form* (1) *or* (2)*. We refer to the members of* $\mathcal{H}_\Theta$ *as test functions. The associated families of* $\mathcal{H}_\Theta$*-weighted bivariate prediction errors* $\{\mathcal{I}^{h_\theta}_{\xi_\theta}(A; \mathbb{z}, \mathbb{y}) : A \subseteq S, \mathbb{y}, \mathbb{z} \in \mathcal{X}\}_{\theta\in\Theta}$ *and univariate prediction errors* $\{\mathcal{I}^{h_\theta}_{\xi_\theta}(A; \mathbb{y}) : A \subseteq S, \mathbb{y} \in \mathcal{X}\}_{\theta\in\Theta}$ *are defined as the signed Borel measures*

$$\mathcal{I}^{h_\theta}_{\xi_\theta}(A; \mathbb{z}, \mathbb{y}) = \sum_{x \in \mathbb{z} \cap A} h_\theta(x; \mathbb{y} \setminus \{x\}) - \int_A h_\theta(u; \mathbb{y}) \xi_\theta(u; \mathbb{y}) \, \mathrm{d}u \tag{4}$$

and $\mathcal{I}^{h_\theta}_{\xi_\theta}(A; \mathbb{y}) = \mathcal{I}^{h_\theta}_{\xi_\theta}(A; \mathbb{y}, \mathbb{y})$*. In particular, when* Ξ_Θ *and* $\mathcal{H}_\Theta$ *are of the form* (2) *then* $\mathcal{I}^{h_\theta}_{\xi_\theta}(A; \mathbb{z}, \mathbb{y}) = \mathcal{I}^{h_\theta}_{\xi_\theta}(A; \mathbb{z}) = \int_A h_\theta(u) \mathcal{I}^1_{\xi_\theta}(\mathrm{d}u; \mathbb{z})$ *for any* $\mathbb{y}, \mathbb{z} \in \mathcal{X}$ *and* $A \subseteq S$*.*

Regarding the interpretation of (4), which will be motivated in § 4.3, for two point processes Z and Y, the random signed measure $\mathcal{I}^{h_\theta}_{\xi_\theta}(A; Z, Y)$, $A \subseteq S$, represents an empirical measure of how well Y predicts points of Z in A via ξ_θ and h_θ; univariate prediction errors thus correspond to autoprediction. The test function h_θ weights the associated contributions of distinct points, ξ_θ is intended to describe the distributional properties of the superposition $Z \cup Y$ and $\mathcal{I}^{h_\theta}_{\xi_\theta}(A; Z, Y)$ estimates how well the specific choice $\theta \in \Theta$ does in predicting points of Z from Y. In some sense, this is a coupling idea. Furthermore, by replacing the test function in (4) by either $h_\theta(u; W \cap \mathbb{y} \setminus \{u\})$ or $\mathbb{1}(\mathbb{y} \subseteq W) h_\theta(u; \mathbb{y} \setminus \{u\})$ for some possibly bounded $W \subseteq S$, we ensure that $\mathbb{y} \in \mathcal{X}$ is contained in W. Indeed, a prediction error family

forms a signed transition kernel (Kallenberg, 2017) and it is straightforward to extend (4) to the nth-order case, where we sum over $\mathbb{z}^n_{\neq} \cap A \subseteq S^n$ and $\xi_\theta, h_\theta \colon S^n \times \mathcal{X} \to \mathbb{R}$.

To compute prediction errors in practice, we need to numerically approximate the integral in (4). Given any quadrature rule with quadrature points $\mathbb{v} \in \mathcal{X}$, $\mathbb{z} \subseteq \mathbb{v}$, and quadrature weights $\{w_v \colon v \in \mathbb{v}\}$, $\sum_{v \in \mathbb{v}} w_v = |S|$, we may exploit the Berman–Turner device (Berman & Turner, 1992) to approximate the integral in (4), i.e., $\mathcal{I}^{h_\theta}_{\xi_\theta}(A; \mathbb{z}, \mathbb{y}) \approx \sum_{v \in \mathbb{v} \cap A} h_\theta(v; \mathbb{y})\{\mathbb{1}(v \in \mathbb{z}) - \xi_\theta(v; \mathbb{y}) w_v\}$.

4.2. *Autoprediction: innovations and classical statistical approaches*

Before we turn to studying different properties of our prediction errors, we point out a connection between them and the so-called innovations of Baddeley et al. (2005, 2008), as well as the related estimating equation approaches considered in the literature; see Møller & Waagepetersen (2017), Coeurjolly & Lavancier (2019) and the references therein. An innovation coincides with a univariate prediction error $\mathcal{I}^{h_\theta}_{\lambda_\theta}(A; X)$, where X is some point process and $\xi_\theta(\cdot;\cdot) = \lambda_\theta(\cdot;\cdot)$ belongs to a parametric family of conditional intensity functions. It should be emphasized that Baddeley et al. (2005, 2008) used innovations to define point process residuals, obtained by replacing θ by an estimate $\hat{\theta}$ in a univariate prediction error.

As an immediate consequence of the Georgii–Nguyen–Zessin formula and the Campbell formula, we obtain the following. If X has conditional intensity $\lambda = \lambda_{\theta_0} \in \Xi_\Theta$, where Ξ_Θ is of the form (1), then $E\{\mathcal{I}^h_\lambda(W; X)\} = 0$ for any test function $h \colon S \times \mathcal{X} \to \mathbb{R}$. Also, if X has intensity $\rho = \rho_{\theta_0} \in \Xi_\Theta$, where Ξ_Θ is of the form (2), then $E\{\mathcal{I}^h_\rho(W; X)\} = 0$ for any test function satisfying $h(\cdot; \mathbb{x}) = h(\cdot)$ for any $\mathbb{x} \in \mathcal{X}$. In the case of conditional intensities, variance and covariance expressions can be found in Baddeley et al. (2008), Daley & Vere-Jones (2008) and Coeurjolly & Rubak (2013), while extension to the nth-order case is straightforward. These observations indicate that univariate prediction errors may sensibly be exploited as loss functions for parameter estimation. In particular, we may use $\mathcal{L}(\theta; \mathbb{x}) = \mathcal{I}^{h_\theta}_{\xi^n_\theta}(W; \mathbb{x})^2$, $\theta \in \Theta$, to obtain an estimate. As we shall see, a particularly interesting choice for $\mathcal{H}_\Theta$ is $h_\theta(\cdot; \mathbb{y}) = f\{\xi^n_\theta(\cdot; \mathbb{y})\}$, $\theta \in \Theta$, for some suitable $f \colon \mathbb{R} \to \mathbb{R}$. In fact, by using different test functions, univariate prediction errors yield many existing statistical approaches as particular cases, e.g., quasilikelihood, Poisson process likelihood, Palm-likelihood, Takacs–Fiksel, pseudolikelihood, nonparametric product density/intensity and K-function-based minimum contrast estimation; see, e.g., Diggle (2014), Møller & Waagepetersen (2017), Coeurjolly & Lavancier (2019) and the references therein.

4.3. *Properties of point process prediction errors*

Below, in Theorem 2, which is proved and stated in nth-order form in the Supplementary Material, we derive expectations for our prediction errors, together with necessary and sufficient conditions for the prediction errors to have mean zero.

THEOREM 2. *Given a point process X in S, let Z be an arbitrary thinning of X, with $Y = X \setminus Z$, and let $\breve{X}$ be the associated bivariate point process representation in Definition 1. Furthermore, let $\Xi_\Theta = \{\xi\}$ and $\mathcal{H}_\Theta = \{h\}$ consist of one element each.*

When $\xi, h \colon S \to \mathbb{R}$ are of the form (2), $\mathcal{I}^h_\xi(\cdot; Z, Y) = \mathcal{I}^h_\xi(\cdot; Z)$ satisfies

$$E\{\mathcal{I}^h_\xi(A; Z)\} = \int_A h(u)\{\rho_Z(u) - \xi(u)\}\,\mathrm{d}u$$

for any $A \subseteq S$, where $\rho_Z(\cdot)$ denotes the intensity of Z. Moreover, this expectation is 0 *for any $A \subseteq S$ and any test function h of the form* (2) *if and only if*

$$\xi(u) = \rho_Z(u) \quad \text{almost everywhere.} \tag{5}$$

If, instead, $\xi, h\colon S \times \mathcal{X} \to \mathbb{R}$ are of the form (1)*, when $\breve{X}$ admits a conditional intensity $\breve{\lambda}(\cdot; \breve{X})$, for any $A \subseteq S$, we have*

$$E\{\mathcal{I}_\xi^h(A; Z, Y)\} = \int_A E(h(u; Y)[\breve{\lambda}\{(u, 1); \breve{X}\} - \xi(u; Y)])\, \mathrm{d}u.$$

Assume further that $E[\breve{\lambda}\{(u, 1); \breve{X}\}^2] < \infty$ for $|\cdot|$-almost any $u \in S$. Then, for any $A \subseteq S$ and test function h such that $E\{h(u; Y)^2\} < \infty$, we have $E\{\mathcal{I}_\xi^h(A; Z, Y)\} = 0$ if and only if

$$\xi(u; Y) = E[\breve{\lambda}\{(u, 1); \breve{X}\} \mid Y] \quad \text{almost everywhere.} \tag{6}$$

While Theorem 2 provides expressions for expectations of prediction errors, variance expressions can be found in the Supplementary Material. These indicate that the variances are governed by the dependence structure of $(Y, Z) = (X \setminus Z, Z)$ and the test function. Moreover, the general expressions in Theorem 2 are of limited practical use, but Corollary 1, which is proved and stated in nth-order form in the Supplementary Material, shows that they become explicit when Z is an independent thinning.

Corollary 1. *Assume that the setting of Theorem 2 holds. When Z is an independent thinning of X, based on a retention probability function $p(u) \in (0, 1)$, $u \in S$, then* (5) *reads $\xi(u) = p(u)\rho_X(u)$ almost everywhere and the right-hand side of* (6) *is given by*

$$p(u)E\{\lambda_X(u; X) \mid Y\} = w(u, Z, Y)\lambda_X(u; Y), \tag{7}$$

where $w(u, Z, Y) = p(u)\lambda_X(u; Y)^{-1}E\{\lambda_X(u; X) \mid Y\}$; $w(u, Z, Y) \leqslant p(u)$ if X is repulsive, $w(u, Z, Y) \geqslant p(u)$ if X is attractive and $w(u, Z, Y) = p(u)$ if X is a Poisson process, almost surely.

5. The new statistical approach

5.1. *Point process learning*

Given the definitions in § 3.2 and § 4.1, we can now specify our new statistical approach. The philosophical argument here is that a good approach should result in a model that, given the current, i.e. training, data, does well in predicting unseen, i.e. validation data.

Definition 5. *Generate training-validation pairs $(\mathbb{x}_i^T, \mathbb{x}_i^V) \subseteq W^2 \subseteq S^2$, $i = 1, \ldots, k$, from at least one realization of a point process $X \subseteq S$, in accordance with § 3.2. Given either the form* (1) *or* (2)*, consider further a general parameterized estimator family $\Xi_\Theta = \{\xi_\theta : \theta \in \Theta\}$ and $k_i \geqslant 1$ test function families $\mathcal{H}_\Theta^{ij} = \{h_\theta^{ij} : \theta \in \Theta\}$, $j = 1, \ldots, k_i$, for each $i = 1, \ldots, k$. Let*

$$\mathcal{I}_{ij}(\theta) = \mathcal{I}_{\xi_\theta}^{h_\theta^{ij}}(W; \mathbb{x}_i^V, \mathbb{x}_i^T) = \sum_{x \in \mathbb{x}_i^V} h_\theta^{ij}(x; \mathbb{x}_i^T) - \int_W h_\theta^{ij}(u; \mathbb{x}_i^T)\xi_\theta(u; \mathbb{x}_i^T)\, \mathrm{d}u \tag{8}$$

when Ξ_Θ *and* $\mathcal{H}_\Theta^{ij}$ *are of the form* (1)*, or let*

$$\mathcal{I}_{ij}(\theta) = \mathcal{I}_{\xi_\theta}^{h_\theta^{ij}}(W; \mathbb{x}_i^T) = \sum_{x \in \mathbb{x}_i^T} h_\theta^{ij}(x) - \int_W h_\theta^{ij}(u)\xi_\theta(u)\,\mathrm{d}u \tag{9}$$

when Ξ_Θ *and* $\mathcal{H}_\Theta^{ij}$ *are of the form* (2)*. We say that any method that generates estimates in* Θ *by exploiting* (8) *or* (9) *belongs to the field of point process learning.*

The use of (9) in fact results in a point process subsampling approach, very much akin to that proposed by Moradi et al. (2019), in the sense that it does not make explicit use of $\mathbb{x}_i^V$, $i = 1, \ldots, k$, which is different from actual cross-validation-based approaches.

Once we have made a choice for $\Xi_\Theta = \{\xi_\theta : \theta \in \Theta\}$, which governs what we are interested in fitting, some further choices remain to be made: the cross-validation approach with associated parameters, how to combine the prediction errors to carry out the estimation and the test function families employed. These, as well as other choices, may be viewed as hyperparameter choices.

When we do not want a training-validation pair with $\mathbb{x}_i^V$ and/or $\mathbb{x}_i^T$ being empty to influence the estimation, we must require that $1 \leqslant \#\mathbb{x}_i^V$ and/or $1 \leqslant \#\mathbb{x}_i^T$. This may be achieved by replacing $\mathcal{I}_{ij}(\theta)$ with $\widetilde{\mathcal{I}}_{ij}(\theta) = I_i\mathcal{I}_{ij}(\theta)$ for a suitable indicator function I_i, which in the case of (8) may be absorbed into the test function. For (8), having $\mathbb{x}_i^T$ empty makes no sense, as this results in using $\mathbb{x}_i^T = \emptyset$ to predict $\mathbb{x}_i^V = \mathbb{x}$. Hence, in most cases, one would use $I_i = \mathbb{1}(1 \leqslant \#\mathbb{x}_i^T \leqslant \#\mathbb{x} - 1)$ if we consider (8) and $I_i = \mathbb{1}(1 \leqslant \#\mathbb{x}_i^T \leqslant \#\mathbb{x})$ if we consider (9), but sometimes the preferred choice may be to set $I_i = 1$ for all $i = 1, \ldots, k$. We write $\mathcal{T}_k = \{i \in \{1, \ldots, k\} : I_i = 1\}$. As usual, nth-order extensions are straightforward.

How to combine the prediction errors in Definition 5 depends on the statistical analysis undertaken, but, motivated by Theorem 2, the essential idea is that all of them should be close to 0. We emphasize that a prediction error does not necessarily attain the value 0; cf. the 'leave-one-out' discussion in Cronie & van Lieshout (2018). Considering $\widetilde{\mathcal{I}}_i(\theta) = \{\widetilde{\mathcal{I}}_{i1}(\theta), \ldots, \widetilde{\mathcal{I}}_{ik_i}(\theta)\}^{\mathrm{T}}$ and $\mathcal{I}_i(\theta) = \{\mathcal{I}_{i1}(\theta), \ldots, \mathcal{I}_{ik_i}(\theta)\}^{\mathrm{T}}$, $i = 1, \ldots, k$, we see two natural choices.

A. Assuming that $k_i = k_0 \geqslant 1$, $i = 1, \ldots, k$, we generate a point estimate in Θ by minimizing a loss function $\mathcal{L}(\theta) = f_{\mathcal{L}}[\{\mathcal{I}_i(\theta) : i \in \mathcal{T}_k\}]$ for some suitable $f_{\mathcal{L}} : (\mathbb{R}^{k_0})^{\#\mathcal{T}_k} \to \mathbb{R}^{k_0}$.
B. Denote the estimate resulting from minimizing $\theta \mapsto \mathcal{I}_{ij}(\theta)^2$, $\theta \in \Theta$, by $\hat{\theta}_{ij} = \hat{\theta}\{(\mathbb{x}_i^T, \mathbb{x}_i^V), p, W, \Xi_\Theta, \mathcal{H}_\Theta^{ij}\} \in \Theta$ for $i \in \mathcal{T}_k$, $j = 1, \ldots, k_i$. The sample median and mean of these estimates may serve as point estimates, while empirical quantiles may serve as confidence/uncertainty regions for the true parameter $\theta_0 \in \Theta$. In addition, the mean/median of $\xi_{\hat{\theta}_{ij}}$, $i \in \mathcal{T}_k$, $j = 1, \ldots, k_i$, may serve as a point estimate of ξ_{θ_0} (cf. Moradi et al., 2019).

In § 5.2, where we focus on point process learning under p-thinning-based cross-validation, we restrict ourselves to the case where $\mathcal{H}_\Theta^{ij} = \mathcal{H}_\Theta = \{h_\theta : \theta \in \Theta\}$ for all i and j, i.e., we use one single test function family for all training-validation pairs, whereby $\widetilde{\mathcal{I}}_i(\theta) = \widetilde{\mathcal{I}}_{i1}(\theta) = I_i\mathcal{I}_{i1}(\theta) = I_i\mathcal{I}_i(\theta)$ and in case B we have $\hat{\theta}_i = \hat{\theta}_{i1}$, $i \in \mathcal{T}_k$. Moreover, we

consider the following choices for $f_{\mathcal{L}}$ in case A:

$$\mathcal{L}_j(\theta) = \frac{1}{k}\sum_{i=1}^{k} |\widetilde{\mathcal{I}}_i(\theta)|^j \propto \frac{1}{\#\mathcal{T}_k}\sum_{i\in\mathcal{T}_k} |\mathcal{I}_i(\theta)|^j, \qquad j = 1, 2, \tag{10}$$

$$\mathcal{L}_3(\theta) = \left\{\frac{1}{k}\sum_{i=1}^{k} \widetilde{\mathcal{I}}_i(\theta)\right\}^2 \propto \left\{\frac{1}{\#\mathcal{T}_k}\sum_{i\in\mathcal{T}_k} \mathcal{I}_i(\theta)\right\}^2; \tag{11}$$

that is, we find a parameter θ such that all prediction error terms are close to 0 in an average sense. We have observed that even if $\theta \mapsto |\mathcal{I}_i(\theta)|^j$ is unidentifiable, i.e., is flat over a region of Θ around the minimum, when averaging such functions as in (10), we generate loss functions with seemingly smaller flat regions, which implicitly mitigates the unidentifiability. Moreover, since positively weighted sums of convex functions and $x \mapsto |x|^j$, $j = 1, 2$, are convex, the form of $\mathcal{I}_i(\theta)$, which is governed by the test function, influences the convexity of, e.g., (10). We further have $\mathcal{L}_1(\theta)^2 \leqslant \mathcal{L}_2(\theta) \leqslant k\mathcal{L}_3(\theta)$, by Hölder's and Jensen's inequalities, with equality when $k = 1$.

Concerning test function choices, motivated by Baddeley et al. (2005) and Cronie & van Lieshout (2018), a simple recommendation is $h_\theta(\cdot) = f\{\xi_\theta(\cdot)\}$, where $f(x) = x^{-\gamma}$ and $\gamma = 1/2, 1$. We have indications that $\gamma = 1$, which conveniently sets the integrals in (8) and (9) to $|W|$ if ξ_θ is positive on W, yields the estimators with the lowest variances and mean (integrated) square errors of the two. In the Supplementary Material we provide an in-depth discussion on hyperparameter choices, in particular test function choices, and we introduce an algorithm for data-driven hyperparameter selection. We also present results on consistency and asymptotic normality of the estimators generated by (10) under p-thinning-based cross-validation.

5.2. *Numerical evaluation: bandwidth selection using p-thinning-based cross-validation*

We now apply our new theory to the problem of optimal bandwidth selection and numerically compare it to the approach of Cronie & van Lieshout (2018). In the Supplementary Material we apply our new bandwidth selection approach to the two datasets illustrated in Fig. 1, i.e., a point pattern of tree locations on Barro Colorado Island, Panama, and a point pattern of spines on one branch of the dendritic tree of a rat neuron.

We consider the p-thinning-based cross-validation approaches in Definition 3 and combine the loss functions $\mathcal{L}_j(\theta)$, $j = 1, 2, 3$, in (10)–(11) with the prediction errors in (8) and the indicator $I_i = \mathbb{1}(1 \leqslant \#\mathbb{x}_i^T \leqslant \#\mathbb{x} - 1)$. We here let $\xi_\theta(u; \mathbb{x}_i^T) = w\lambda_\theta(u; \mathbb{x}_i^T) = w\hat{\rho}_\theta(u; \mathbb{x}_i^T)$, where $w = p/(1-p)$ is an approximation of a parameterized form of the weight function $w(\cdot)$ in (7):

$$w_\theta(u, X_i^V, X_i^T) = \frac{E\{\hat{\rho}_\theta(u; X) \mid X_i^T\}}{\hat{\rho}_\theta(u; X_i^T)/p} = \frac{\hat{\rho}_\theta(u; X_i^T) + E\{\hat{\rho}_\theta(u; X_i^V) \mid X_i^T\}}{\hat{\rho}_\theta(u; X_i^T)/p} \approx p + \frac{p^2}{1-p},$$

since $\hat{\rho}_\theta(u; X) = \hat{\rho}_\theta(u; X_i^T) + \hat{\rho}_\theta(u; X_i^V)$. Following our general recommendation, we further let $h_\theta(u, \mathbb{x}_i^T) = f\{w\hat{\rho}_\theta(u; \mathbb{x}_i^T)\} = \{p\hat{\rho}_\theta(u; \mathbb{x}_i^T)/(1-p)\}^{-\gamma}$, $\gamma = 1/2, 1$, whereby

$$\mathcal{I}_i(\theta) = \sum_{x\in\mathbb{x}_i^V} \left\{\frac{p\hat{\rho}_\theta(x, \mathbb{x}_i^T)}{1-p}\right\}^{-\gamma} - \frac{p}{1-p}\int_W \left\{\frac{p\hat{\rho}_\theta(u, \mathbb{x}_i^T)}{1-p}\right\}^{-\gamma} \hat{\rho}_\theta(u, \mathbb{x}_i^T)\,\mathrm{d}u. \tag{12}$$

Table 1. *Results for the Cronie & van Lieshout (2018) approach for different processes*

	IAB	ISB	IV	MISE
Log-Gaussian Cox	19.48	963.47	17597.99	18561.47
Poisson	15.80	921.82	4408.21	5330.04
Determinantal point	9.14	276.75	2002.55	2279.31

IAB, integrated absolute bias; ISB, integrated square bias; IV, integrated variance; MISE, mean integrated square error.

We first focus on $\gamma = 1$, which sets the integral in (12) to $|W|$. In both this and the approach of Cronie & van Lieshout (2018), which is implemented in the R package `spatstat` (Baddeley et al., 2015), for a given realization $\mathbb{x}$, we let κ be the Gaussian kernel and $e_\theta(\cdot) \equiv 1$ in (3), and once the bandwidth $\hat{\theta}$ has been selected, the final intensity estimate $\hat{\rho}_{\hat{\theta}}$ is generated using local edge correction, i.e., $e_{\hat{\theta}}(u, x) = \int_W \kappa_{\hat{\theta}}(v - x)\,\mathrm{d}v$. We simulate 100 realizations $\mathbb{x}$ on $W = [0, 1]^2$ and find estimates $\hat{\rho}_{\hat{\theta}}(u)$ of the true intensity $\rho(u)$, $u = (u_1, u_2) \in W$, for a collection of models, a subset of the models in Cronie & van Lieshout (2018). For a given model and approach, we report estimates of the integrated absolute bias $\textsc{iab} = \int_W |\hat{E}\{\hat{\rho}_{\hat{\theta}}(u, X)\} - \rho(u)|\,\mathrm{d}u$, the integrated square bias $\textsc{isb} = \int_W [\hat{E}\{\hat{\rho}_{\hat{\theta}}(u, X)\} - \rho(u)]^2\,\mathrm{d}u$, the integrated variance $\textsc{iv} = \int_W \hat{\mathrm{var}}\{\hat{\rho}_{\hat{\theta}}(u, X)\}\,\mathrm{d}u$ and the mean integrated square error $\textsc{mise} = \textsc{isb} + \textsc{iv}$.

The models we consider, which represent different kinds of spatial interaction, are as follows.

(i) A log-Gaussian Cox process (aggregation) with random intensity $\Lambda(u) = \exp\{Z(u)\}$, where the Gaussian random field Z has mean function $(u_1, u_2) \mapsto 10 + 80u_1$ and covariance function $(u, v) \mapsto \sigma^2 \exp\{-r\|u - v\|_2\}$, $(\sigma^2, r) = (2 \log 5, 50)$; here $\rho(u) = (10 + 80u_1)\exp(\sigma^2/2)$ and $E\{X(W)\} = 250$.

(ii) A Poisson process (complete spatial randomness) with intensity function $\rho(u) = 10 + 480u_1$ and $E\{X(W)\} = 250$.

(iii) A homogeneous determinantal point process (inhibition) with kernel $(u, v) \mapsto \sigma^2 \exp\{-r\|u - v\|_2\}$, $(\sigma^2, \beta) = (250, 50)$, independently thinned with retention probability $u \mapsto (10 + 80u_1)/90$ to obtain an inhomogeneous version with $\rho(u) = \sigma^2(10 + 80u_1)/90$ and $E\{X(W)\} \approx 138.9$.

See Table 1 for the results for the Cronie & van Lieshout (2018) approach.

Turning to the new approach, with $\gamma = 1$, in Figs. 2 and 3 we respectively present the results for Monte Carlo cross-validation, with $p = 0.1, 0.3, 0.5, 0.7, 0.9$ and $k = 400$, and multinomial cross-validation, with $k = 2, 3, \ldots, 10$, based on the loss functions $\mathcal{L}_j(\theta)$, $j = 1, 2, 3$.

Comparing Figs. 2 and 3 with the results for Cronie & van Lieshout (2018)'s method, we see that, regardless of the choices of p, k and the model, all point process learning approaches outperform Cronie & van Lieshout (2018)'s method in terms of MISE. Although Cronie & van Lieshout (2018)'s method performs slightly better in terms of bias, it performs comparatively poorly in terms of variance, which is consequently the reason for its higher MISE; it is further precisely the lower variance that ensures that Cronie & van Lieshout (2018)'s approach outperforms its predecessors (Moradi et al., 2019). We do, however, hypothesize that if $p \to 0$ in the Monte Carlo cross-validation case, possibly in combination with $k \to \infty$, or $k \to \infty$ in the multinomial cross-validation case, e.g., in combination with $\mathcal{L}_3$, we would reach the same bias level as Cronie & van Lieshout (2018), but still with a significantly lower MISE. In the Monte Carlo cross-validation case, we further emphasize

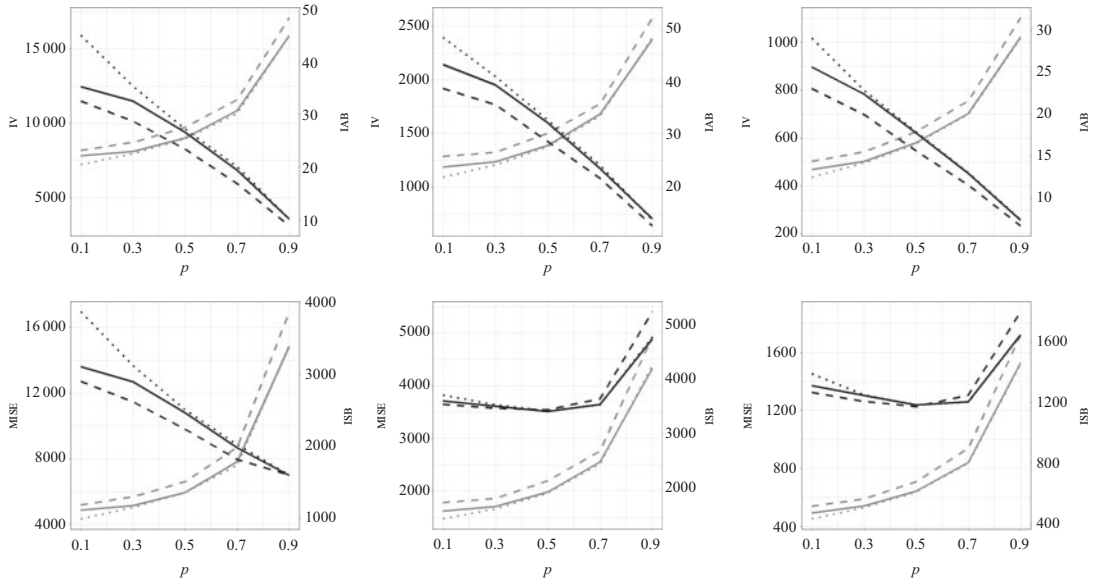

Fig. 2. Performance of the loss functions $\mathcal{L}_1$ (solid), $\mathcal{L}_2$ (dashed) and $\mathcal{L}_3$ (dotted), together with the test function $f(x) = 1/x$, using Monte Carlo cross-validation with $p = 0.1, 0.3, 0.4, 0.7, 0.9$ and $k = 400$. Columns: log-Gaussian Cox process (left), Poisson process (middle) and determinantal point process (right). Top row: integrated absolute bias (grey curve, right axis) and integrated variance (black curve, left axis). Bottom row: integrated square bias (grey curve, right axis) and mean integrated square error (black curve, left axis).

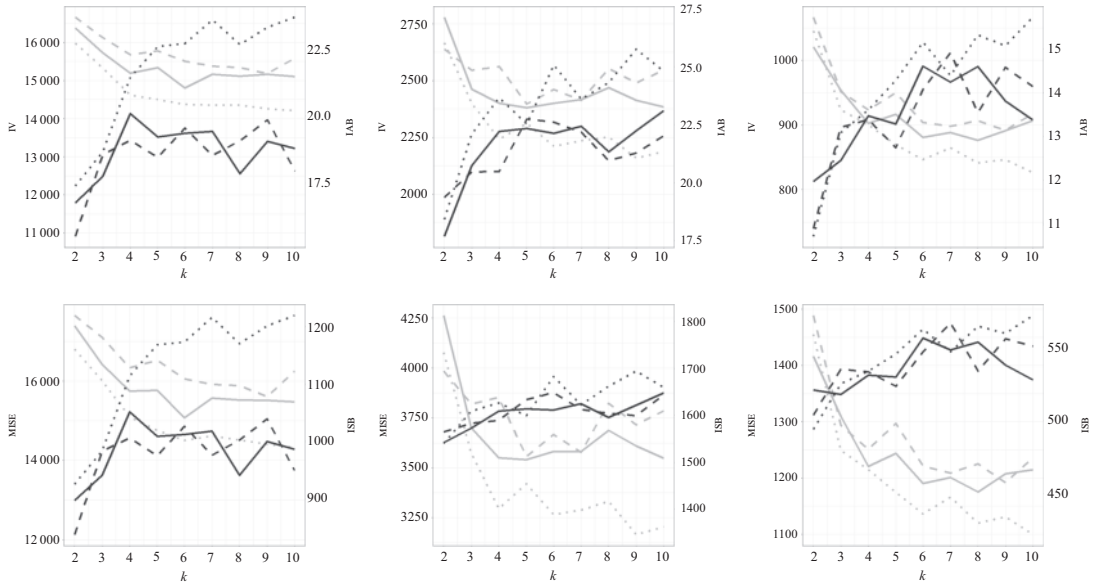

Fig. 3. The same structure as in Fig. 2, using multinomial cross-validation with $k = 2, 3, \ldots, 10$.

that increasing k beyond 100 essentially has little or no effect on the chosen performance measures, so our general suggestion is to fix $k \geqslant 100$. One alternative here is to sequentially increase k and stop once the loss function shows signs of convergence, guaranteed by the law of large numbers. We further conclude that multinomial cross-validation is the go-to

method if computational costs are the main priority, whereas Monte Carlo cross-validation is the go-to method if precision is prioritized; e.g., two-fold multinomial cross-validation is roughly 400 times faster than Monte Carlo cross-validation with $k = 400$ and $p = 0.5$. To exemplify, on a 2021 Apple MacBook with 64 GB RAM and an M1 processor, selecting the bandwidth for one realization of the aggregated model in the two-fold multinomial cross-validation case takes approximately 2.5 s with our current implementation, versus 0.02 s for the `spatstat` implementation of the Cronie & van Lieshout (2018) approach; neither makes use of parallelization.

It seems that $\mathcal{L}_3$ favours a lower bias over a lower variance or mean integrated square error, whereas $\mathcal{L}_2$ favours the opposite and $\mathcal{L}_1$ seems to offer some middle ground between the two. In Fig. 2 we further see that in the case of Monte Carlo cross-validation, $p \in [0.5, 0.7]$ tends to be a safe choice, which balances the trade-off between bias and variance, irrespectively of the degree of aggregation of the underlying model. Moreover, it seems that the performance of multinomial cross-validation in terms of mean integrated square error is the best when $k = 2$, see Fig. 3, which is equivalent to Monte Carlo cross-validation with $p = 0.5, k = 1$.

To shed some light on the choice of test function, in the Supplementary Material we also explore $\gamma = 1/2$. The overall conclusions regarding the hyperparameters p and k are the same as for $\gamma = 1$. Moreover, in Fig. S1 in the Supplementary Material, which illustrates the same set-up as in Fig. 2, but with only the loss function $\mathcal{L}_2$, we see that the new approach performs much better than the state-of-the-art method in terms of MISE. Comparing the two test function choices, Fig. 2 and Fig. S1 within the Supplementary Material, reveals that $\gamma = 1/2$ reduces the bias with respect to $\gamma = 1$, but at the cost of increased variance, and thus also mean integrated square error. In the Supplementary Material we let our data-driven hyperparameter selection algorithm select k in the case of multinomial cross-validation, which yields a performance essentially on par with our rule of thumb, $k = 2$. On the other hand, in the case of Monte Carlo cross-validation, the performance essentially corresponds to keeping $p = 0.9$ fixed; for computational reasons, we here fixed $k = 100$. This is suboptimal to our rule of thumb, i.e., $p \in [0.5, 0.7]$; the gain in the mean integrated square error from performing well with the log-Gaussian Cox process, which corresponds to fixing $p = 0.9$, is much bigger than the loss in the mean integrated square error from performing relatively poorly with the other models. We thus see that there is merit to our algorithm in the Supplementary Material, in particular since we in practice have no knowledge of the underlying process.

6. Discussion

As our new approach outperforms the method of Cronie & van Lieshout (2018) in non-parametric intensity estimation, framed as a conditional intensity estimation problem, it will likely have a central role in the future of point process statistics. We have indicated a few classical statistical approaches for parametric modelling of conditional intensities, which are based on univariate prediction errors. We believe that these could be improved by reframing them within our new framework.

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Supplementary material

The Supplementary Material includes additional background material, alternative cross-validation procedures, additional plots for the simulation study, hyperparameter selection, higher-order statements, proofs of the results in the main text, examples of kernel intensity estimation for two datasets, in a Euclidean domain and on a linear network, and different asymptotic results.

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