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# Nowhere scattered $C^*$ -algebras

Hannes Thiel and Eduard Vilalta

**Abstract.** We say that a  $C^*$ -algebra is *nowhere scattered* if none of its quotients contains a minimal open projection. We characterize this property in various ways, by topological properties of the spectrum, by divisibility properties in the Cuntz semigroup, by the existence of Haar unitaries for states, and by the absence of nonzero ideal-quotients that are elementary, scattered or type I. Under the additional assumption of real rank zero or stable rank one, we show that nowhere scatteredness implies even stronger divisibility properties of the Cuntz semigroup.

## 1. Introduction

A topological space is said to be *scattered* if each of its nonempty closed subsets contains an isolated point. Analogously, a  $C^*$ -algebra is *scattered* if each of its nonzero quotients contains a minimal open projection; see Section 2. Here, a *minimal open projection* in a  $C^*$ -algebra  $A$  is a projection  $p \in A$  such that  $pAp = \mathbb{C}p$ ; see Paragraph 2.1. In this paper, we consider  $C^*$ -algebras that are very far from scattered.

**Definition A.** A  $C^*$ -algebra is *nowhere scattered* if none of its quotients contains a minimal open projection.

Similarly, we say that a topological space is *nowhere scattered* if none of its closed subsets contains an isolated point; see Definition 5.1. Thus, one-element sets cannot be closed, and nowhere scattered spaces are far from being  $T_1$ , let alone Hausdorff. In Section 5, we study this topological notion and the relation to its noncommutative counterpart. We prove that a separable  $C^*$ -algebra is nowhere scattered if and only if its spectrum is; see Theorem 5.3.

Besides this topological description, we also show that nowhere scatteredness admits various further characterizations: in terms of the structure of ideal-quotients (Section 3), by the existence of Haar unitaries and maximal abelian subalgebras with diffuse spectrum (Section 6), and by divisibility properties of the Cuntz semigroup (Section 8). We present a number of these characterizations in Theorem B below.

For  $C^*$ -algebras of real rank zero, nowhere scatteredness can be described in terms of divisibility properties of the Murray–von Neumann semigroup of projections; see The-

orem 9.1. However, in contrast with the Cuntz semigroup, this invariant does not contain enough information to characterize nowhere scatteredness in the general setting; see Remark 9.2. In Theorem 9.11, we also provide additional characterizations of nowhere scatteredness for  $C^*$ -algebras of stable rank one.

**Theorem B** (Theorem 3.1, Theorem 6.2, Theorem 8.9). Let  $A$  be a  $C^*$ -algebra. Then the following are equivalent:

- (1)  $A$  is nowhere scattered;
- (2) every quotient of  $A$  is antiliminal;
- (3)  $A$  has no nonzero elementary/scattered/type I ideal-quotients;
- (4) no hereditary sub- $C^*$ -algebra of  $A$  admits a one-dimensional (a finite-dimensional) irreducible representation;
- (5) every positive functional (pure state) on  $A$  is nowhere scattered;
- (6) for every pure state  $\varphi$  on  $A$  and every ideal  $I \subseteq A$ , there exists a Haar unitary in  $\tilde{I}$  for  $\varphi$  (there exists a masa  $C_0(X) \subseteq I$  such that  $\varphi$  induces a diffuse measure on  $X$  with total mass  $\|\varphi|_I\|$ );
- (7) for every positive functional  $\varphi$  on  $A$  and every hereditary sub- $C^*$ -algebra  $B \subseteq A$ , there exists a Haar unitary in  $\tilde{B}$  for  $\varphi$  (there exists a masa  $C_0(X) \subseteq B$  such that  $\varphi$  induces a diffuse measure on  $X$  with total mass  $\|\varphi|_B\|$ );
- (8) every element in  $\text{Cu}(A)$  is weakly  $(2, \omega)$ -divisible;
- (9) every element in  $\text{Cu}(A)$  is weakly  $(k, \omega)$ -divisible for every  $k \geq 2$ .

In Section 4, we study permanence properties of such  $C^*$ -algebras.

**Proposition C** (Proposition 4.1, Proposition 4.6). Nowhere scatteredness passes to hereditary sub- $C^*$ -algebras (in particular, ideals), to quotients, and to inductive limits.

Further, nowhere scatteredness satisfies the Löwenheim–Skolem condition: For every nowhere scattered  $C^*$ -algebra, the collection of *separable*, nowhere scattered sub- $C^*$ -algebras is  $\sigma$ -complete and cofinal among all separable sub- $C^*$ -algebras; see Proposition 4.11.

The concept of nowhere scattered  $C^*$ -algebras has implicitly appeared in the literature before, but not under this name (or any name for that matter). For example, our characterizations in terms of divisibility properties of  $\text{Cu}(A)$  can also be derived from results in [36]; see Remark 8.10. Nowhere scattered, real rank zero  $C^*$ -algebras have been considered in [17, 33].

The term ‘nowhere scattered’ was first used in [44] to name positive functionals on  $C^*$ -algebras that give no weight to scattered ideal-quotients. It is the purpose of this paper to initiate a systematic study of nowhere scatteredness by collecting, systematizing and complementing existing results.

### Terminology and notation

An ideal in a  $C^*$ -algebra means a closed, two-sided ideal. Given a  $C^*$ -algebra  $A$ , we use  $A_+$  to denote the positive elements in  $A$ . Given a Hilbert space  $H$ , we let  $\mathcal{B}(H)$  denote the  $C^*$ -algebra of bounded, linear operators on  $H$ , and  $\mathcal{K}(H)$  is the ideal of compact operators.

## 2. Scattered $C^*$ -algebras

Following [22, Definition 2.1], a  $C^*$ -algebra is said to be *scattered* if each of its states is atomic, that is, a countable weighted sum of pure states. This definition is inspired by a result of Pełczyński and Semadeni, [31], which states that a compact, Hausdorff space  $X$  is scattered if and only if every regular Borel probability measure on  $X$  is atomic. Scatteredness admits various characterizations, both for topological spaces and for  $C^*$ -algebras – and several of these descriptions are remarkably similar in the two settings; see Paragraph 2.2.

Note that a locally compact, Hausdorff space  $X$  is scattered if and only if its one-point compactification is, and this is in turn equivalent to  $C_0(X)$  being a scattered  $C^*$ -algebra. Thus, it follows from [27, Lemma 2.2] that a  $C^*$ -algebra  $A$  is scattered if and only if for each commutative sub- $C^*$ -algebra  $C_0(X) \subseteq A$  the space  $X$  is scattered.

**2.1.** By a *minimal open projection* in a  $C^*$ -algebra  $A$  we mean a nonzero projection  $p \in A$  such that  $pAp = \mathbb{C}p$ . In some parts of the literature, such projections are simply called ‘minimal projections’, for example [19, Definition 1.1]. We do not follow this convention since it could potentially lead to confusion: one could understand a minimal projection in a  $C^*$ -algebra to be a nonzero projection  $p$  such that every subprojection  $q$  of  $p$  satisfies  $q = 0$  or  $q = p$ , and this would include projections like the unit of the Jiang–Su algebra.

Let us justify our terminology: An *open projection* in a  $C^*$ -algebra  $A$  is a projection  $p \in A^{**}$  such that there exists an increasing net in  $A_+$  that converges in the weak\*-topology of  $A^{**}$  to  $p$ . Then  $pA^{**}p \cap A$  is a hereditary sub- $C^*$ -algebra of  $A$ , and this induces a natural bijection between open projections in  $A$  and hereditary sub- $C^*$ -algebras of  $A$ . For details we refer to [30, p. 77f].

*Claim:* Let  $p \in A^{**}$  be an open projection. Then  $p$  belongs to  $A$  and satisfies  $pAp = \mathbb{C}p$  if and only if  $p$  is minimal in the sense that for every open projection  $q \in A^{**}$  with  $q \leq p$  we have  $q = 0$  or  $q = p$ . Indeed, the forward implication is clear. Conversely, assume that  $p$  is minimal in the above sense and set  $B := pA^{**}p \cap A$ . If  $x, y \in B$  satisfy  $xy = 0$ , then  $x^*x$  and  $yy^*$  are orthogonal, positive elements, whose support projections are orthogonal open subprojections of  $p$ . Using minimality of  $p$ , we deduce that  $x^*x = 0$  or  $yy^* = 0$ , which shows that  $B$  has no zero divisors and thus  $B \cong \mathbb{C}$  by the Gelfand–Mazur theorem. This implies that  $B$  is unital and thus  $p \in A$ , and also  $pAp = \mathbb{C}p$ .

**2.2.** Let  $A$  be a  $C^*$ -algebra ( $X$  be a locally compact, Hausdorff space). Then, the following are equivalent:

- (1)  $A$  is *scattered*, that is, every state on  $A$  is atomic; (every regular Borel probability measure on  $X$  is atomic, [31]);
- (2)  $A^{**}$  is an atomic von Neumann algebra, that is,  $A^{**}$  is isomorphic to a product of type I factors, [22, Theorem 2.2];
- (3)  $A^*$  has the Radon–Nikodým property, [12, Theorem 3];
- (4) for every separable sub- $C^*$ -algebra  $B \subseteq A$ , the dual space  $B^*$  is separable – to see this, note that (3) implies (4) by [12, Theorem 1], and that (4) implies (12) since  $C_0((0, 1])$  is a separable  $C^*$ -algebra with nonseparable dual space;
- (5)  $A$  admits a composition series  $(I_\lambda)_\lambda$  such that each successive quotient  $I_{\lambda+1}/I_\lambda$  is elementary, [23, Theorem 2];
- (6)  $A$  is type I and  $\widehat{A}$  is scattered, [23, Corollary 3];
- (7)  $A$  is type I and the center of  $A$  is scattered, [26, Theorem 2.2];
- (8) every sub- $C^*$ -algebra of  $A$  has real rank zero, [27, Theorem 2.3]; (every continuous image of  $X$  is zero-dimensional, [31]);
- (9) every nonzero sub- $C^*$ -algebra of  $A$  contains a minimal open projection, [19, Theorem 1.4];
- (10) the spectrum of every selfadjoint element in  $A$  is countable, [21]; (every continuous function  $X \rightarrow \mathbb{R}$  has countable range, [39, Corollary 8.5.6]);
- (11) every quotient of  $A$  contains a minimal open projection, [19, Theorem 1.4]; ( $X$  is *scattered*, that is, every closed subset of  $X$  contains an isolated point);
- (12) there exists no sub- $C^*$ -algebra  $C_0((0, 1]) \subseteq A$ , [19, Theorem 1.4]; (there is no continuous, surjective map  $X \rightarrow [0, 1]$ , [31]).

### 3. Nowhere scattered $C^*$ -algebras

In this section, we prove basic characterizations of nowhere scatteredness; see Theorem 3.1. We observe that (weakly) purely infinite  $C^*$ -algebras are nowhere scattered, and we show that a von Neumann algebra is nowhere scattered if and only if its type I summand is zero; see Example 3.3 and Proposition 3.4 respectively.

A  $C^*$ -algebra is *elementary* if it is isomorphic to the algebra of compact operators on some Hilbert space. An *ideal-quotient* of a  $C^*$ -algebra  $A$  is a (closed, two-sided) ideal of a quotient of  $A$ . A relatively open subset of a closed subset of a topological space is said to be *locally closed*. Using the correspondence between ideals (quotients) of  $A$  and open (closed) subsets of its primitive ideal space  $\text{Prim}(A)$ , it follows that ideal-quotients of  $A$  correspond to locally closed subsets of  $\text{Prim}(A)$ .

An element  $a \in A_+$  is *abelian* if the hereditary sub- $C^*$ -algebra  $\overline{aAa}$  is commutative. A  $C^*$ -algebra is *type I* if every nonzero quotient contains a nonzero, abelian element; see [8, Definition IV.1.1.6]. A  $C^*$ -algebra is *antiliminal* if it contains no nonzero abelian positive elements; see [8, Definition IV.1.1.6]. An irreducible representation  $\pi: A \rightarrow \mathcal{B}(H)$  is said to be *GCR* if  $\pi(A) \cap \mathcal{K}(H) \neq \{0\}$ .

**Theorem 3.1.** *Let  $A$  be a  $C^*$ -algebra. Then the following are equivalent:*

- (1)  *$A$  is nowhere scattered (no quotient contains a minimal open projection);*
- (2) *every quotient of  $A$  is antiliminal;*
- (3)  *$A$  has no nonzero ideal-quotients of type I;*
- (4)  *$A$  has no nonzero scattered ideal-quotients;*
- (5)  *$A$  has no nonzero elementary ideal-quotients;*
- (6)  *$A$  has no nonzero irreducible GCR representation;*
- (7) *no hereditary sub- $C^*$ -algebra of  $A$  admits a finite-dimensional irreducible representation;*
- (8) *no hereditary sub- $C^*$ -algebra of  $A$  admits a one-dimensional irreducible representation.*

*Proof.* Let us first prove that (1) implies (2). Assume that there exists an ideal  $I \subseteq A$  such that  $A/I$  is not antiliminal. Set  $B := A/I$  and choose a nonzero, abelian element  $b \in B_+$ . This allows us to obtain an ideal  $J \subseteq \overline{bBb}$  such that  $\overline{bBb}/J \cong \mathbb{C}$ . Let  $K \subseteq B$  be the ideal generated by  $J$ . Since  $\overline{bBb} \subseteq B$  is hereditary, we have  $K \cap \overline{bBb} = J$ . It follows that the image of  $\overline{bBb}$  under the quotient map  $B \rightarrow B/K$  is isomorphic to  $\overline{bBb}/J$ , and so  $B/K$  contains a minimal open projection, which contradicts (1), as required.

To show that (2) implies (3), let  $I \subseteq J \subseteq A$  be ideals such that  $J/I$  is nonzero and of type I. Then  $J/I$  contains a nonzero, abelian element, whence  $A/I$  is not antiliminal. The implications '(3) $\Rightarrow$ (4) $\Rightarrow$ (5)' follow using that every elementary  $C^*$ -algebra is scattered, and that every scattered  $C^*$ -algebra is type I.

Now assume that there exists a nonzero irreducible GCR representation  $\pi: A \rightarrow \mathcal{B}(H)$ . Then  $\mathcal{K}(H) \subseteq \pi(A)$  by [8, Corollary IV.1.2.5]. Let  $I$  be the kernel of  $\pi$ , and set  $J := \pi^{-1}(\mathcal{K}(H))$ . Then  $I \subseteq J \subseteq A$  are ideals such that  $J/I \cong \mathcal{K}(H)$ . This proves that (5) implies (6).

To see that (6) implies (7), let  $B \subseteq A$  be a hereditary sub- $C^*$ -algebra, and let  $\pi_0: B \rightarrow \mathcal{B}(H_0)$  be a nonzero, finite-dimensional irreducible representation. Extend  $\pi_0$  to an irreducible representation  $\pi: A \rightarrow \mathcal{B}(H)$  on some Hilbert space  $H$  containing  $H_0$ , such that  $H_0$  is invariant under  $\pi(B)$  and such that  $\pi(b)\xi = \pi_0(b)\xi$  for all  $b \in B$  and  $\xi \in H_0$ ; see [8, Proposition II.6.4.11].

Let  $H_1$  be the essential subspace of  $\pi|_B$ , that is, the closed linear subspace of  $H$  generated by  $\pi(B)H$ . By [8, Proposition II.6.1.9], the restriction of  $\pi|_B$  to  $H_1$  is irreducible. Since  $H_0$  is a closed subspace of  $H_1$  that is invariant under  $\pi(B)$ , it follows that  $H_0 = H_1$ .

Let  $p \in \mathcal{B}(H)$  denote the orthogonal projection onto  $H_0$ . Then  $p\pi(b) = \pi(b)$  and therefore  $\pi(b) = \pi(b)p = p\pi(b)p$  for every  $b \in B$ . Thus,  $\pi(B) \subseteq \mathcal{K}(H)$ , and it follows that  $\pi$  is GCR.

The implication '(7) $\Rightarrow$ (8)' is clear. Finally, to see that (8) implies (1), let  $I \subseteq A$  be an ideal and let  $p \in A/I$  be a minimal open projection. Let  $\pi: A \rightarrow A/I$  be the quotient map. Set  $B := \pi^{-1}(Cp)$ . Then  $B$  is a hereditary sub- $C^*$ -algebra of  $A$  that admits a nonzero, one-dimensional representation. ■

**Example 3.2.** A simple  $C^*$ -algebra is nowhere scattered if and only if it is not elementary. In particular, a unital simple  $C^*$ -algebra is nowhere scattered if and only if it is infinite-dimensional.

A  $C^*$ -algebra  $A$  is *purely infinite* (in the sense of Kirchberg–Rørdam, [24, Definition 4.1]) if and only if every element  $x$  of its Cuntz semigroup  $\text{Cu}(A)$ , as defined in Paragraph 7.1, satisfies  $2x = x$ ; see [24, Theorem 4.16] and [4, Proposition 7.2.8].

More generally,  $A$  is said to be *weakly purely infinite* if there exists  $n \in \mathbb{N}$  such that every  $x \in \text{Cu}(A)$  satisfies  $2nx = nx$ ; see [25, Definition 4.3]. Every purely infinite  $C^*$ -algebra is weakly purely infinite, and it is an open problem if the converse holds.

A  $C^*$ -algebra  $A$  is *traceless* if no algebraic ideal of  $A$  admits a nonzero quasitrace; see [25, Definition 4.2]. Equivalently, every lower-semicontinuous 2-quasitrace  $A_+ \rightarrow [0, \infty]$  (in the sense of [16]) takes only values in  $\{0, \infty\}$ ; see [9, Remark 2.27 (viii)]. Using the correspondence between quasitraces on  $A$  and functionals on  $\text{Cu}(A)$ , this is also equivalent to the property that every  $x \in \text{Cu}(A)$  satisfies  $2\hat{x} = \hat{x}$ . Consequently, every weakly purely infinite  $C^*$ -algebra is traceless; see [25, Theorem 4.8].

**Example 3.3.** Since every elementary  $C^*$ -algebra admits a nontrivial quasitrace, it follows that a traceless  $C^*$ -algebra cannot have nonzero elementary ideal-quotients. Using Theorem 3.1, we deduce that every traceless  $C^*$ -algebra is nowhere scattered. In particular, every (weakly) purely infinite  $C^*$ -algebra is nowhere scattered.

**Proposition 3.4.** A von Neumann algebra is nowhere scattered if and only if its type I summand is zero.

*Proof.* Let  $M$  be a von Neumann algebra. To show the forward implication, assume that  $M$  is nowhere scattered. If  $p \in M$  is a nonzero abelian projection, then the hereditary sub- $C^*$ -algebra  $pMp$  is commutative and therefore admits a one-dimensional irreducible representation. Thus, it follows from Theorem 3.1 that  $M$  contains no nonzero abelian projections, and thus its type I summand is zero.

To show the converse implication, assume that the type I summand of  $M$  is zero. To reach a contradiction, assume that  $B \subseteq M$  is a hereditary sub- $C^*$ -algebra that admits a finite-dimensional, irreducible representation  $\pi$ ; see Theorem 3.1. Choose a projection  $p \in B$  such that  $\pi(p) \neq 0$ . Since  $M$  has no type I summand, for each  $n \geq 1$  we can find  $2^n$  pairwise orthogonal and equivalent projections whose sum is equal to  $p$ ; see [42, Proposition V.1.35]. This contradicts that  $\pi(p)$  has nonzero, finite rank. ■

## 4. Permanence properties

In this section, we show that nowhere scatteredness enjoys many permanence properties. In particular, it passes to hereditary sub- $C^*$ -algebras (hence, ideals), quotients, and inductive limits.

**Proposition 4.1.** *Let  $A$  be a nowhere scattered  $C^*$ -algebra. Then every quotient and every hereditary sub- $C^*$ -algebra of  $A$  is nowhere scattered.*

*Proof.* This follows from Theorem 3.1, since condition (2) passes to quotients, and condition (7) passes to hereditary sub- $C^*$ -algebras. ■

**Proposition 4.2.** *Let  $A$  be a  $C^*$ -algebra, and let  $I \subseteq A$  be an ideal. Then  $A$  is nowhere scattered if (and only if)  $I$  and  $A/I$  are nowhere scattered.*

*Proof.* Assume that  $I$  and  $A/I$  are nowhere scattered. To verify condition (4) of Theorem 3.1, let  $J \subseteq K \subseteq A$  be ideals such that  $K/J$  is scattered. Then  $I \cap J \subseteq I \cap K \subseteq I$  are ideals such that  $(I \cap K)/(I \cap J)$  is isomorphic to an ideal of  $K/J$  and is therefore scattered. Since  $I$  is nowhere scattered, we get  $I \cap J = I \cap K$ .

Similarly,  $J/I \cap J \subseteq K/I \cap K \subseteq A/I$  are ideals such that  $(J/I \cap J)/(K/I \cap K)$  is isomorphic to a quotient of  $J/K$  and therefore scattered. Since  $A/I$  is nowhere scattered, we get  $J/I \cap J = K/I \cap K$ . It follows that  $J = K$ . ■

A  $C^*$ -algebra is scattered if and only if each of its sub- $C^*$ -algebras has real rank zero; see Paragraph 2.2. It follows that every sub- $C^*$ -algebra of a scattered  $C^*$ -algebra is again scattered. The analog for nowhere scatteredness does not hold: Take, for example,  $\mathbb{C} \subseteq A$  in any unital, nowhere scattered  $C^*$ -algebra  $A$ .

Conversely, given a full sub- $C^*$ -algebra  $B \subseteq A$ , it is natural to ask if  $A$  is nowhere scattered whenever  $B$  is. Without extra assumptions, this fails: Consider for example a type II factor  $M \subseteq \mathcal{B}(H)$ . By Proposition 3.4,  $M$  is nowhere scattered, but  $\mathcal{B}(H)$  contains an elementary ideal and therefore is *not* nowhere scattered. The ‘right’ additional condition is for  $B$  to *separate the ideals* of  $A$ , that is, two ideals  $I, J \subseteq A$  satisfy  $I = J$  whenever  $I \cap B = J \cap B$ .

**Proposition 4.3.** *Let  $A$  be  $C^*$ -algebra, and let  $B \subseteq A$  be a nowhere scattered sub- $C^*$ -algebra that separates the ideals of  $A$ . Then  $A$  is nowhere scattered.*

*Proof.* Let  $I \subseteq J \subseteq A$  be ideals with  $J/I$  scattered. In order to prove condition (4) in Theorem 3.1, we show that this ideal-quotient is zero.

Note that  $(J \cap B)/(I \cap B)$  is scattered since it is isomorphic to a subalgebra of  $J/I$ . Since  $B$  is nowhere scattered, it follows that  $J \cap B = I \cap B$ , and thus  $J = I$ , as desired. ■

**Example 4.4.** Consider an action of an exact, discrete group  $G$  on a nowhere scattered  $C^*$ -algebra  $A$ . Assume that the induced action  $G \curvearrowright \widehat{A}$  is essentially free. Then, by [41, Theorem 1.20],  $A$  separates the ideals of the reduced crossed product  $A \rtimes_{\text{red}} G$ . Hence,  $A \rtimes_{\text{red}} G$  is nowhere scattered by Proposition 4.3.

Given a  $C^*$ -algebra  $A$ , recall that a family  $(A_\lambda)_{\lambda \in \Lambda}$  of sub- $C^*$ -algebras  $A_\lambda \subseteq A$  is said to *approximate*  $A$  if for every finite subset  $\{a_1, \dots, a_n\} \subseteq A$  and  $\varepsilon > 0$  there exists  $\lambda \in \Lambda$  and  $b_1, \dots, b_n \in A_\lambda$  such that  $\|a_k - b_k\| < \varepsilon$  for  $k = 1, \dots, n$ .

**Proposition 4.5.** *Let  $A$  be  $C^*$ -algebra, and let  $(A_\lambda)_{\lambda \in \Lambda}$  be a family of nowhere scattered sub- $C^*$ -algebras of  $A$  that approximates  $A$ . Then  $A$  is nowhere scattered.*

*Proof.* As above, let us prove condition (4) of Theorem 3.1. Take  $I \subseteq J \subseteq A$  ideals such that  $J/I$  is scattered. For each  $\lambda$ , the ideal-quotient  $(A_\lambda \cap J)/(A_\lambda \cap I)$  of  $A_\lambda$  is isomorphic to a subalgebra of  $J/I$  and therefore scattered. Since  $A_\lambda$  is nowhere scattered, we get  $A_\lambda \cap J = A_\lambda \cap I$ .

To verify that  $J = I$ , let  $a \in J_+$ . Given  $\varepsilon > 0$ , use that  $(A_\lambda)_{\lambda \in \Lambda}$  approximates  $A$  and a functional calculus argument to obtain  $\lambda$  and a positive element  $b \in A_\lambda$  such that  $\|a - b\| < \varepsilon$ . Consider the  $\varepsilon$ -cut-down  $(b - \varepsilon)_+$  as in Paragraph 7.1. Then there exists  $r \in A$  such that  $(b - \varepsilon)_+ = rar^*$ ; see, for example, [43, Lemma 2.29]. Since  $J$  is an ideal, and since  $A_\lambda$  is closed under functional calculus, we deduce that  $(b - \varepsilon)_+$  belongs to  $A_\lambda \cap J$ , and thus to  $I$ . We have  $\|a - (b - \varepsilon)_+\| < 2\varepsilon$ , and since  $\varepsilon > 0$  was arbitrary, we deduce that  $a \in I$ , as desired. ■

**Proposition 4.6.** *An inductive limit of nowhere scattered  $C^*$ -algebras is nowhere scattered.*

*Proof.* Let  $\Lambda$  be a directed set, let  $(A_\lambda)_{\lambda \in \Lambda}$  be a family of nowhere scattered  $C^*$ -algebras, and let  $\varphi_{\kappa, \lambda}: A_\lambda \rightarrow A_\kappa$  be coherent connecting morphisms for  $\lambda \leq \kappa$  in  $\Lambda$ . The inductive limit of this system is given by a  $C^*$ -algebra  $A$  together with morphisms into the limit  $\varphi_\lambda: A_\lambda \rightarrow A$ . For each  $\lambda$ , set  $B_\lambda := \varphi_\lambda(A_\lambda) \subseteq A$ . Then  $B_\lambda$  is a quotient of  $A_\lambda$ , and therefore is nowhere scattered by Proposition 4.1. It follows from standard properties of the inductive limit that  $(B_\lambda)_\lambda$  approximates  $A$ . Hence,  $A$  is nowhere scattered by Proposition 4.5. ■

**Lemma 4.7.** *Let  $A$  be a  $C^*$ -algebra, and let  $a \in A_+$ . Then  $\overline{aAa}$  has no one-dimensional irreducible representations if and only if there exists a countable subset  $G \subseteq A$  such that  $(axa)^2 = 0$  for each  $x \in G$ , and such that  $a \in C^*(\{axa : x \in G\})$ .*

*Proof.* Without loss of generality, we may assume that  $a$  is strictly positive and thus  $A = \overline{aAa}$ . To show the backward implication, assume that  $G$  has the stated properties, and let  $\pi: A \rightarrow \mathbb{C}$  be a one-dimensional representation. Given  $x \in G$ , we have  $\pi(axa)^2 = \pi((axa)^2) = 0$ , and therefore  $\pi(axa) = 0$ . It follows that  $\pi(a) = 0$ , and thus  $\pi = 0$ , as desired.

To show the forward implication, assume that  $A$  has no one-dimensional irreducible representations. Set  $N_2 := \{y \in A : y^2 = 0\}$ . An additive commutator in  $A$  is an element of the form  $cd - dc$  for some  $c, d \in A$ . Let  $L$  denote the linear span of the additive commutators in  $A$ . Then,  $A$  is generated by  $L$  as a (closed, two-sided) ideal. By [35, Theorem 1.3], the  $C^*$ -algebra generated by  $L$  agrees with the (closed, two-sided) ideal generated by  $L$ . Hence,  $A = C^*(L)$ . By [35, Corollary 2.3], the closure of  $L$  agrees with the closed, linear span of  $N_2$ , which implies that  $A = C^*(N_2)$ . This allows us to choose a countable subset  $F \subseteq N_2$  such that  $a \in C^*(F)$ . The statement now follows from the following claim.

*Claim:* Let  $y \in N_2$  and  $\gamma > 0$ . Then there exists  $x \in A$  such that  $axa \in N_2$  and  $\|y - axa\| < \gamma$ .

We may assume that  $\|y\| \leq 1$ . By [40, Theorem 5], the universal  $C^*$ -algebra generated by a contractive, square-zero element is projective. This implies that the corresponding relations are stable (see for example [28, Theorem 14.1.4]), that is, for every  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that if  $z$  is an element in a  $C^*$ -algebra satisfying  $\|z\| \leq 1$  and  $\|z^2\| \leq \delta$ , then there exists a contractive, square-zero element  $z'$  with  $\|z - z'\| < \varepsilon$ .

For each  $n \geq 1$ , let  $f_n: \mathbb{R} \rightarrow [0, 1]$  be a continuous function that takes the value 0 on  $(-\infty, \frac{1}{n}]$  and that takes the value 1 on  $[\frac{2}{n}, \infty)$ . Then  $(f_n(a))_n$  is an approximate unit for  $A$ . Thus,  $\lim_{n \rightarrow \infty} \|(f_n(a)yf_n(a))^2\| = 0$ . Thus, we can find  $m \geq 1$  such that

$$\|(f_m(a)yf_m(a))^2\| < \delta\left(\frac{\gamma}{2}\right) \quad \text{and} \quad \|y - f_m(a)yf_m(a)\| < \frac{\gamma}{2}.$$

We view  $f_m(a)yf_m(a)$  as an element of the sub- $C^*$ -algebra  $\overline{f_m(a)Af_m(a)}$ . Using also that  $f_m(a)yf_m(a)$  is contractive, we obtain  $z \in \overline{f_m(a)Af_m(a)}$  such that  $z^2 = 0$  and  $\|z - f_m(a)yf_m(a)\| < \frac{\gamma}{2}$ . Then  $\|y - z\| < \gamma$ .

It remains to show that  $z = axa$  for some  $x \in A$ . Note that  $f_{2m}(a)$  acts as a unit on  $\overline{f_m(a)Af_m(a)}$ . Using functional calculus, choose  $b \in A$  such that  $f_{2m}(a) = ab = ba$ . We obtain

$$z = f_{2m}(a)zf_{2m}(a) = abzba,$$

and thus  $x := bzb$  has the claimed properties. ■

**Proposition 4.8.** *Let  $A$  be a  $C^*$ -algebra. Then the following are equivalent:*

- (1)  $A$  is nowhere scattered;
- (2) for every  $a \in A_+$  there exists  $b \in A_+$  with  $\|a - b\| < \frac{1}{2}$  and such that

$$\overline{(b - \frac{1}{2})_+ A (b - \frac{1}{2})_+}$$

*has no one-dimensional irreducible representations;*

- (3) for every  $a \in A_+$  there exist  $b \in A_+$  with  $\|a - b\| < \frac{1}{2}$  and a countable subset  $G \subseteq A$  such that  $((b - \frac{1}{2})_+ x (b - \frac{1}{2})_+)^2 = 0$  for each  $x \in G$  and such that  $(b - \frac{1}{2})_+ \in C^*(\{(b - \frac{1}{2})_+ x (b - \frac{1}{2})_+ : x \in G\})$ .

*Proof.* If  $A$  is nowhere scattered, then by Theorem 3.1 every hereditary sub- $C^*$ -algebra of  $A$  has no one-dimensional irreducible representations. This shows that (1) implies (2). The equivalence between (2) and (3) follows from Lemma 4.7.

It remains to show that (2) implies (1). Thus, assume that (2) holds, and note that it suffices to verify statement (8) of Theorem 3.1. So let  $B \subseteq A$  be a hereditary sub- $C^*$ -algebra. To reach a contradiction, let  $\pi: B \rightarrow \mathbb{C}$  be a one-dimensional irreducible representation. Extend  $\pi$  to an irreducible representation  $\tilde{\pi}$  of  $A$  on some Hilbert space  $H$ , such that there is a one-dimensional subspace  $H_0 \subseteq H$  that is invariant under  $\tilde{\pi}(B)$ , and such that  $\tilde{\pi}|_B$  agrees with  $\pi$  on  $H_0$ .

Choose  $a \in B_+$  with  $\pi(a) = 1$ . By assumption, we obtain  $b \in A_+$  such that  $\|a - b\| < \frac{1}{2}$ , and such that  $(b - \frac{1}{2})_+ A (b - \frac{1}{2})_+$  has no one-dimensional irreducible representations. By [25, Lemma 2.2], there exists  $y \in A$  such that  $(b - \frac{1}{2})_+ = yay^*$ . Set  $x := ya^{1/2}$ . Then

$$(b - \frac{1}{2})_+ = xx^* \quad \text{and} \quad x^*x \in \overline{aAa} \subseteq B.$$

This implies that  $\overline{x^*Ax}$  is isomorphic (as a  $C^*$ -algebra) to  $\overline{xAx^*}$ . We have  $\overline{xAx^*} = \overline{(b - \frac{1}{2})_+ A (b - \frac{1}{2})_+}$ , which does not have one-dimensional irreducible representations. It follows that  $\overline{x^*Ax}$  has no one-dimensional irreducible representations, and thus  $\pi$  vanishes on  $\overline{x^*Ax}$ . Hence,  $\pi(x^*x) = 0$ , and  $\tilde{\pi}(x^*x) = 0$ . This implies  $\tilde{\pi}(x) = 0$ , and so

$$\tilde{\pi}((b - \frac{1}{2})_+) = \tilde{\pi}(xx^*) = 0.$$

But

$$\|a - (b - \frac{1}{2})_+\| \leq \|a - b\| + \|b - (b - \frac{1}{2})_+\| < 1,$$

and thus  $\|\pi(a)\| = \|\tilde{\pi}(a)\| < 1$ , a contradiction. ■

**Remark 4.9.** In condition (2) of Proposition 4.8, it is not enough to require  $\overline{bAb}$  to have no one-dimensional irreducible representations. Indeed, in any unital  $C^*$ -algebra  $A$  without one-dimensional irreducible representations (for example  $A = M_2$ ), for every  $a \in A_+$  and  $\varepsilon > 0$  we can set  $b = a + \frac{\varepsilon}{2} \in A_+$ , which satisfies  $\|a - b\| < \varepsilon$  and  $\overline{bAb} = A$ .

**Proposition 4.10.** *Let  $A$  be a nowhere scattered  $C^*$ -algebra, and let  $B_0 \subseteq A$  be a separable sub- $C^*$ -algebra. Then there exists a separable, nowhere scattered sub- $C^*$ -algebra  $B \subseteq A$  such that  $B_0 \subseteq B$ .*

*Proof.* We will inductively choose:

- an increasing sequence  $B_0 \subseteq B_1 \subseteq \dots$  of separable sub- $C^*$ -algebras of  $A$ ;
- for each  $k \geq 0$ , a countable, dense subset  $F_k \subseteq (B_k)_+$ ;
- for each  $b \in F_k$  a countable set  $G_{k,b} \subseteq A$  such that  $((b - \frac{1}{2})_+ x (b - \frac{1}{2})_+)^2 = 0$  for every  $x \in G_{k,b}$ , and such that  $(b - \frac{1}{2})_+$  belongs to the sub- $C^*$ -algebra generated by  $\{(b - \frac{1}{2})_+ x (b - \frac{1}{2})_+ : x \in G_{k,b}\}$ .

We will also ensure that  $G_{k,b} \subseteq B_{k+1}$  for each  $k \geq 0$  and  $b \in F_k$ .

Let  $k \geq 0$ , and assume that we have chosen  $B_k$ . We will describe how to choose  $F_k$ ,  $G_{k,b}$  and  $B_{k+1}$ . First, let  $F_k$  be any countable, dense subset of  $(B_k)_+$ . For each  $b \in F_k$ , using that  $(b - \frac{1}{2})_+ A (b - \frac{1}{2})_+$  has no irreducible one-dimensional representations (since  $A$  is nowhere scattered), we can apply Lemma 4.7 to obtain a countable subset  $G_{k,b} \subseteq A$  with the claimed properties. Then let  $B_{k+1}$  be the sub- $C^*$ -algebra of  $A$  generated by  $B_k$  together with  $\bigcup_{b \in F_k} G_{k,b}$ .

Set  $B := \bigcup_k B_k$ . Then  $B$  is a separable sub- $C^*$ -algebra of  $A$  containing  $B_0$ . To see that  $B$  is nowhere scattered, we verify statement (3) of Proposition 4.8. So let  $a \in B_+$ . Using that  $\bigcup_k F_k$  is dense in  $B_+$ , we can find  $k$  and  $b \in F_k$  such that  $\|a - b\| < \frac{1}{2}$ . By construction,  $B$  contains the set  $G_{k,b}$ , which satisfies the desired conditions. ■

A property  $\mathcal{P}$  for  $C^*$ -algebras satisfies the *Löwenheim–Skolem condition* if for every  $C^*$ -algebra  $A$  with property  $\mathcal{P}$  there exists a family  $\mathcal{S}$  of separable sub- $C^*$ -algebras of  $A$  that each have property  $\mathcal{P}$ , and such that  $\mathcal{S}$  is  $\sigma$ -complete (for every countable, directed subfamily  $\mathcal{D} \subseteq \mathcal{S}$ , the  $C^*$ -algebra  $\overline{\bigcup \mathcal{D}} \subseteq A$  belongs to  $\mathcal{S}$ ) and cofinal (for every separable sub- $C^*$ -algebra  $B_0 \subseteq A$  there exists  $B \in \mathcal{S}$  with  $B_0 \subseteq B$ ).

It is known that many interesting properties of  $C^*$ -algebras (such as real rank zero, stable rank one, nuclearity, simplicity) satisfy the Löwenheim–Skolem condition.

For properties of Cu-semigroups, the Löwenheim–Skolem condition was considered in [46], where it was also shown that properties like (O5), (O6) and weak cancellation each satisfy it; see Sections 7 and 9 for definitions.

**Proposition 4.11.** *Let  $A$  be a nowhere scattered  $C^*$ -algebra. Then the family  $\mathcal{S}$  of separable, nowhere scattered sub- $C^*$ -algebras of  $A$  is  $\sigma$ -complete and cofinal. In particular, nowhere scatteredness satisfies the Löwenheim–Skolem condition.*

*Proof.* To show that  $\mathcal{S}$  is  $\sigma$ -complete, let  $\mathcal{D} \subseteq \mathcal{S}$  be a countable, directed subfamily. Then  $\overline{\bigcup \mathcal{D}}$  is the inductive limit of  $\mathcal{D}$ , considered as a net indexed over itself. Hence, it follows from Proposition 4.6 that  $\overline{\bigcup \mathcal{D}}$  is nowhere scattered and thus belongs to  $\mathcal{S}$ , as desired. Further, by Proposition 4.10,  $\mathcal{S}$  is cofinal. ■

**Proposition 4.12.** *Let  $A$  and  $B$  be Morita equivalent  $C^*$ -algebras. Assume that  $A$  is nowhere scattered. Then so is  $B$ .*

*In particular, a  $C^*$ -algebra  $D$  is nowhere scattered if and only if its stabilization  $D \otimes \mathcal{K}$  is.*

*Proof.* By [8, Theorem II.7.6.9],  $A$  and  $B$  are isomorphic to complementary full corners in another  $C^*$ -algebra, that is, there exists a  $C^*$ -algebra  $C$  and a projection  $p \in M(C)$  such that  $pCp$  and  $(1 - p)C(1 - p)$  are full (hereditary) sub- $C^*$ -algebras of  $C$  satisfying  $A \cong pCp$  and  $B \cong (1 - p)C(1 - p)$ .

Assuming that  $A$  is nowhere scattered, it follows that  $pCp$  is as well. Since  $pCp$  is a full hereditary sub- $C^*$ -algebra, it separates the ideals of  $C$ . Hence,  $C$  is nowhere scattered by Proposition 4.3. Using that  $(1 - p)C(1 - p)$  is a hereditary sub- $C^*$ -algebra of  $C$ , it is nowhere scattered by Proposition 4.1. Thus,  $B$  is nowhere scattered. ■

**Proposition 4.13.** *Let  $(A_j)_{j \in J}$  be a family of nowhere scattered  $C^*$ -algebras. Then the direct sum  $\bigoplus_{j \in J} A_j$  is nowhere scattered.*

*Proof.* By Proposition 4.2, nowhere scatteredness passes to sums of finitely many summands. Thus, for every finite subset  $F \subseteq J$ , the sum  $\bigoplus_{j \in F} A_j$  is nowhere scattered. Now the result follows from Proposition 4.6, using that  $\bigoplus_{j \in J} A_j$  is the inductive limit of  $\bigoplus_{j \in F} A_j$ , indexed over the finite subsets of  $J$  ordered by inclusion. ■

The next example shows that nowhere scatteredness does not pass to products (of infinitely many summands).

**Example 4.14.** By [36, Corollary 8.6], there exists a sequence  $(A_k)_{k \in \mathbb{N}}$  of unital, simple, infinite-dimensional  $C^*$ -algebras such that their product  $\prod_k A_k$  has a one-dimensional, irreducible representation. Thus, while each  $A_k$  is nowhere scattered, their product is not.

This example also shows that nowhere scatteredness does not pass to multiplier algebras of separable  $C^*$ -algebras. As an example, consider  $A := \bigoplus_k A_k$  with  $A_k$  as above. Then  $A$  is separable and nowhere scattered by Proposition 4.13. Further, it is well known that  $M(A)$  is canonically isomorphic to  $\prod_k A_k$ ; see, for example, [8, II.8.1.3].

**Example 4.15.** Let  $M$  be a type  $\text{II}_1$  factor, let  $\varphi: M \rightarrow \mathbb{C}$  be a pure state, and let  $A$  be the associated hereditary kernel, that is,

$$A = \{a \in M : \varphi(aa^*) = \varphi(a^*a) = 0\}.$$

By [38, Theorem 1],  $A$  is a simple  $C^*$ -algebra such that  $D(A)/A \cong \mathbb{C}$ , where  $D(A)$  is Sakai’s derived algebra. Pedersen showed that the derived algebra of a simple  $C^*$ -algebra is naturally isomorphic with its multiplier algebra; see the remarks after Proposition 2.6 in [29].

Thus,  $A$  is a simple, nowhere scattered  $C^*$ -algebra with  $M(A)/A \cong \mathbb{C}$ , whence  $M(A)$  is not nowhere scattered.

Examples 4.14 and 4.15 above show that for a nonunital, nowhere scattered  $C^*$ -algebra  $A$  the multiplier algebra  $M(A)$  may have a one-dimensional irreducible representation (and hence  $M(A)$  is not nowhere scattered) even if we additionally assume that  $A$  is separable or simple. We suspect that there are also examples for the case that  $A$  is separable and simple.

**Question 4.16.** Does there exist a nonunital, separable, simple, nonelementary  $C^*$ -algebra  $A$  such that  $M(A)$  has a one-dimensional irreducible representation?

Sufficient conditions which ensure that the multiplier algebra of a  $C^*$ -algebra is nowhere scattered are studied in [49].

## 5. Topological characterizations

A topological space  $X$  is said to be *scattered* (or *dispersed*) if every nonempty closed subset  $C$  of  $X$  contains a point that is isolated relative to  $C$ .

**Definition 5.1.** We say that a topological space  $X$  is *nowhere scattered* if no closed subset of  $X$  contains an isolated point.

A subset of a topological space is said to be *perfect* if it is closed and contains no isolated points. Thus, a topological space is nowhere scattered if and only if each of its closed subsets is perfect. It follows that nowhere scatteredness passes to closed subspaces. Further, using that an isolated point in an open subset is also isolated in the whole space, we see that nowhere scatteredness passes to open subspaces, and thus to locally closed subspaces. Considering one-element subsets shows that nowhere scatteredness does not pass to every subspace.

**Proposition 5.2.** *Let  $X$  be a topological space. Then the following are equivalent:*

- (1)  $X$  is nowhere scattered;
- (2)  $X$  has no nonempty, scattered, locally closed subsets;
- (3)  $X$  has no nonempty, locally closed subset that is  $T_1$ ;
- (4)  $X$  has no locally closed subset containing only one element.

*Proof.* Since nowhere scatteredness passes to locally closed subsets, and since a nowhere scattered space is not scattered, we see that (1) implies (2). Since a singleton is scattered, it follows that (2) implies (3). It is clear that (3) implies (4). Finally, it follows directly from the definition that (4) implies (1). ■

Let  $A$  be a  $C^*$ -algebra. We use  $\widehat{A}$  to denote the spectrum of  $A$ , that is, the set of unitary equivalence classes of irreducible representations of  $A$ , equipped with the hull-kernel topology. We refer to [8, Paragraph II.6.5.13] for details.

By [23, Corollary 3],  $A$  is scattered if and only if  $A$  is of type I and  $\widehat{A}$  is scattered as a topological space. A separable  $C^*$ -algebra is of type I if and only if its spectrum is a  $T_0$ -space; see [8, Theorem IV.1.5.7]. Since scattered spaces are  $T_0$ , it follows that a separable  $C^*$ -algebra  $A$  is scattered if and only if  $\widehat{A}$  is scattered. The assumption of separability is necessary: Akemann and Weaver's counterexample to the Naimark problem, [1], is a  $C^*$ -algebra that is nonelementary and simple (hence, not scattered), but whose spectrum is a one-point space (hence, scattered). This also shows that the forward implication in Theorem 5.3 below does not hold for general nonseparable  $C^*$ -algebras, although the backwards implication does.

**Theorem 5.3.** *A separable  $C^*$ -algebra is nowhere scattered if and only if its spectrum is.*

*Proof.* Let  $A$  be a separable  $C^*$ -algebra. By Theorem 3.1,  $A$  is nowhere scattered if and only if it has no nonzero scattered ideal-quotients. On the other hand, by Proposition 5.2,

$\widehat{A}$  is nowhere scattered if and only if it has no nonempty scattered locally closed subsets. Using that a separable  $C^*$ -algebra is scattered if and only if its spectrum is (see [23, Corollary 3]), the result now follows from the natural correspondence between ideal-quotients of  $A$  and locally closed subsets of  $\widehat{A}$ . ■

### 6. Diffuse masas and Haar unitaries

In this section, we observe that a  $C^*$ -algebra is nowhere scattered if and only if each of its positive functionals is. We use this result to connect nowhere scatteredness of a  $C^*$ -algebra to the existence of Haar unitaries and diffuse masas (maximal abelian sub- $C^*$ -algebras) for positive functionals.

Let  $A$  be a  $C^*$ -algebra, and let  $\varphi: A \rightarrow \mathbb{C}$  be a positive functional. We say that  $\varphi$  is *nowhere scattered* if it gives no weight to scattered ideal-quotients of  $A$ , that is, if  $\|\varphi|_I\| = \|\varphi|_J\|$  whenever  $I \subseteq J \subseteq A$  are ideals such that  $J/I$  is scattered; see [44, Definition 3.5].

If  $A$  is unital, then a unitary  $u \in A$  is said to be a *Haar unitary* with respect to  $\varphi$  if  $\varphi(u^k) = 0$  for all  $k \in \mathbb{Z} \setminus \{0\}$ ; see [44, Definition 4.8]. This definition is a generalization to the setting of positive functionals of the well-established notion of Haar unitaries with respect to traces. By [44, Proposition 4.9],  $\varphi$  admits a Haar unitary if and only if there exists a unital (maximal) abelian sub- $C^*$ -algebra  $C(X) \subseteq A$  such that  $\varphi$  induces a diffuse measure on  $X$ . We extend this to the nonunital setting in Lemma 6.1 below.

**Lemma 6.1.** *Let  $\varphi$  be a positive functional on a nonunital  $C^*$ -algebra  $A$ , and let  $\widetilde{A} := A + \mathbb{C}1 \subseteq A^{**}$  be the minimal unitization of  $A$ . Set  $\widetilde{\varphi} := \varphi^{**}|_{\widetilde{A}}: \widetilde{A} \rightarrow \mathbb{C}$ , which is the canonical extension of  $\varphi$  to a positive functional on  $\widetilde{A}$ . Then  $\widetilde{\varphi}$  admits a Haar unitary if and only if there exists a maximal abelian sub- $C^*$ -algebra  $C_0(X) \subseteq A$  such that  $\varphi$  induces a diffuse measure  $\mu$  on  $X$  with  $\mu(X) = \|\varphi\|$ .*

*Proof.* To show the backward implication, assume that we have an abelian sub- $C^*$ -algebra  $C_0(X) \subseteq A$  with the stated properties. We identify  $C_0(X) + \mathbb{C}1 \subseteq \widetilde{A}$  with  $C(\widetilde{X})$ , where  $\widetilde{X}$  is the forced one-point compactification of  $X$ . Recall that, if  $X$  is already compact, then  $\widetilde{X}$  is the disjoint union of  $X$  and one extra point.

Let  $\widetilde{\mu}$  be the measure on  $\widetilde{X}$  induced by  $\widetilde{\varphi}$ . Then

$$\widetilde{\mu}(\widetilde{X}) = \|\widetilde{\varphi}\| = \|\varphi\| = \mu(X).$$

Hence,  $\widetilde{\mu}(\widetilde{X} \setminus X) = 0$ , which implies that  $\widetilde{\mu}$  is diffuse. It follows that a Haar unitary for  $\widetilde{\varphi}$  can be found in  $C_0(X) + \mathbb{C}1 \subseteq \widetilde{A}$ .

Conversely, assume that  $\widetilde{\varphi}$  admits a Haar unitary. By [44, Proposition 4.9], we obtain a maximal abelian sub- $C^*$ -algebra  $C(Y) \subseteq \widetilde{A}$  such that  $\widetilde{\varphi}$  induces a diffuse measure  $\mu$  on  $Y$ . We claim that  $A \cap C(Y)$  has the desired properties.

Note that  $1 \in C(Y)$ . Let  $\pi: \widetilde{A} \rightarrow \mathbb{C}$  be the canonical one-dimensional representation such that  $\ker(\pi) = A$ . The restriction of  $\pi$  to  $C(Y)$  corresponds to the evaluation at a

point  $y \in Y$ , and the ideal  $A \cap C(Y)$  of  $C(Y)$  naturally corresponds to the open subset  $X := Y \setminus \{y\}$ . We identify  $C_0(X)$  with  $A \cap C(Y)$ , and we note that the measure on  $X$  induced by  $\varphi|_{C_0(X)}$  is the restriction of  $\mu$  to  $X$ . This measure on  $X$  is therefore diffuse.

Using that  $\mu$  is diffuse on  $Y$ , we have  $\mu(X) = \mu(Y)$  and thus

$$\mu(X) = \mu(Y) = \|\tilde{\varphi}\| = \|\varphi\|.$$

Finally, using that  $C(Y) \subseteq \tilde{A}$  is maximal abelian, it follows that  $C_0(X) \subseteq A$  is maximal abelian as well. ■

**Theorem 6.2.** *Let  $A$  be a  $C^*$ -algebra. Then the following are equivalent:*

- (1)  $A$  is nowhere scattered;
- (2) every positive functional on  $A$  is nowhere scattered;
- (3) every pure state on  $A$  is nowhere scattered;
- (4) for every positive functional  $\varphi: A \rightarrow \mathbb{C}$  and every hereditary sub- $C^*$ -algebra  $B \subseteq A$  there exists a Haar unitary in  $\tilde{B}$  with respect to  $\varphi$ ;
- (5) for every pure state  $\varphi: A \rightarrow \mathbb{C}$  and every ideal  $I \subseteq A$  there exists a Haar unitary in  $\tilde{I}$  with respect to  $\varphi$ ;
- (6) for every positive functional  $\varphi: A \rightarrow \mathbb{C}$  and every hereditary sub- $C^*$ -algebra  $B \subseteq A$  there exists a maximal abelian sub- $C^*$ -algebra  $C_0(X) \subseteq B$  such that  $\varphi$  induces a diffuse measure  $\mu$  on  $X$  with  $\mu(X) = \|\varphi|_B\|$ ;
- (7) for every pure state  $\varphi: A \rightarrow \mathbb{C}$  and every ideal  $I \subseteq A$  there exists a maximal abelian sub- $C^*$ -algebra  $C_0(X) \subseteq I$  such that  $\varphi$  induces a diffuse measure  $\mu$  on  $X$  with  $\mu(X) = \|\varphi|_I\|$ .

*Proof.* By Theorem 3.1,  $A$  is nowhere scattered if and only if  $A$  has no nonzero scattered ideal-quotients. This shows that (1) implies (2), which in turn implies (3). To see that (3) implies (1), assume for the sake of contradiction that  $A$  is not nowhere scattered. Then there exist ideals  $I \subseteq J \subseteq A$  such that  $J/I$  is nonzero and scattered. Choose a nonzero pure state  $\varphi_0$  on  $J/I$ . Composing with the quotient map  $J \rightarrow J/I$  we obtain a pure state  $\varphi$  on  $J$ , which we can extend to a pure state  $\tilde{\varphi}$  on  $A$ . Then  $\tilde{\varphi}|_I = 0$ , while  $\tilde{\varphi}|_J \neq 0$ , which shows that  $\tilde{\varphi}$  is a pure state on  $A$  that is *not* nowhere scattered, a contradiction.

By [44, Theorem 4.11], (2) is equivalent to (4). Similarly, (3) is equivalent to (5). Finally, it follows from Lemma 6.1 that (4) is equivalent to (6), and that (5) is equivalent to (7). ■

## 7. A new property of Cuntz semigroups

We introduce a new property for Cu-semigroups, called (O8); see Definition 7.2. This new property can be thought of as the Cu-semigroup version of the projectivity of  $C_0((0, 1]) \oplus C_0((0, 1])$ , which allows one to lift orthogonal positive elements in a quotient of a  $C^*$ -

algebra. We show that the Cuntz semigroup of every  $C^*$ -algebra satisfies (O8); see Theorem 7.4. We also deduce a result that can be interpreted as the Cu-version of the projectivity of  $M_n(C_0((0, 1]))$ ; see Proposition 7.8.

**7.1.** Let  $A$  be a  $C^*$ -algebra. Given  $a, b \in A_+$ , one says that  $a$  is *Cuntz subequivalent* to  $b$ , denoted  $a \lesssim b$ , if there is a sequence  $(r_n)_n$  in  $A$  such that  $\lim_n \|a - r_n b r_n^*\| = 0$ . Further,  $a$  is said to be *Cuntz equivalent* to  $b$ , in symbols  $a \sim b$ , if  $a \lesssim b$  and  $b \lesssim a$ . The *Cuntz semigroup* of  $A$  is the set of Cuntz equivalence classes of positive elements in the stabilization of  $A$ , that is,

$$\text{Cu}(A) := (A \otimes \mathcal{K})_+ / \sim,$$

where the class of a positive element  $a$  is denoted by  $[a]$ . One equips  $\text{Cu}(A)$  with the addition induced by the orthogonal sum, and with the partial order given by  $[a] \leq [b]$  if  $a \lesssim b$ . This turns  $\text{Cu}(A)$  into a partially ordered, abelian monoid.

As shown in [14], the Cuntz semigroup of any  $C^*$ -algebra enjoys additional order-theoretic properties; see also [6] and [4]. To state them, recall that given two elements  $x, y$  in a partially ordered set, one says that  $x$  is *way-below*  $y$ , denoted  $x \ll y$ , if for every increasing sequence  $(y_k)_k$  with supremum satisfying  $y \leq \sup_k y_k$  there exists  $k'$  such that  $x \leq y_{k'}$ . Given a  $C^*$ -algebra  $A$ ,  $[a] \ll [b]$  in  $\text{Cu}(A)$  if and only if there exists  $\varepsilon > 0$  such that  $a \lesssim (b - \varepsilon)_+$ , where  $(b - \varepsilon)_+$  is the  $\varepsilon$ -cut-down of  $b$  obtained by applying functional calculus for the function  $t \mapsto \max\{0, t - \varepsilon\}$  to  $b$ .

One says that a positively ordered, abelian monoid  $S$  is a *Cu-semigroup*, also called *abstract Cuntz semigroup*, if the following conditions are satisfied:

- (O1) Every increasing sequence in  $S$  has a supremum.
- (O2) For every  $x \in S$  there exists a  $\ll$ -increasing sequence  $(x_n)_n$  with  $x = \sup_n x_n$ .
- (O3) If  $x' \ll x$  and  $y' \ll y$  in  $S$ , then  $x' + y' \ll x + y$ .
- (O4) If  $(x_n)_n$  and  $(y_n)_n$  are increasing sequences in  $S$ , then

$$\sup_n (x_n + y_n) = \sup_n x_n + \sup_n y_n.$$

By [14],  $\text{Cu}(A)$  is a Cu-semigroup for every  $C^*$ -algebra  $A$ .

Since the introduction of Cu-semigroups, the Cuntz semigroup of any  $C^*$ -algebra has been shown to satisfy additional properties, such as the two stated below; see [4] and [34] respectively, and also [2].

- (O5) Given  $x + y \leq z$ ,  $x' \ll x$  and  $y' \ll y$  in  $S$  there exists  $c \in S$  such that  $x' + c \leq z \leq x + c$  and  $y' \ll c$ . (This property is often applied with  $y' = y = 0$ , in which case it says that for  $x' \ll x \leq z$  there exists an ‘almost complement’  $c$  such that  $x' + c \leq z \leq x + c$ .)
- (O6) Given  $x' \ll x \leq y + z$  in  $S$  there exist  $e, f \in S$  such that  $x' \leq e + f$  with  $e \leq x, y$ , and  $f \leq x, z$ .

**Definition 7.2.** Let  $S$  be a Cu-semigroup. We say that  $S$  satisfies (O8) if for all  $x', x, y', y, z, w \in S$  satisfying  $2w = w$  and

$$x + y \ll z + w, \quad x' \ll x, \quad \text{and} \quad y' \ll y$$

there exist  $z_1, z_2 \in S$  such that

$$z_1 + z_2 \ll z, \quad x' \ll z_1 + w, \quad y' \ll z_2 + w, \quad z_1 \ll x + w, \quad \text{and} \quad z_2 \ll y + w.$$

**Remark 7.3.** A Cu-semigroup  $S$  satisfies (O8) if and only if for all  $x', x, y', y, z, w$  in  $S$  satisfying  $2w = w$  and

$$x + y \ll z + w, \quad x' \ll x, \quad \text{and} \quad y' \ll y$$

there exist  $z_1, z_2 \in S$  such that

$$z_1 + z_2 \leq z, \quad x' \leq z_1 + w, \quad y' \leq z_2 + w, \quad z_1 \leq x + w, \quad \text{and} \quad z_2 \leq y + w.$$

Indeed, the forward implication is trivial. To see the backward implication, let  $x', x, y', y, z, w$  satisfy the conditions in the statement, and choose  $\tilde{x}, \tilde{y}, \tilde{z}$  such that

$$x + y \ll \tilde{z} + w, \quad x' \ll \tilde{x} \ll x, \quad y' \ll \tilde{y} \ll y, \quad \text{and} \quad \tilde{z} \ll z.$$

Then, there exist  $t_1, t_2 \in S$  with

$$t_1 + t_2 \leq \tilde{z}, \quad \tilde{x} \leq t_1 + w, \quad \tilde{y} \leq t_2 + w, \quad t_1 \leq x + w, \quad \text{and} \quad t_2 \leq y + w.$$

In particular, since  $x' \ll t_1 + w$  and  $y' \ll t_2 + w$ , we can find elements  $z_1 \ll t_1$  and  $z_2 \ll t_2$  such that  $x' \ll z_1 + w$  and  $y' \ll z_2 + w$ . It is easy to check that such elements satisfy the desired conditions.

**Theorem 7.4.** *The Cuntz semigroup of every  $C^*$ -algebra satisfies (O8).*

*Proof.* Let  $A$  be a  $C^*$ -algebra, and let  $x', x, y', y, z, w \in \text{Cu}(A)$  satisfy  $2w = w$  and

$$x + y \ll z + w, \quad x' \ll x, \quad \text{and} \quad y' \ll y.$$

We may assume that  $A$  is stable. Let  $I$  be the ideal of  $A$  corresponding to the ideal  $\{s \in \text{Cu}(A) : s \leq w\}$  of  $\text{Cu}(A)$ , and let  $\pi: A \rightarrow A/I$  denote its quotient map. Choose  $a, b, c \in A_+$  and  $\varepsilon > 0$  such that  $a$  and  $b$  are orthogonal,

$$x = [a], \quad y = [b], \quad z = [c], \quad x' \leq [(a - \varepsilon)_+], \quad \text{and} \quad y' \leq [(b - \varepsilon)_+].$$

By [13, Proposition 3.3], two elements  $e, f \in A_+$  satisfy  $[e] \leq [f] + w$  if and only if  $\pi(e) \lesssim \pi(f)$ . Thus,

$$\pi(a) + \pi(b) = \pi(a + b) \lesssim \pi(c).$$

Using Rørdam’s lemma (see for example [43, Theorem 2.30]), there exists  $r \in A/I$  such that

$$(\pi(a) - \varepsilon)_+ + (\pi(b) - \varepsilon)_+ = ((\pi(a) + \pi(b)) - \varepsilon)_+ = r^*r$$

and  $rr^* \in \overline{\pi(c)(A/I)\pi(c)}$ .

Set

$$e := r(\pi(a) - \varepsilon)_+r^* \quad \text{and} \quad f := r(\pi(b) - \varepsilon)_+r^*.$$

Note that  $e$  and  $f$  are orthogonal positive elements contained in the hereditary sub- $C^*$ -algebra  $\overline{\pi(c)(A/I)\pi(c)}$ . Since  $\pi$  maps the hereditary sub- $C^*$ -algebra  $\overline{cAc}$  onto  $\overline{\pi(c)(A/I)\pi(c)}$ , and since the  $C^*$ -algebra  $C_0((0, \|e\|]) \oplus C_0((0, \|f\|])$  is projective (see [15, Section 4]), we can choose orthogonal positive elements  $\tilde{e}, \tilde{f} \in \overline{cAc}$  such that  $\pi(\tilde{e}) = e$  and  $\pi(\tilde{f}) = f$ . Set

$$z_1 := [\tilde{e}] \quad \text{and} \quad z_2 := [\tilde{f}].$$

Using that  $\tilde{e}$  and  $\tilde{f}$  are orthogonal, and that  $\tilde{e} + \tilde{f} \in \overline{cAc}$ , we get  $z_1 + z_2 \leq [c] = z$ . Also note that, since  $\pi(\tilde{e}) = e \lesssim (\pi(a) - \varepsilon)_+ \lesssim \pi(a)$ , we have  $z_1 \leq x + w$ . Similarly,  $z_2 \leq y + w$ .

Moreover, one also gets

$$r^*er = r^*r(\pi(a) - \varepsilon)_+r^*r = (\pi(a) - \varepsilon)_+^3 \sim (\pi(a) - \varepsilon)_+,$$

which shows that  $x' \leq z_1 + w$ . An analogous argument proves  $y' \leq z_2 + w$ .

It follows from Remark 7.3 that  $\text{Cu}(A)$  satisfies (O8), as desired. ■

One can think of (O8) as a weak form of Riesz refinement. In this sense, Proposition 7.5 below can be seen as a Cu-version of the fact that a cancellative, algebraically ordered semigroup with Riesz decomposition has Riesz refinement.

A Cu-semigroup  $S$  is said to be *weakly cancellative* if for all  $x, y, z \in S$  with  $x + z \ll y + z$  we have  $x \ll y$ . It follows from [37, Theorem 4.3] that the Cuntz semigroup of every stable rank one  $C^*$ -algebra is weakly cancellative.

**Proposition 7.5.** *Let  $S$  be a weakly cancellative Cu-semigroup satisfying (O5) and (O6). Then  $S$  satisfies (O8).*

*Proof.* Let  $x', x, y', y, z, w$  satisfy

$$x + y \ll z + w, \quad x' \ll x, \quad y' \ll y, \quad \text{and} \quad 2w = w.$$

Applying (O6) for  $x' \ll x \ll z + w$ , we obtain  $\tilde{z}_1$  such that

$$x' \ll \tilde{z}_1 + w \quad \text{and} \quad \tilde{z}_1 \ll x, z.$$

Choose  $z_1 \in S$  such that

$$x' \ll z_1 + w \quad \text{and} \quad z_1 \ll \tilde{z}_1.$$

Applying (O5) for  $z_1 \ll \tilde{z}_1 \leq z$ , we find  $c \in S$  with

$$z_1 + c \leq z \leq \tilde{z}_1 + c.$$

Then,

$$x + y \ll z + w \leq \tilde{z}_1 + c + w$$

and, since  $S$  is weakly cancellative and  $\tilde{z}_1 \leq x$ , we get  $y \ll c + w$ .

Applying (O6) for  $y' \ll y \ll c + w$ , we obtain  $z_2$  such that

$$y' \ll z_2 + w \quad \text{and} \quad z_2 \ll y, c.$$

We have  $z_1 + z_2 \leq z_1 + c \leq z$ . It now follows from Remark 7.3 that  $z_1$  and  $z_2$  have the desired properties. ■

**Lemma 7.6.** *Let  $S$  be a Cu-semigroup satisfying (O8), let  $w \in S$  satisfy  $2w = w$ , and let  $x'_1, \dots, x'_n, x_1, \dots, x_n, z \in S$  satisfy*

$$x_1 + \dots + x_n \ll z + w, \quad x'_1 \ll x_1, \dots \quad \text{and} \quad x'_n \ll x_n.$$

*Then there exist  $z_1, \dots, z_n \in S$  such that*

$$z_1 + \dots + z_n \ll z, \quad x'_j \ll z_j + w, \quad \text{and} \quad z_j \ll x_j + w$$

*for  $j = 1, \dots, n$ .*

*Proof.* We prove the result by induction over  $n$ . The case  $n = 1$  is clear, and the case  $n = 2$  holds by definition of (O8).

Thus, let  $n \geq 3$ , and assume that the statement holds for  $n - 1$ . Let  $x'_j, x_j, z, w$  for  $j = 1, \dots, n$  be as in the statement, and choose  $x''_{n-1}, x''_n$  such that

$$x'_{n-1} \ll x''_{n-1} \ll x_{n-1} \quad \text{and} \quad x'_n \ll x''_n \ll x_n.$$

Applying the assumption for the  $n - 1$  pairs  $x'_1 \ll x_1, \dots, x'_{n-2} \ll x_{n-2}$  and  $x''_{n-1} + x''_n \ll x_{n-1} + x_n$ , we obtain  $z_1, \dots, z_{n-2}, v$  such that

$$z_1 + \dots + z_{n-2} + v \ll z, \quad x'_j \ll z_j + w, \quad \text{and} \quad z_j \ll x_j + w$$

for  $j = 1, \dots, n - 2$  and

$$x''_{n-1} + x''_n \ll v + w \quad \text{and} \quad v \ll x_{n-1} + x_n + w.$$

Applying (O8) for

$$x''_{n-1} + x''_n \ll v + w, \quad x'_{n-1} \ll x''_{n-1}, \quad \text{and} \quad x'_n \ll x''_n,$$

we obtain  $z_{n-1}, z_n$  with  $z_{n-1} + z_n \ll v$  and such that

$$x'_{n-1} \ll z_{n-1} + w, \quad x'_n \ll z_n + w, \quad z_{n-1} \ll x''_{n-1} + w, \quad \text{and} \quad z_n \ll x''_n + w.$$

Then  $z_1, \dots, z_n$  have the desired properties. ■

**Proposition 7.7.** *Let  $S$  be a Cu-semigroup satisfying (O6) and (O8), let  $w \in S$  be such that  $2w = w$ , and let  $x'_1, \dots, x'_n, x_1, \dots, x_n, z \in S$  satisfy*

$$x_1 + \dots + x_n \ll z + w, \quad \text{and} \quad x'_1 \ll x_1 \ll x'_2 \ll x_2 \ll \dots \ll x'_n \ll x_n.$$

*Then, there exist  $z_1, \dots, z_n \in S$  such that*

$$z_1 + z_2 + \dots + z_n \ll z, \quad z_1 \ll \dots \ll z_n, \quad x'_j \ll z_j + w, \quad \text{and} \quad z_j \ll x_j + w$$

*for  $j = 1, \dots, n$ .*

*Proof.* Applying Lemma 7.6, we obtain  $y_1, \dots, y_n$  such that

$$y_1 + \dots + y_n \ll z, \quad x'_j \ll y_j + w, \quad \text{and} \quad y_j \ll x_j + w$$

for  $j = 1, \dots, n$ .

For every  $j$ , let  $y'_j \ll y_j$  be such that  $x'_j \ll y'_j + w$ . Set  $z_n := y_n$ , and note that

$$y'_{n-1} \ll y_{n-1} \leq x_{n-1} + w \leq x'_n + w \leq y_n + w = z_n + w.$$

Applying (O6), we obtain  $z_{n-1}$  such that

$$y'_{n-1} \ll z_{n-1} + w, \quad \text{and} \quad z_{n-1} \ll y_{n-1}, z_n,$$

where we note that one also has

$$y'_{n-2} \ll y_{n-2} \leq x_{n-2} + w \leq x'_{n-1} + w \leq y'_{n-1} + w \leq z_{n-1} + w.$$

Proceeding in this manner, we obtain elements  $z_1, \dots, z_n$  such that

$$y'_j \ll z_j + w \quad \text{and} \quad z_j \ll y_j, z_{j+1}$$

for every  $j \leq n - 1$ .

It is easy to see that such elements satisfy the required properties. ■

The next result can be interpreted as the Cu-semigroup version of the projectivity of  $C_0((0, 1], M_n)$ .

**Proposition 7.8.** *Let  $S$  be a Cu-semigroup satisfying (O6) and (O8), let  $n \geq 1$ , and let  $x', x, y, w \in S$  satisfy*

$$nx \ll y + w, \quad x' \ll x, \quad \text{and} \quad 2w = w.$$

*Then there exists  $z \in S$  such that*

$$nz \ll y, \quad x' \ll z + w, \quad \text{and} \quad z \ll x + w.$$

*Proof.* Choose  $x'_1, x_1, x'_2, x_2, \dots, x_n$  such that

$$x' \ll x'_1 \ll x_1 \ll x'_2 \ll x_2 \ll \dots \ll x'_n \ll x_n \ll x.$$

Applying Proposition 7.7, we obtain  $z_1, \dots, z_n$  satisfying

$$z_1 + \dots + z_n \ll y, \quad z_1 \ll \dots \ll z_n, \quad x'_j \ll z_j + w, \quad \text{and} \quad z_j \ll x_j + w$$

for  $j = 1, \dots, n$ .

Set  $z := z_1$ . Then

$$nz \leq z_1 + \dots + z_n \ll y, \quad x' \leq x'_1 \ll z_1 + w = z + w,$$

and  $z = z_1 \leq z_n \ll x_n + w \leq x + w$ . ■

### 8. $C^*$ -algebras and Cu-semigroups without elementary ideal-quotients

In this section, we prove that a Cu-semigroup  $S$  satisfying (O5), (O6) and (O8) has no nonzero elementary ideal-quotients if and only if  $S$  is weakly  $(2, \omega)$ -divisible; see Proposition 8.8. In Paragraph 8.1 we provide a tailored definition of *elementary Cu-semigroup*, which is justified by Lemma 8.2. We deduce in Theorem 8.9 that a  $C^*$ -algebra is nowhere scattered if and only if every element in its Cuntz semigroup is weakly  $(2, \omega)$ -divisible. This can also be deduced from results in [36]; see Remark 8.10.

**8.1.** A  $C^*$ -algebra is said to be *elementary* if it is isomorphic to the compact operators on some Hilbert space.

In [4, Paragraph 5.1.16], a nonzero Cu-semigroup was said to be ‘elementary’ if it is simple and contains a minimal nonzero element. This definition includes  $\{0, \infty\}$ , which is the Cuntz semigroup of simple, purely infinite  $C^*$ -algebras – and these  $C^*$ -algebras are very far from elementary. To amend this, we will instead say that a nonzero Cu-semigroup is *elementary* if it is simple and contains a minimal, nonzero element  $x$  that is *finite* (that is,  $x \neq 2x$ ). Lemma 8.2 below shows that this refined definition fits the established terminology in  $C^*$ -algebras.

The next result is shown in [18, Theorem 4.4.4]. We include a proof for the convenience of the reader.

**Lemma 8.2.** *Let  $A$  be a (nonzero)  $C^*$ -algebra. Then the following are equivalent:*

- (1)  $A$  is elementary;
- (2)  $\text{Cu}(A) \cong \overline{\mathbb{N}}$ ;
- (3)  $\text{Cu}(A)$  is elementary.

*Proof.* To show that (1) implies (2), assume that  $A$  is elementary. Upon stabilization, we may assume that  $A \cong \mathcal{K}(H)$  for some infinite-dimensional Hilbert space  $H$ . Using the spectral theorem for compact operators, one can show that  $A_+ \rightarrow \overline{\mathbb{N}}$ , mapping  $a \in A_+$  to its rank, induces the desired isomorphism  $\text{Cu}(A) \cong \overline{\mathbb{N}}$ .

Alternatively, one sees that every projection in  $\mathcal{K}(H)$  has finite-rank, and that two projections are Murray–von Neumann equivalent if and only if their ranks agree. It follows that the Murray–von Neumann semigroup  $V(A)$  is isomorphic to  $\mathbb{N}$ . Using that  $A$  has real rank zero, we obtain that  $\text{Cu}(A)$  is isomorphic to the sequential ideal completion of  $\mathbb{N}$ , and thus  $\text{Cu}(A) \cong \overline{\mathbb{N}}$ ; see, for example, [4, Remark 5.5.6].

It is clear that (2) implies (3). To show that (3) implies (1), assume that  $\text{Cu}(A)$  is elementary. Then  $\text{Cu}(A)$  is simple, and consequently so is  $A$  (see [4, Corollary 5.1.12]). Choose a minimal, nonzero element  $x \in \text{Cu}(A)$  with  $x \neq 2x$ , and let  $a \in A_+$  be a nonzero element.

Since  $x$  is minimal, we have  $x \ll x$ . Using that  $A$  is simple, it follows from [11, Theorem 5.8] that we can choose a projection  $p \in A \otimes \mathcal{K}$  with  $[p] = x$ . Since  $a \neq 0$ , we have  $[a] \neq 0$  and therefore  $x \leq [a]$ . Then  $p \lesssim a$ , whence we obtain a projection  $q \in A$  with  $[q] = x$ . Thus,  $q$  is a minimal open projection, that is,  $qAq = \mathbb{C}q$ . This is well-known to imply that  $A$  is elementary. ■

Given a Cu-semigroup  $S$ , a downward-hereditary submonoid  $I \subseteq S$  is an *ideal* if it is closed under suprema of increasing sequences. For each ideal  $I$  of  $S$ , one can consider its corresponding quotient  $S/I$ , which is again a Cu-semigroup; see [4, Lemma 5.1.2].

We omit the proof of the following lemma.

**Lemma 8.3.** *Let  $S$  be a Cu-semigroup satisfying (O8) and let  $I$  be an ideal of  $S$ . Then,  $I$  and  $S/I$  satisfy (O8).*

**Lemma 8.4.** *Let  $S$  be a Cu-semigroup satisfying (O5) and (O8). Let  $I$  be an ideal of  $S$  and denote by  $\pi$  the canonical map  $S \rightarrow S/I$ . Given  $y \in S$  and  $e, e' \in S/I$  such that*

$$e' \ll e \quad \text{and} \quad 2e \leq \pi(y),$$

*there exists  $z \in S$  such that  $e' \ll \pi(z) \ll e$  and  $2z \ll y$ .*

*Proof.* Let  $x \in S$  be such that  $\pi(x) = e$  and take  $x' \ll x$  such that  $e' \ll \pi(x')$ . Since  $\pi(2x) \leq \pi(y)$ , there exists  $w \in I$  satisfying  $2x \leq y + w$ . Thus, we have  $2x \leq y + \infty w$ , where we note that  $\infty w \in I$ .

Applying Proposition 7.8, we obtain an element  $z \in S$  such that

$$2z \ll y, \quad x' \ll z + \infty w, \quad \text{and} \quad z \ll x + \infty w.$$

Passing to the quotient, and using that  $\infty w \in I$ , we get

$$e' \ll \pi(x') \ll \pi(z) \ll \pi(x) = e,$$

as required. ■

By an *ideal-quotient* in a Cu-semigroup  $S$  we mean a quotient  $J/I$  for some ideals  $I \subseteq J$  of  $S$ .

**Lemma 8.5.** *Let  $S$  be a Cu-semigroup satisfying (O5), (O6) and (O8). Assume that  $S$  has no nonzero elementary ideal-quotients. Let  $x \in S$  be nonzero. Then there exists  $z \in S$  with  $0 \neq 2z \leq x$ .*

*Proof.* Using that  $x$  is nonzero, we can choose nonzero elements  $x'', x' \in S$  such that  $x'' \ll x' \ll x$ .

Applying (O5), we obtain  $c \in S$  with

$$x'' + c \leq x \leq x' + c.$$

Let  $I \subseteq S$  be the ideal generated by  $c$ . Then,  $x$  is either in  $I$  or not. We study each case separately.

*Case 1:* We have  $x \in I$ . In this case, and since ideals are downward hereditary, we have  $x'' \in I$ , and therefore  $x'' \ll \infty c$ . Thus, there exists  $n \in \mathbb{N}$  such that  $x'' \leq nc$ .

Let  $x''' \ll x''$  be a nonzero element. Following the proof of [34, Proposition 5.2.1], we can apply (O6) to  $x''' \ll x'' \leq nc$  to obtain elements  $c_1, \dots, c_n \leq x'', c$  such that

$$x''' \leq c_1 + \dots + c_n.$$

Using that  $x''' \neq 0$ , it follows that there exists some  $j$  such that the element  $c_j$  is nonzero. Setting  $z = c_j$ , one has  $2z \leq x'' + c \leq x$ , as required.

*Case 2:* We have  $x \notin I$ . Let  $K \subseteq S$  denote the ideal generated by  $x$ . Then the image of  $x$  in  $K/I$  is a nonzero, compact, full element.

It follows that there exists a maximal ideal  $J \subseteq K$  containing  $I$ . Consequently, the quotient  $K/J$  is simple. Let  $\pi: K \rightarrow K/J$  be the quotient map.

By [4, Proposition 5.1.3], (O5) and (O6) pass to ideals and quotients. Thus,  $K/J$  is a simple, nonelementary Cu-semigroup satisfying (O5) and (O6). Applying [34, Proposition 5.2.1], we obtain  $e \in K/J$  with  $0 \neq 2e \leq \pi(x)$ . Choose  $e' \in K/J$  with  $0 \neq e' \ll e$ . By Lemmas 8.3 and 8.4, we obtain  $z \in K \subseteq S$  such that  $e' \ll \pi(z)$  and  $2z \leq x$ . It follows that  $z$  is nonzero and has the desired properties. ■

**8.6.** Let  $S$  be a Cu-semigroup and let  $k \geq 1$ . An element  $x \in S$  is said to be  $(k, \omega)$ -divisible if for every  $x' \in S$  satisfying  $x' \ll x$  there exist  $n \in \mathbb{N}$  and  $y \in S$  such that  $ky \leq x$  and  $x' \leq ny$ . Further,  $x$  is said to be *weakly*  $(k, \omega)$ -divisible if for every  $x' \in S$  satisfying  $x' \ll x$  there exist  $n \in \mathbb{N}$  and  $y_1, \dots, y_n \in \text{Cu}(A)$  such that  $ky_1, \dots, ky_n \leq x$  and  $x' \leq y_1 + \dots + y_n$ ; see [36, Definition 5.1] or [3, Paragraph 5.1].

We say that  $S$  is (weakly)  $(k, \omega)$ -divisible if each of its elements is.

**Lemma 8.7.** *Let  $S$  be a weakly  $(2, \omega)$ -divisible Cu-semigroup. Then,  $S$  is weakly  $(k, \omega)$ -divisible for every  $k \in \mathbb{N}$ .*

*Proof.* It suffices to show that every element in  $S$  is weakly  $(2^k, \omega)$ -divisible for each  $k \geq 1$ . We prove this by induction over  $k$ , where we note that the case  $k = 1$  holds by assumption.

Thus, let  $k \geq 1$  and assume that every element is weakly  $(2^k, \omega)$ -divisible. To verify that every element is weakly  $(2^{k+1}, \omega)$ -divisible, let  $x', x \in S$  satisfy  $x' \ll x$ . Choose  $x'' \in S$  such that  $x' \ll x'' \ll x$ . Using that  $x$  is weakly  $(2, \omega)$ -divisible, we obtain  $m \in \mathbb{N}$  and  $y_1, \dots, y_m \in S$  such that

$$2y_1, \dots, 2y_m \leq x, \quad \text{and} \quad x'' \leq y_1 + \dots + y_m.$$

Choose  $y'_1, \dots, y'_m \in S$  such that

$$y'_1 \ll y_1, \dots, y'_m \ll y_m, \quad \text{and} \quad x' \leq y'_1 + \dots + y'_m.$$

Applying the induction assumption for each pair  $y'_i \ll y_i$ , we obtain  $n(i) \in \mathbb{N}$  and  $z_{i,1}, \dots, z_{i,n(i)} \in S$  such that

$$2^k z_{i,1}, \dots, 2^k z_{i,n(i)} \leq y_i, \quad \text{and} \quad y'_i \leq z_{i,1} + \dots + z_{i,n(i)}.$$

Consequently,

$$2^{k+1} z_{i,j} \leq 2y_i \leq x,$$

for each  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, n(i)\}$ . Further,

$$x' \leq y'_1 + \dots + y'_m = \sum_{i=1}^m y'_i \leq \sum_{i=1}^m \sum_{j=1}^{n(i)} z_{i,j},$$

which shows that the elements  $z_{i,j}$  have the desired properties. ■

**Proposition 8.8.** *Let  $S$  be a Cu-semigroup satisfying (O5), (O6) and (O8). Then the following are equivalent:*

- (1)  $S$  has no nonzero elementary ideal-quotients;
- (2) every element in  $S$  is weakly  $(2, \omega)$ -divisible;
- (3) every element in  $S$  is weakly  $(k, \omega)$ -divisible for every  $k \geq 2$ .

*Proof.* The equivalence between (2) and (3) follows from Lemma 8.7. To show that (2) implies (1), assume that  $S$  is weakly  $(2, \omega)$ -divisible, and note that a nonzero elementary Cu-semigroup  $T$  is not weakly  $(2, \omega)$ -divisible. Indeed, if  $x \in T$  is minimal, nonzero and  $x \neq 2x$ , then one can see that  $x$  is not weakly  $(2, \omega)$ -divisible. Since weak  $(2, \omega)$ -divisibility passes to ideals and quotients, it follows that  $S$  has no nonzero elementary ideal-quotients.

Conversely, to show that (1) implies (2), assume now that  $S$  has no nonzero elementary ideal-quotients and let  $x \in S$ . The proof is inspired by that of [7, Theorem 6.7]. Set

$$D := \{z \in S : 2z \leq x\}.$$

Let  $I \subseteq S$  be the ideal generated by  $D$ . Note that  $s \in S$  belongs to  $I$  if and only if there exists a sequence  $(z_k)_k$  in  $D$  such that  $s \leq \sum_{k=0}^{\infty} z_k$ .

*Claim:* We have  $x \in I$ .

To prove the claim, let  $\pi: S \rightarrow S/I$  denote the quotient map. To reach a contradiction, assume that  $\pi(x) \neq 0$ . By [4, Proposition 5.1.3] and Lemma 8.3, we know that (O5), (O6) and (O8) pass to quotients.

Hence, we can apply Lemma 8.5 to  $S/I$ . Thus, since  $\pi(x) \neq 0$ , we obtain  $e \in S/I$  with  $0 \neq 2e \leq \pi(x)$ . Choose  $e' \in S/I$  with  $0 \neq e' \ll e$ . By Lemma 8.4, we obtain  $z \in S$  such that  $e' \ll \pi(z)$  and  $2z \leq x$ . Consequently,  $z \in D \subseteq I$ , which implies  $\pi(z) = 0$  and therefore  $e' = 0$ , a contradiction. Thus, we have  $\pi(x) = 0$  and so  $x \in I$ , which proves the claim.

Now, given any  $x' \in S$  such that  $x' \ll x$ , take  $u \in S$  with  $x' \ll u \ll x$ . Using that  $u \ll x \in I$ , we obtain  $z_1, \dots, z_n \in D$  such that  $u \leq z_1 + \dots + z_n$ .

Note that we have  $2z_j \leq x$  for every  $j$  and  $x' \ll u \leq z_1 + \dots + z_n$ . It follows that  $x$  is weakly  $(2, \omega)$ -divisible, as desired. ■

**Theorem 8.9.** *Let  $A$  be a  $C^*$ -algebra. Then the following are equivalent:*

- (1)  $A$  is nowhere scattered;
- (2) every element in  $\text{Cu}(A)$  is weakly  $(2, \omega)$ -divisible;
- (3) every element in  $\text{Cu}(A)$  is weakly  $(k, \omega)$ -divisible for every  $k \geq 2$ ;

*Proof.* It follows from [4, Proposition 5.1.10] that ideal-quotients in  $A$  naturally correspond to ideal-quotients in  $\text{Cu}(A)$ . Thus, by Lemma 8.2,  $A$  has no nonzero elementary ideal-quotients if and only if  $\text{Cu}(A)$  has no nonzero elementary ideal-quotients. Applying Theorem 3.1, we see that  $A$  is nowhere scattered if and only if  $\text{Cu}(A)$  has no nonzero elementary ideal-quotients.

As noted in Paragraph 7.1,  $\text{Cu}(A)$  is a Cu-semigroup satisfying (O5) and (O6). By Theorem 7.4,  $\text{Cu}(A)$  also satisfies (O8). Now the result follows from Proposition 8.8. ■

**Remark 8.10.** Let us indicate an alternative proof of Theorem 8.9 that is based on the results by Rørdam and Robert in [36].

Let  $A$  be a  $C^*$ -algebra. By Proposition 4.12,  $A$  is nowhere scattered if and only if its stabilization is. We may therefore assume that  $A$  is stable.

First, assume that  $A$  is nowhere scattered. Let  $x \in \text{Cu}(A)$  and  $k \geq 2$ . Choose  $a \in A_+$  with  $x = [a]$ , and let  $B := \overline{aAa}$  be the generated hereditary sub- $C^*$ -algebra. By [36, Theorem 5.3 (iii)],  $B$  has no irreducible representations of dimension less than or equal to  $k - 1$  if and only if  $[a]$  is weakly  $(k, \omega)$ -divisible in  $\text{Cu}(B)$  (equivalently, in  $\text{Cu}(A)$ ). Thus, it follows from Theorem 3.1 that  $x$  is weakly  $(k, \omega)$ -divisible.

Conversely, assume that every element of  $\text{Cu}(A)$  is weakly  $(2, \omega)$ -divisible. To reach a contradiction, assume that  $A$  is not nowhere scattered. We need to find  $x \in \text{Cu}(A)$  that is not weakly  $(2, \omega)$ -divisible. Applying Theorem 3.1 we obtain a hereditary sub- $C^*$ -algebra  $B \subseteq A$  that admits a one-dimensional irreducible representation  $\pi$ . Choose  $c \in B_+$  with  $\pi(c) \neq 0$ , and consider the hereditary sub- $C^*$ -algebra  $C := cAc$ . Then  $\pi$  restricts to a one-dimensional irreducible representation of  $C$ . Using [36, Theorem 5.3 (iii)] again, it follows that  $[c]$  is not weakly  $(2, \omega)$ -divisible in  $\text{Cu}(C)$ , and therefore also not in  $\text{Cu}(A)$ .

### 9. Real rank zero and stable rank one

In this section, we establish characterizations of nowhere scatteredness among  $C^*$ -algebras of real rank zero or stable rank one.

A unital  $C^*$ -algebra has *real rank zero* if the invertible, selfadjoint elements are dense in the set of selfadjoint elements. A nonunital  $C^*$ -algebra has real rank zero if its minimal unitization does. This important property is well studied and enjoys many nice permanence properties. We refer to [8, p. 453ff] and [10].

The *Murray–von Neumann semigroup*  $V(A)$  of a  $C^*$ -algebra  $A$  is defined as the set of equivalence classes of projections in  $A \otimes \mathcal{K}$ , where two projections  $p$  and  $q$  are equivalent if there exists a partial isometry  $v \in A \otimes \mathcal{K}$  with  $p = vv^*$  and  $q = v^*v$ . Equipped with the addition induced by orthogonal sum and the algebraic order  $\leq_{\text{alg}}$ ,  $V(A)$  becomes a pre-ordered monoid.

A semigroup  $S$  is *weakly divisible* if for every  $x \in S$  there exist  $y, z \in S$  such that  $x = 2y + 3z$ .

**Theorem 9.1.** *Let  $A$  be a  $C^*$ -algebra of real rank zero. Then the following are equivalent:*

- (1)  $A$  is nowhere scattered;
- (2)  $V(A)$  is weakly divisible;
- (3)  $\text{Cu}(A)$  is weakly divisible.

*Proof.* To show that (1) implies (2), assume that  $A$  is nowhere scattered. By Proposition 4.12, we may assume  $A$  to be stable. Thus, any element in  $V(A)$  is of the form  $[p]$  with  $p$  a projection in  $A$ .

Then, given any  $[p] \in V(A)$ , it follows from Theorem 3.1 (7) that  $pAp$  has no one-dimensional irreducible representation. As shown in the proof of [7, Corollary 6.8], this implies that  $[p] = 2y + 3z$  for some  $y, z \in V(A)$ .

Assume now that (2) is satisfied. In this case, if  $x' \leq_{\text{alg}} x$  and  $x' = 2y' + 3z'$  in  $V(A)$ , then there exist  $y, z \in V(A)$  with  $y' \leq_{\text{alg}} y$  and  $z' \leq_{\text{alg}} z$ , and such that  $x = 2y + 3z$ . Indeed, given  $w \in V(A)$  such that  $x' + w = x$ , we can find  $u, v$  satisfying  $w = 2u + 3v$ . Setting  $y = y' + u$  and  $z = z' + v$ , the result follows.

Since  $A$  has real rank zero, there exists an order preserving, monoid morphism  $\alpha: V(A) \rightarrow \text{Cu}(A)$  with sup-dense image; see, for example, [47, Theorem 5.7]. That is, every element  $x \in \text{Cu}(A)$  can be written as the supremum of an increasing sequence in  $\alpha(V(A))$ . By the remark above, this implies that

$$x = \sup_n (2y_n + 3z_n)$$

with  $(y_n)_n, (z_n)_n$  increasing sequences in  $\alpha(V(A))$ .

Thus, we get  $x = 2 \sup_n (y_n) + 3 \sup_n (z_n)$ , as desired.

Finally, to show that (3) implies (1), assume that  $\text{Cu}(A)$  is weakly divisible. By Theorem 8.9 it suffices to show that every element in  $\text{Cu}(A)$  is weakly  $(2, \omega)$ -divisible. So

let  $x', x \in \text{Cu}(A)$  satisfy  $x' \ll x$ . By assumption, we obtain  $y, z \in \text{Cu}(A)$  such that  $x = 2y + 3z$ . Set  $s := x + y$ . Then, one gets  $2s = 2x + 2y \leq x$  and  $x' \leq x \leq 3y + 3z = 3s$ , as required. ■

**Remark 9.2.** If  $A$  does not have real rank zero, then weak divisibility of  $V(A)$  is not equivalent to nowhere scatteredness.

Take, for example, the Jiang–Su algebra  $\mathcal{Z}$ . Then, since  $1 \in \mathbb{N}$  cannot be decomposed as  $2x + 3y$  in  $\mathbb{N}$ , it follows that  $V(\mathcal{Z}) \cong \mathbb{N}$  is not weakly divisible.

However, it follows from Example 3.2 that  $\mathcal{Z}$  is nowhere scattered.

A unital  $C^*$ -algebra has *stable rank one* if its invertible elements are dense. A non-unital  $C^*$ -algebra has stable rank one if its minimal unitization does. Just as real rank zero, the property of stable rank one enjoys many permanence properties. We refer to [8, Section V.3.1].

Cuntz semigroups of stable rank one  $C^*$ -algebras have additional regularity characteristics; see [3, 45]. In particular, by [3, Theorem 3.5], they satisfy the *Riesz interpolation property*: If  $x_1, x_2, y_1, y_2$  are such that  $x_j \leq y_k$  for each  $j, k \in \{1, 2\}$ , then there exists  $z$  with  $x_1, x_2 \leq z \leq y_1, y_2$ .

A Cu-semigroup  $S$  is *countably based* if it contains a countable subset  $B$  such that every element in  $S$  is the supremum of an increasing sequence of elements in  $B$ . If  $S$  is a countably based Cu-semigroup, then every upward directed subset of  $S$  has a supremum. Separable  $C^*$ -algebras have countably based Cuntz semigroups.

Given a separable, stable rank one  $C^*$ -algebra  $A$ , it follows from [3, Theorem 3.8] that  $\text{Cu}(A)$  is *inf-semilattice ordered*. That is, infima of finite sets exists and, for every triple  $x, y, z$ , one has

$$(x + z) \wedge (y + z) = (x \wedge y) + z.$$

**Definition 9.3.** We say that a Cu-semigroup  $S$  satisfies the *interval axiom* if for all  $x', x, y, u, v \in S$  satisfying

$$x' \ll x, \quad x \ll y + u, \quad \text{and} \quad x \ll y + v,$$

there exists  $w \in S$  such that

$$x' \ll y + w \quad \text{and} \quad w \ll u, v.$$

**Remark 9.4.** The interval axiom as defined in Definition 9.3 above is the Cu-version of the ‘algebraic interval axiom’ considered by Wehrung for positively ordered monoids in [50, Paragraph 1.3].

**Proposition 9.5.** *Let  $S$  be a countably based Cu-semigroup. Then  $S$  is inf-semilattice ordered if and only if  $S$  has the Riesz interpolation property and satisfies the interval axiom.*

*Proof.* To show that forward implication, assume that  $S$  is inf-semilattice ordered. Given  $x_j, y_k \in S$  with  $x_j \leq y_k$  for  $i, j \in \{1, 2\}$ , we have  $x_1, x_2 \leq (y_1 \wedge y_2) \leq y_1, y_2$ , which shows that  $S$  has the Riesz interpolation property. To verify the interval axiom, let  $x', x, y, u, v \in S$  satisfy

$$x' \ll x, \quad x \ll y + u, \quad \text{and} \quad x \ll y + v.$$

Then

$$x' \ll x \leq (y + u) \wedge (y + v) = y + (u \wedge v),$$

which allows us to choose  $w \in S$  such that  $x' \ll y + w$  and  $w \ll (u \wedge v)$ . Thus,  $w$  has the desired properties.

Let us show the backward implication. Given  $x, y \in S$ , it follows from the Riesz interpolation property that the set  $L := \{z \in S : z \leq x, y\}$  is upward directed. Since  $S$  is countably based, the supremum of  $L$  exists and  $x \wedge y = \sup L$ . Thus,  $S$  is an inf-semilattice.

To show that addition distributes over infima, let  $x, y, z \in S$  and note that the inequality  $(x + z) \wedge (y + z) \geq (x \wedge y) + z$  is clear. For the other inequality, set  $w = (x + z) \wedge (y + z)$ , and let  $w' \in S$  satisfy  $w' \ll w$ . Applying the interval axiom, we obtain  $s$  such that  $w' \leq z + s$  and  $s \leq x, y$ . Then  $s \leq x \wedge y$ , and therefore  $w' \leq (x \wedge y) + z$ . Since this holds for every  $w'$  way-below  $w$ , we get  $w \leq (x \wedge y) + z$ . ■

A submonoid  $T$  of a Cu-semigroup  $S$  is said to be a *sub-Cu-semigroup* if  $T$  is a Cu-semigroup with the induced order, and if the inclusion  $T \rightarrow S$  preserves suprema of increasing sequences and the way-below relation.

In analogy to its definition for  $C^*$ -algebras (as defined in Section 4), we say that a property  $\mathcal{P}$  for Cu-semigroups satisfies the *Löwenheim–Skolem condition* if for every Cu-semigroup with property  $\mathcal{P}$  there exists a family  $\mathcal{S}$  of countably based sub-Cu-semigroups of  $S$  each having property  $\mathcal{P}$ , and such that  $\mathcal{S}$  is  $\sigma$ -complete and cofinal; see [46, Paragraph 5.2]. The next result can be proved with the methods that are used to prove [46, Proposition 5.3]. We omit the details.

**Proposition 9.6.** *The interval axiom and the Riesz interpolation property both satisfy the Löwenheim–Skolem condition.*

If  $A$  is a separable  $C^*$ -algebra of stable rank one, then  $\text{Cu}(A)$  is inf-semilattice ordered by [3, Theorem 3.8], and thus satisfies the interval axiom. With the techniques of [46], we can show that this also holds in the nonseparable case.

**Corollary 9.7.** *Let  $A$  be a  $C^*$ -algebra of stable rank one. Then  $\text{Cu}(A)$  satisfies weak cancellation, the Riesz interpolation property, and the interval axiom.*

*Proof.* It follows from [37, Theorem 4.3] and [3, Theorem 3.5] that  $\text{Cu}(A)$  is weakly cancellative and has the Riesz interpolation property. To show that  $\text{Cu}(A)$  satisfies the interval axiom, let  $x', x, y, u, v \in \text{Cu}(A)$  satisfy

$$x' \ll x, \quad x \ll y + u, \quad \text{and} \quad x \ll y + v.$$

Applying [46, Proposition 6.1], we obtain a  $\sigma$ -complete and cofinal collection of separable sub- $C^*$ -algebras  $B \subseteq A$  such that the inclusion  $B \rightarrow A$  induces an order-embedding  $\text{Cu}(B) \rightarrow \text{Cu}(A)$  whose image contains  $x', x, y, u, v$ . Using that stable rank one satisfies the Löwenheim–Skolem condition, we may choose such separable sub- $C^*$ -algebras  $B \subseteq A$  with stable rank one. Thus, each Cuntz semigroup  $\text{Cu}(B)$  is inf-semilattice ordered by [3, Theorem 3.8]. It follows from Proposition 9.5 that  $\text{Cu}(B)$  satisfies the interval axiom. Hence, an element  $w$  with the desired properties can be found in  $\text{Cu}(B)$  and, since  $\text{Cu}(B)$  can be identified with a sub-Cu-semigroup of  $\text{Cu}(A)$ , we deduce that  $\text{Cu}(A)$  satisfies the interval axiom. ■

**Remark 9.8.** Cuntz semigroups of separable  $C^*$ -algebras of real rank zero do not necessarily satisfy the interval axiom. Indeed, in [20] Goodearl constructs a separable (stably finite, nuclear)  $C^*$ -algebra  $A$  of real rank zero such that  $K_0(A)$  does not have the Riesz decomposition property. Thus,  $K_0(A)$  does not have the Riesz interpolation property and, by [32, Lemma 4.2], it follows that  $V(A)$  does not have the Riesz interpolation property either. However, as noted in [50], if a refinement monoid satisfies the (algebraic) interval axiom, then it satisfies Riesz interpolation. By [5, Lemma 2.3], which is based on [51],  $V(A)$  is a refinement monoid and, therefore,  $V(A)$  does not satisfy the algebraic interval axiom. Using that  $\text{Cu}(A)$  is isomorphic to the sequential ideal completion of  $V(A)$  (see, for example, [4, Remark 5.5.6]), we deduce that  $\text{Cu}(A)$  does not satisfy the interval axiom.

**Question 9.9.** For which  $C^*$ -algebras does the Cuntz semigroup satisfy the interval axiom?

**Lemma 9.10.** *Let  $S$  be a weakly cancellative Cu-semigroup satisfying (O5), the interval axiom, and the Riesz interpolation property, and let  $x \in S$  and  $k \geq 2$ . Then  $x$  is weakly  $(k, \omega)$ -divisible if and only if  $x$  is  $(k, \omega)$ -divisible.*

*Proof.* It suffices to show the forward implication. Thus, assume that  $x$  is weakly  $(k, \omega)$ -divisible. Then there exists a sequence  $(y_n)_n$  in  $S$  such that  $ky_n \leq x$  for each  $n$ , and such that  $x \leq \sum_{n=1}^\infty y_n$ .

By [46, Propositions 5.3, 5.4] and Proposition 9.6, the properties (O5), weak cancellation, the interval axiom, and the Riesz interpolation property each satisfy the Löwenheim–Skolem condition. Using this, we find a countably based sub-Cu-semigroup  $T \subseteq S$  containing  $x, y_1, y_2, \dots$ , and such that  $T$  satisfies (O5), weak cancellation, the interval axiom, and the Riesz interpolation property. By Proposition 9.5,  $T$  is inf-semilattice ordered.

We note that  $x$  is weakly  $(k, \omega)$ -divisible in  $T$ . Applying [3, Theorem 5.5], it follows that  $x$  is  $(k, \omega)$ -divisible in  $T$ , and hence also in  $S$ . ■

**Theorem 9.11.** *Let  $A$  be a  $C^*$ -algebra of stable rank one. Then the following are equivalent:*

- (1)  $A$  is nowhere scattered;
- (2) every element in  $\text{Cu}(A)$  is  $(2, \omega)$ -divisible;

(3) every element in  $\text{Cu}(A)$  is  $(k, \omega)$ -divisible for every  $k \geq 2$ .

*Proof.* As observed in Paragraph 7.1,  $\text{Cu}(A)$  is a Cu-semigroup satisfying (O5). By Corollary 9.7,  $\text{Cu}(A)$  satisfies weak cancellation, the Riesz interpolation property, and the interval axiom. Thus, Lemma 9.10 shows that an element in  $\text{Cu}(A)$  is weakly  $(k, \omega)$ -divisible if and only if it is  $(k, \omega)$ -divisible. Now the result follows from Theorem 8.9. ■

The question of whether this stronger divisibility property in  $\text{Cu}(A)$  characterizes nowhere scatteredness is connected to the global Glimm problem, which will be studied in [48].

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