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## Research Article

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# Galton–Watson Theta-Processes in a Varying Environment

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**Abstract:** We consider a special class of Galton–Watson theta-processes in a varying environment fully defined by four parameters, with two of them  $(\theta, r)$  being fixed over time  $n$ , and the other two  $(a_n, c_n)$  characterizing the altering reproduction laws. We establish a sequence of transparent limit theorems for the theta-processes with possibly defective reproduction laws. These results may serve as a stepping stone towards incisive general results for the Galton–Watson processes in a varying environment.

**Keywords:** Branching Process, Varying Environment, Limit Theorem

**MSC 2020:** 60J80

## 1 Introduction

The basic version of the Galton–Watson process (GW-process) was conceived as a stochastic model of the population growth or extinction of a single species of individuals [3, 7]. The GW-process  $\{Z_n\}_{n \geq 0}$  unfolds in the discrete time setting, with  $Z_n$  standing for the population size at the generation  $n$  under the assumption that each individual is replaced by a random number of offspring. It is assumed that the offspring numbers are independent random variables having the same distribution  $\{p(j)\}_{j \geq 0}$ .

By allowing the offspring number distribution  $\{p_n(j)\}_{j \geq 0}$  to depend on the generation number  $n$ , we arrive at the GW-process in a varying environment [4]. This more flexible model is fully described by a sequence of probability generating functions

$$f_n(s) = \sum_{j \geq 0} p_n(j) s^j, \quad 0 \leq s \leq 1, \quad n \geq 1.$$

Introduce the composition of generating functions

$$F_n(s) = f_1 \circ \dots \circ f_n(s), \quad 0 \leq s \leq 1, \quad n \geq 1.$$

Given that the GW-process starts at time zero with a single individual, we get

$$E(s^{Z_n}) = F_n(s), \quad P(Z_n = 0) = F_n(0).$$

The state 0 of the GW-process is absorbing and the extinction probability for the modeled population is determined by

$$q = \lim F_n(0)$$

(here and throughout, all limits are taken as  $n \rightarrow \infty$ , unless otherwise specified). In the case of *proper* reproduction laws with  $f_n(1) = 1$  for all  $n \geq 1$ , we get

$$E(Z_n) = F_n'(1) = f_1'(1) \cdots f_n'(1), \quad E(Z_n | Z_n > 0) = \frac{F_n'(1)}{1 - F_n(0)}.$$

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In [5], the usual ternary classification of the GW-processes into supercritical, critical, and subcritical processes [1], was adapted to the framework of the varying environment. Given  $0 < f'_n(1) < \infty$  for all  $n$ , it was shown that under a regularity condition (A) in [5], it makes sense to distinguish among four classes of the GW-processes in a varying environment: supercritical, asymptotically degenerate, critical, and subcritical processes. In a more recent paper [10] devoted to the Markov theta-branching processes in a varying environment, the quaternary classification of [5] was further refined into a quinary classification, which can be adapted to the discrete time setting as follows:

- *supercritical case*:  $q < 1$  and  $\lim E(Z_n) = \infty$ ,
- *asymptotically degenerate case*:  $q < 1$  and  $\liminf E(Z_n) < \infty$ ,
- *critical case*:  $q = 1$  and  $\lim E(Z_n|Z_n > 0) = \infty$ ,
- *strictly subcritical case*:  $q = 1$  and a finite  $\lim E(Z_n|Z_n > 0)$  exists,
- *loosely subcritical case*:  $q = 1$  and  $\lim E(Z_n|Z_n > 0)$  does not exist.

Our paper is build upon the properties of a special parametric family of generating functions [9] leading to what will be called here the Galton–Watson theta-processes or  $GW^\theta$ -processes. The remarkable property of the  $GW^\theta$ -processes in a varying environment is that the generating functions  $F_n(s)$  have explicit expressions presented in Section 2. An important feature of the  $GW^\theta$ -processes is that they allow for defective reproduction laws. If the generating function  $f_i(s)$  is *defective*, in that  $f_i(1) < 1$ , then  $F_n(1) < 1$  for all  $n \geq i$ . In the defective case [6, 11], a single individual, with probability  $1 - f_i(1)$  may force the entire GW-process to visit to an ancillary absorbing state  $\Delta$  by the observation time  $n$  with probability

$$P(Z_n = \Delta) = 1 - F_n(1).$$

In Sections 3 and 4, we state ten limit theorems for the  $GW^\theta$ -processes in a varying environment. These results are illuminated in Section 5 by ten examples describing different growth and extinction patterns under environmental variation. The proofs are collected in Section 6.

## 2 Proper and Defective Reproduction Laws

**Definition 1.** Consider a sequence  $(\theta, r, a_n, c_n)_{n \geq 1}$  satisfying one of the following sets of conditions:

- (a)  $\theta \in (0, 1]$ ,  $r = 1$ , and for  $n \geq 1$ ,  $0 < a_n < \infty$ ,  $c_n > 0$ ,  $c_n \geq 1 - a_n$ ,
- (b)  $\theta \in (0, 1]$ ,  $r > 1$ , and for  $n \geq 1$ ,  $0 < a_n < 1$ ,  $(1 - a_n)r^{-\theta} \leq c_n \leq (1 - a_n)(r - 1)^{-\theta}$ ,
- (c)  $\theta \in (-1, 0)$ ,  $r = 1$ , and for  $n \geq 1$ ,  $0 < a_n < 1$ ,  $0 < c_n \leq 1 - a_n$ ,
- (d)  $\theta \in (-1, 0)$ ,  $r > 1$ , and for  $n \geq 1$ ,  $0 < a_n < 1$ ,  $(1 - a_n)(r - 1)^{-\theta} \leq c_n \leq (1 - a_n)r^{-\theta}$ ,
- (e)  $\theta = 0$ ,  $r = 1$ , and for  $n \geq 1$ ,  $0 < a_n < 1$ ,  $0 \leq c_n < 1$ ,
- (f)  $\theta = 0$ ,  $r > 1$ , and for  $n \geq 1$ ,  $0 < a_n < 1$ ,  $0 \leq c_n \leq 1$ .

A  $GW^\theta$ -process with parameters  $(\theta, r, a_n, c_n)_{n \geq 1}$  is a GW-process in a varying environment characterized by a sequence of probability generating functions  $(f_n(s))_{n \geq 1}$  defined by

$$f_n(s) = r - (a_n(r - s)^{-\theta} + c_n)^{-\frac{1}{\theta}}, \quad 0 \leq s < r, \quad f_n(r) = r, \quad (2.1)$$

for  $\theta \neq 0$ , and for  $\theta = 0$ , defined by

$$f_n(s) = r - (r - c_n)^{1-a_n}(r - s)^{a_n}, \quad 0 \leq s \leq r. \quad (2.2)$$

Definition 1 is motivated by [9, Definitions 14.1 and 14.2], which also mentions a trivial case of  $\theta = -1$  not included here. Observe that in the setting of varying environment, the key parameters  $\theta \in (-1, 1]$  and  $r \geq 1$  stay constant over time, while the parameters  $(a_n, c_n)$  may vary. The case  $\theta = r = 1$  is the well studied case of the linear-fractional reproduction law.

This section contains two key lemmas. Lemma 1 gives the explicit expressions for the generating functions  $F_n(s)$  in terms of positive constants  $A_n, C_n, D_n = D_n(r)$  defined by

$$A_0 = 1, \quad A_n = \prod_{i=1}^n a_i, \quad C_n = \sum_{i=1}^n A_{i-1} c_i, \quad D_n = \prod_{i=1}^n (r - c_i)^{A_{i-1} - A_i}.$$

Lemmas 2 presents the asymptotic properties of the constants  $A_n, C_n, D_n$  leading to the limit theorems stated in Sections 3 and 4.

**Lemma 1.** Consider a  $\text{GW}^\theta$ -process with parameters  $(\theta, r, a_n, c_n)$ . If  $\theta \neq 0$ , then

$$F_n(s) = r - (A_n(r-s)^{-\theta} + C_n)^{-\frac{1}{\theta}}, \quad 0 \leq s < r, \quad F_n(r) = r, \quad n \geq 1,$$

and if  $\theta = 0$ , then

$$F_n(s) = r - (r-s)^{A_n} D_n, \quad 0 \leq s \leq r, \quad n \geq 1.$$

Here:

(a) for  $\theta \in (0, 1], r = 1$ ,

$$0 < A_n < \infty, \quad C_n > 0, \quad C_n \geq 1 - A_n, \quad F_n(1) = 1, \quad F'_n(1) = A_n^{-\frac{1}{\theta}}, \quad n \geq 1,$$

(b) for  $\theta \in (0, 1], r > 1$ ,

$$0 < A_n < 1, \quad (1 - A_n)r^{-\theta} \leq C_n \leq (1 - A_n)(r-1)^{-\theta}, \quad F_n(1) \leq 1, \quad n \geq 1,$$

with  $F_n(1) = 1$  if and only if  $c_k = (1 - a_k)(r-1)^{-\theta}$ ,  $1 \leq k \leq n$ , implying  $F'_n(1) = A_n$ ,

(c) for  $\theta \in (-1, 0), r = 1$ ,

$$0 < A_n < \infty, \quad 0 < C_n \leq 1 - A_n, \quad F_n(1) = 1 - C_n^{-\frac{1}{\theta}}, \quad n \geq 1,$$

(d) for  $\theta \in (-1, 0), r > 1$ ,

$$0 < A_n < 1, \quad (1 - A_n)(r-1)^{-\theta} \leq C_n \leq (1 - A_n)r^{-\theta}, \quad F_n(1) \leq 1, \quad n \geq 1,$$

with  $F_n(1) = 1$  if and only if  $c_k = (1 - a_k)(r-1)^{-\theta}$ ,  $1 \leq k \leq n$ , implying  $F'_n(1) = A_n$ ,

(e) for  $\theta = 0, r = 1$ ,

$$0 < A_n < 1, \quad 0 < D_n \leq 1, \quad F_n(1) = 1, \quad F'_n(1) = \infty, \quad n \geq 1,$$

(f) for  $\theta = 0, r > 1$ ,

$$0 < A_n < 1, \quad (r-1)^{1-A_n} \leq D_n \leq r^{1-A_n}, \quad F_n(1) \leq 1, \quad n \geq 1,$$

with  $F_n(1) = 1$  if and only if  $c_k = 1$ ,  $1 \leq k \leq n$ , implying  $F'_n(1) = A_n$ .

**Lemma 2.** Denote the limits  $A = \lim A_n, C = \lim C_n, D = \lim D_n$ , whenever they exist, whether finite or infinite.

(a) If  $\theta \in (0, 1], r = 1$ , then  $C \in [1, \infty]$ , and if  $C < \infty$ , then  $A \in [0, \infty]$ .

(b) If  $\theta \in (0, 1], r > 1$ , then  $A \in [0, 1)$  and  $(1 - A)r^{-\theta} \leq C \leq (1 - A)(r-1)^{-\theta}$ .

(c) If  $\theta \in (-1, 0), r = 1$ , then  $A \in [0, 1)$  and  $0 < C \leq 1 - A$ .

(d) If  $\theta \in (-1, 0), r > 1$ , then  $A \in [0, 1)$  and  $(1 - A)(r-1)^{-\theta} \leq C \leq (1 - A)r^{-\theta}$ .

(e) If  $\theta = 0, r = 1$ , then  $A \in [0, 1)$  and  $D = \prod_{n \geq 1} (1 - c_n)^{A_{n-1} - A_n}$  with  $D \in [0, 1]$ .

(f) If  $\theta = 0, r > 1$ , then  $A \in [0, 1)$  and  $D = \prod_{n \geq 1} (r - c_n)^{A_{n-1} - A_n}$  with  $(r-1)^{1-A} \leq D \leq r^{1-A}$ .

### 3 Limit Theorems for the Proper $\text{GW}^\theta$ -Processes

Theorems 1–5 deal with the  $\text{GW}^\theta$ -process in the case  $\theta \in (0, 1], r = 1$ , when by Lemma 1,

$$E(Z_n) = A_n^{-\frac{1}{\theta}}, \quad P(Z_n > 0) = (A_n + C_n)^{-\frac{1}{\theta}}.$$

Putting  $B_n = \frac{C_n}{A_n}$ , we obtain

$$E(Z_n | Z_n > 0) = (1 + B_n)^{\frac{1}{\theta}}.$$

These five theorems fully cover the five regimes of reproduction in a varying environment and could be summarized as follows. Let  $\theta \in (0, 1], r = 1$ ,

- given  $C < \infty$ , the  $\text{GW}^\theta$ -process is
  - supercritical if  $A_n \rightarrow 0$ , see Theorem 1,
  - asymptotically degenerate if  $A_n \rightarrow A \in (0, \infty)$ , see Theorem 2,
  - strictly subcritical if  $A_n \rightarrow \infty$ , see Theorem 4,

- given  $C = \infty$ , the  $\text{GW}^\theta$ -process is
  - critical if  $B_n \rightarrow \infty$ , see Theorem 3,
  - strictly subcritical if  $B_n \rightarrow B \in [0, \infty)$ , see Theorem 4,
  - loosely subcritical if the  $\lim B_n$  does not exist, see Theorem 5.

This section also includes Theorem 6 addressing the proper case  $\theta = 0$ ,  $r = 1$ . Notice that Theorem 6 deals with the case of infinite mean values, when the above mentioned quinary classification does not apply.

**Theorem 1.** *Let  $\theta \in (0, 1]$ ,  $r = 1$ , and  $C < \infty$ . If  $A_n \rightarrow 0$ , then  $q = 1 - C^{-\frac{1}{\theta}}$  and  $A_n^{\frac{1}{\theta}} Z_n$  almost surely converges to a random variable  $W$  such that*

$$E(e^{-\lambda W}) = 1 - (\lambda^{-\theta} + C)^{-\frac{1}{\theta}}, \quad \lambda \geq 0.$$

**Theorem 2.** *Let  $\theta \in (0, 1]$ ,  $r = 1$ , and  $C < \infty$ . If  $A_n \rightarrow A \in (0, \infty)$ , then*

$$q = 1 - (A + C)^{-\frac{1}{\theta}}, \quad E(Z_n) \rightarrow A^{-\frac{1}{\theta}},$$

and  $Z_n$  almost surely converges to a random variable  $Z_\infty$  such that

$$E(Z_\infty) = A^{-\frac{1}{\theta}}, \quad E(s^{Z_\infty}) = 1 - (A(1-s)^{-\theta} + C)^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

**Theorem 3.** *Let  $\theta \in (0, 1]$ ,  $r = 1$ , and  $C = \infty$ . If  $B_n \rightarrow \infty$ , then  $q = 1$ ,*

$$P(Z_n > 0) \sim C_n^{-\frac{1}{\theta}}, \quad E(Z_n | Z_n > 0) \sim B_n^{\frac{1}{\theta}},$$

and with  $\lambda_n = \lambda B_n^{-\frac{1}{\theta}}$ ,

$$E(e^{-\lambda_n Z_n} | Z_n > 0) \rightarrow 1 - (1 + \lambda^{-\theta})^{-\frac{1}{\theta}}, \quad \lambda \geq 0.$$

**Theorem 4.** *Let  $\theta \in (0, 1]$  and  $r = 1$ . If  $A_n \rightarrow \infty$  and  $B_n \rightarrow B \in [0, \infty)$ , then  $q = 1$ ,*

$$P(Z_n > 0) \sim (1 + B)^{-\frac{1}{\theta}} A_n^{-\frac{1}{\theta}}, \quad E(Z_n | Z_n > 0) \rightarrow (1 + B)^{\frac{1}{\theta}},$$

and

$$E(s^{Z_n} | Z_n > 0) \rightarrow 1 - ((1 + B)(1 - s)^{-\theta} + B + B^2)^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

**Theorem 5.** *Let  $\theta \in (0, 1]$ ,  $r = 1$ , and assume that  $\lim B_n$  does not exist. Then  $q = 1$  and letting*

$$B_{k_n} \rightarrow B \in [0, \infty]$$

along a subsequence  $k_n \rightarrow \infty$ , we get:

(i) *if  $B = \infty$ , then*

$$P(Z_{k_n} > 0) \sim C_{k_n}^{-\frac{1}{\theta}}, \quad E(Z_{k_n} | Z_{k_n} > 0) \sim B_{k_n}^{\frac{1}{\theta}},$$

and with  $\lambda_n = \lambda B_{k_n}^{-\frac{1}{\theta}}$ ,

$$E(e^{-\lambda_{k_n} Z_{k_n}} | Z_{k_n} > 0) \rightarrow 1 - (1 + \lambda^{-\theta})^{-\frac{1}{\theta}}, \quad \lambda \geq 0,$$

(ii) *if  $B \in [0, \infty)$ , then  $A_{k_n} \rightarrow \infty$ ,*

$$P(Z_{k_n} > 0) \sim (1 + B)^{-\frac{1}{\theta}} A_{k_n}^{-\frac{1}{\theta}}, \quad E(Z_{k_n} | Z_{k_n} > 0) \rightarrow (1 + B)^{\frac{1}{\theta}},$$

and

$$E(s^{Z_{k_n}} | Z_{k_n} > 0) \rightarrow 1 - ((1 + B)(1 - s)^{-\theta} + B + B^2)^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

**Theorem 6.** *Suppose  $\theta = 0$  and  $r = 1$ . Then  $P(Z_n > 0) = D_n$ , so that  $q = 1 - D$ , with  $D$  given by Lemma 2 (e). Furthermore:*

(i) *if  $A = 0$  and  $D = 0$ , then  $q = 1$  and*

$$P(A_n \ln Z_n \leq x | Z_n > 0) \rightarrow 1 - e^{-x}, \quad x \geq 0,$$

(ii) *if  $A = 0$  and  $D > 0$ , then  $q < 1$  and*

$$P(A_n \ln Z_n \leq x) \rightarrow 1 - e^{-x} D, \quad x \geq 0,$$

(iii) if  $A \in (0, 1)$  and  $D = 0$ , then  $q = 1$  and

$$E(s^{Z_n} | Z_n > 0) \rightarrow 1 - (1 - s)^A, \quad 0 \leq s \leq 1,$$

(iv) if  $A \in (0, 1)$  and  $D > 0$ , then  $q < 1$  and  $Z_n$  almost surely converges to a random variable  $Z_\infty$  such that

$$E(Z_\infty) = \infty, \quad E(s^{Z_\infty}) = 1 - (1 - s)^A D, \quad 0 \leq s \leq 1.$$

## Remarks

We make the following observations.

- (i) It is a straightforward exercise to check that the above mentioned regularity condition (A) in [5] is valid for the  $\text{GW}^\theta$ -process in the case  $\theta \in (0, 1]$ ,  $r = 1$ .
- (ii) The limiting distribution obtained in Theorem 3 coincides with that of [12] obtained for the critical  $\text{GW}$ -processes in a constant environment with a possibly infinite variance for the offspring number.
- (iii) Statement (ii) of Theorem 6 is of the Darling–Seneta-type limit theorem obtained in [2] for  $\text{GW}$ -processes with infinite mean.
- (iv) Part (iv) of Theorem 6 presents the pattern of limit behavior similar to the asymptotically degenerate regime in the case of infinite mean values. The conditions of Theorem 6 (iv) hold if and only if

$$\sum_{n \geq 1} (1 - a_n) < \infty \tag{3.1}$$

and

$$\sum_{n \geq 1} (1 - a_n) \ln \frac{1}{1 - c_n} < \infty. \tag{3.2}$$

## 4 Limit Theorems for the Defective $\text{GW}^\theta$ -Process

In the defective case, there are two kinds of absorption times:

- (i)  $\tau_0$  the absorption time of the  $\text{GW}^\theta$ -process at 0,
- (ii)  $\tau_\Delta$  the absorption time of the  $\text{GW}^\theta$ -process at the state  $\Delta$ .

Let  $\tau = \min(\tau_0, \tau_\Delta)$  be the absorption time of the  $\text{GW}^\theta$ -process either at 0 or at the state  $\Delta$ . Let us recall that  $q = P(\tau_0 < \infty)$  and denote

$$q_\Delta = P(\tau_\Delta < \infty), \quad Q = P(\tau < \infty) = q + q_\Delta.$$

Clearly,

$$P(\tau \leq n) = P(\tau_0 \leq n) + P(\tau_\Delta \leq n) = F_n(0) + 1 - F_n(1),$$

implying

$$P(\tau > n) = F_n(1) - F_n(0).$$

Furthermore,

$$E(Z_n; \tau_\Delta > n) = F_n'(1), \quad E(s^{Z_n}; \tau_\Delta > n) = F_n(s), \quad 0 \leq s \leq 1,$$

so that

$$E(Z_n | \tau > n) = \frac{F_n'(1)}{F_n(1) - F_n(0)}, \quad E(s^{Z_n} | \tau > n) = \frac{F_n(s) - F_n(0)}{F_n(1) - F_n(0)}, \quad 0 \leq s \leq 1.$$

Theorems 7–10 present the transparent asymptotical results on these absorption probabilities and the limit behavior of the  $\text{GW}^\theta$ -process in the four defective cases. Corollaries of Theorems 7–9 deal with the proper subcases, where  $\tau = \tau_0$ . All three corollaries describe a strictly subcritical case, when  $A = 0$ , and an asymptotically degenerate case, when  $A \in (0, 1)$ .

**Theorem 7.** Consider the case  $\theta \in (0, 1]$ ,  $r > 1$ . Then

$$q = r - (Ar^{-\theta} + C)^{-\frac{1}{\theta}}, \quad q_{\Delta} = 1 - r + (A(r-1)^{-\theta} + C)^{-\frac{1}{\theta}},$$

where  $A \in [0, 1)$  and  $(1-A)r^{-\theta} \leq C \leq (1-A)(r-1)^{-\theta}$ .

(i) If  $A = 0$ , then

$$q = 1 - q_{\Delta} = r - C^{-\frac{1}{\theta}} \in [0, 1],$$

so that  $Q = 1$ . Furthermore,

$$\begin{aligned} A_n^{-1}P(\tau > n) &\rightarrow ((r-1)^{-\theta} - r^{-\theta})\theta^{-1}C^{-\frac{1}{\theta}-1}, \\ E(Z_n|\tau > n) &\rightarrow \frac{(r-1)^{-\theta-1}}{(r-1)^{-\theta} - r^{-\theta}}, \quad E(s^{Z_n}|\tau > n) \rightarrow \frac{(r-s)^{-\theta} - r^{-\theta}}{(r-1)^{-\theta} - r^{-\theta}}, \quad 0 \leq s \leq 1. \end{aligned}$$

(ii) If  $A \in (0, 1)$ , then  $Q \in [0, 1)$ ,

$$E(Z_n; \tau_{\Delta} > n) \rightarrow A(A + C(r-1)^{\theta})^{-\frac{1}{\theta}-1},$$

and  $Z_n$  almost surely converges to a random variable  $Z_{\infty}$  taking values in the set  $\{\Delta, 0, 1, 2, \dots\}$ , with

$$\begin{aligned} P(Z_{\infty} = \Delta) &= 1 - r + (A(r-1)^{-\theta} + C)^{-\frac{1}{\theta}}, \\ E(s^{Z_{\infty}}; Z_{\infty} \neq \Delta) &= r - (A(r-s)^{-\theta} + C)^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1. \end{aligned}$$

**Corollary.** Consider the case  $\theta \in (0, 1]$ ,  $r > 1$  assuming

$$c_n = (1 - a_n)(r-1)^{-\theta}, \quad n \geq 1, \quad (4.1)$$

so that  $C = (1-A)(r-1)^{-\theta}$  implying  $q_{\Delta} = 0$ .

(i) If  $A = 0$ , then  $q = 1$  with

$$A_n^{-1}P(Z_n > 0) \rightarrow ((r-1)^{-\theta} - r^{-\theta})\theta^{-1}(r-1)^{\theta+1}.$$

Furthermore,

$$E(Z_n|Z_n > 0) \rightarrow \frac{(r-1)^{-\theta-1}}{\theta((r-1)^{-\theta} - r^{-\theta})}, \quad E(s^{Z_n}|Z_n > 0) \rightarrow \frac{(r-s)^{-\theta} - r^{-\theta}}{(r-1)^{-\theta} - r^{-\theta}}, \quad 0 \leq s \leq 1.$$

(ii) If  $A \in (0, 1)$ , then

$$q = 1 - r + (Ar^{-\theta} + C)^{-\frac{1}{\theta}}, \quad E(Z_n) \rightarrow A,$$

so that  $q \in (0, 1)$ , and  $Z_n$  almost surely converges to a random variable  $Z_{\infty}$  such that

$$E(Z_{\infty}) = A, \quad E(s^{Z_{\infty}}) = r - (A(r-s)^{-\theta} + (1-A)(r-1)^{-\theta})^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

**Theorem 8.** Consider the case  $\theta \in (-1, 0)$ ,  $r > 1$  and put  $\alpha = -\frac{1}{\theta}$ , so that  $\alpha > 1$ . Then

$$q = r - (Ar^{\frac{1}{\alpha}} + C)^{\alpha}, \quad q_{\Delta} = 1 - r + (A(r-1)^{\frac{1}{\alpha}} + C)^{\alpha},$$

where  $A \in [0, 1)$  and  $(1-A)(r-1)^{\frac{1}{\alpha}} \leq C \leq (1-A)r^{\frac{1}{\alpha}}$ .

(i) If  $A = 0$ , then

$$q = 1 - q_{\Delta} = r - C^{\alpha} \in [0, 1],$$

so that  $Q = 1$ . Furthermore,

$$\begin{aligned} A_n^{-1}P(\tau > n) &\rightarrow \alpha C^{\alpha-1}(r^{\frac{1}{\alpha}} - (r-1)^{\frac{1}{\alpha}}), \\ E(Z_n|\tau > n) &\rightarrow \frac{(r-1)^{\frac{1}{\alpha}-1}}{r^{\frac{1}{\alpha}} - (r-1)^{\frac{1}{\alpha}}}, \quad E(s^{Z_n}|\tau > n) \rightarrow \frac{r^{\frac{1}{\alpha}} - (r-s)^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}} - (r-1)^{\frac{1}{\alpha}}}, \quad 0 \leq s \leq 1. \end{aligned}$$

(ii) If  $A \in (0, 1)$ , then  $Q \in [0, 1)$ ,

$$E(Z_n; \tau_{\Delta} > n) \rightarrow A(A + C(r-1)^{-\frac{1}{\alpha}})^{\alpha-1},$$

and  $Z_n$  almost surely converges to a random variable  $Z_{\infty}$  taking values in the set  $\{\Delta, 0, 1, 2, \dots\}$ , with

$$\begin{aligned} P(Z_{\infty} = \Delta) &= 1 - r + (A(r-1)^{\frac{1}{\alpha}} + C)^{\alpha}, \\ E(s^{Z_{\infty}}; Z_{\infty} \neq \Delta) &= r - (A(r-s)^{\frac{1}{\alpha}} + C)^{\alpha}, \quad 0 \leq s \leq 1. \end{aligned}$$

**Corollary.** Consider the case  $\theta \in (-1, 0)$ ,  $r > 1$  assuming (4.1), so that  $C = (1 - A)(r - 1)^{\frac{1}{\alpha}}$  implying  $q_{\Delta} = 0$ .

(i) If  $A = 0$ , then  $q = 1$  with

$$A_n^{-1}P(Z_n > 0) \rightarrow \alpha(r - 1)^{1 - \frac{1}{\alpha}}(r^{\frac{1}{\alpha}} - (r - 1)^{\frac{1}{\alpha}}).$$

Furthermore,

$$E(Z_n | Z_n > 0) \rightarrow \frac{(r - 1)^{-\theta - 1}}{\theta((r - 1)^{-\theta} - r^{-\theta})}, \quad E(s^{Z_n} | Z_n > 0) \rightarrow \frac{(r - s)^{-\theta} - r^{-\theta}}{(r - 1)^{-\theta} - r^{-\theta}}, \quad 0 \leq s \leq 1.$$

(ii) If  $A \in (0, 1)$ , then

$$q = 1 - r + (Ar^{\frac{1}{\alpha}} + (1 - A)(r - 1)^{\frac{1}{\alpha}})^{\alpha}, \quad E(Z_n) \rightarrow A,$$

so that  $q \in (0, 1)$ , and  $Z_n$  almost surely converges to a random variable  $Z_{\infty}$  such that

$$E(Z_{\infty}) = A, \quad E(s^{Z_{\infty}}) = r - (A(r - s)^{\frac{1}{\alpha}} + (1 - A)(r - 1)^{\frac{1}{\alpha}})^{\alpha}, \quad 0 \leq s \leq 1.$$

**Theorem 9.** Consider the case  $\theta = 0$ ,  $r > 1$  implying

$$q = r - r^A D, \quad q_{\Delta} = 1 - r + (r - 1)^A D, \quad Q = 1 - (r^A - (r - 1)^A) D,$$

where  $D$  is given by Lemma 2 (f).

(i) If  $A = 0$ , then  $Q = 1$ , and

$$P(\tau > n) \sim (\ln r - \ln(r - 1))A_n D_n.$$

Moreover,

$$E(Z_n | \tau > n) \rightarrow \frac{(r - 1)^{-1}}{\ln r - \ln(r - 1)}, \quad P(s^{Z_n} | \tau > n) \rightarrow \frac{\ln r - \ln(r - s)}{\ln r - \ln(r - 1)}, \quad 0 \leq s \leq 1.$$

(ii) If  $A \in (0, 1)$ , then  $Q < 1$ ,

$$(r - 1)^{1 - A} \leq D \leq r^{1 - A}, \quad E(Z_n; \tau_{\Delta} > n) \rightarrow A(r - 1)^{A - 1} D,$$

and  $Z_n$  almost surely converges to a random variable  $Z_{\infty}$  taking values in the set  $\{\Delta, 0, 1, 2, \dots\}$ , with

$$P(Z_{\infty} = \Delta) = 1 - r + (r - 1)^A D, \quad E(s^{Z_{\infty}}; Z_{\infty} \neq \Delta) = r - (r - s)^A D, \quad 0 \leq s \leq 1.$$

**Corollary.** Given  $\theta = 0$ ,  $r > 1$ , assume  $c_n \equiv 1$ . Then  $D = (r - 1)^{1 - A}$  implying  $q_{\Delta} = 0$ .

(i) If  $A = 0$ , then  $q = 1$ , and

$$P(Z_n > 0) \sim (\ln r - \ln(r - 1))A_n D_n.$$

Moreover,

$$E(Z_n | Z_n > 0) \rightarrow \frac{(r - 1)^{-1}}{\ln r - \ln(r - 1)}, \quad P(s^{Z_n} | Z_n > 0) \rightarrow \frac{\ln r - \ln(r - s)}{\ln r - \ln(r - 1)}, \quad 0 \leq s \leq 1.$$

(ii) If  $A \in (0, 1)$ , then

$$q = r - r^A (r - 1)^{1 - A}, \quad E(Z_n) \rightarrow A,$$

so that  $q \in (0, 1)$ , and  $Z_n$  almost surely converges to a proper random variable  $Z_{\infty}$ , such that

$$E(Z_{\infty}) = A, \quad E(s^{Z_{\infty}}) = r - (r - s)^A (r - 1)^{1 - A}, \quad 0 \leq s \leq 1.$$

**Theorem 10.** In the case  $\theta \in (-1, 0)$ ,  $r = 1$ , put  $\alpha = -\frac{1}{\theta}$ , so that  $\alpha > 1$ . Then

$$q = 1 - (A + C)^{\alpha}, \quad q_{\Delta} = C^{\alpha}, \quad Q = 1 - (A + C)^{\alpha} + C^{\alpha},$$

where  $A \in [0, 1)$  and  $0 < C \leq 1 - A$ .

(i) If  $A = 0$ , then  $q = 1 - q_{\Delta} = 1 - C^{\alpha}$ ,  $Q = 1$ , and

$$A_n^{-1}P(\tau > n) \rightarrow \alpha C^{\alpha - 1}.$$

Moreover,

$$E(s^{Z_n} | \tau > n) \rightarrow 1 - (1 - s)^{\frac{1}{\alpha}}, \quad 0 \leq s \leq 1.$$

(ii) If  $A \in (0, 1)$ , then  $Q < 1$ ,

$$E(Z_n; \tau_{\Delta} > n) = \infty,$$

and  $Z_n$  almost surely converges to a random variable  $Z_{\infty}$  taking values in the set  $\{\Delta, 0, 1, 2, \dots\}$ , with

$$P(Z_{\infty} = \Delta) = C^{\alpha}, \quad E(s^{Z_{\infty}}; Z_{\infty} \neq \Delta) = 1 - (A(1 - s)^{\frac{1}{\alpha}} + C)^{\alpha}, \quad 0 \leq s \leq 1.$$

## Remarks

We make the following observations.

- (i) Theorem 7 (ii) should be compared to the more general [6, Theorem 1], which allows the limit  $Z_\infty$  to take the value  $\infty$  with a positive probability. The convergence results for the conditional expectation should be compared to the statements of [6, Theorems 3 and 4].
- (ii) The conditional convergence in distribution stated in Theorem 7 (i) should be compared to [11, Theorem 2a ( $k = 0$ )] in the more general setting under the assumption of constant environment.

## 5 Examples

The following ten examples illustrate each of the ten theorems of this paper. Observe that given

$$c_n = (1 - a_n)\sigma, \quad n \geq 1, \quad (5.1)$$

for some suitable positive constant  $\sigma$ , we get  $C_n = (1 - A_n)\sigma$ ,  $n \geq 1$ . Similarly, if

$$c_n = (a_n - 1)\sigma, \quad n \geq 1, \quad (5.2)$$

for some suitable positive constant  $\sigma$ , then  $C_n = (A_n - 1)\sigma$ ,  $n \geq 1$ .

**Example 1.** Suppose  $\theta \in (0, 1]$ ,  $r = 1$ , and

$$a_n = \frac{n}{n+1}, \quad A_n = \frac{1}{n+1}, \quad n \geq 1. \quad (5.3)$$

If (5.1) holds for some  $\sigma \geq 1$ , then by Theorem 1,

$$q = 1 - \sigma^{-\frac{1}{\theta}}, \quad n^{-\frac{1}{\theta}}E(Z_n) \rightarrow 1,$$

and  $n^{-\frac{1}{\theta}}Z_n \rightarrow W$  almost surely, with

$$E(e^{-\lambda W}) = 1 - (\lambda^{-\theta} + \sigma)^{-\frac{1}{\theta}}, \quad \lambda \geq 0.$$

**Example 2.** Suppose  $\theta \in (0, 1]$ ,  $r = 1$ , and

$$a_n = \frac{n(n+3)}{(n+1)(n+2)}, \quad A_n = \frac{n+3}{3(n+1)}, \quad n \geq 1. \quad (5.4)$$

If (5.1) holds for some  $\sigma \geq 1$ , then by Theorem 2,

$$q = 1 - \left(\frac{3}{1+2\sigma}\right)^{\frac{1}{\theta}}, \quad E(Z_n) \rightarrow 3^{\frac{1}{\theta}},$$

and  $Z_n \rightarrow Z_\infty$  almost surely, with

$$E(Z_\infty) = 3^{\frac{1}{\theta}}, \quad E(s^{Z_\infty}) = 1 - 3^{\frac{1}{\theta}}(2\sigma + (1-s)^{-\theta})^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

**Example 3.** Suppose  $\theta \in (0, 1]$  and  $r = 1$ . Let

$$\begin{aligned} a_1 &= \frac{1}{2}, & a_{2n} &= 4, & a_{2n+1} &= \frac{1}{4}, \\ c_{2n-1} &= 1, & c_{2n} &= 2, \\ A_{2n-1} &= \frac{1}{2}, & A_{2n} &= 2, & n &\geq 1. \end{aligned}$$

Then  $C = \infty$  and  $B_n \rightarrow \infty$  implying the conditions of Theorem 3. Observe that for this example,  $\lim A_n$  does not exist.

**Example 4.** Suppose  $\theta \in (0, 1]$  and  $r = 1$ . Recall that Theorem 4 is the only one among Theorems 1–5 which may hold both with  $C < \infty$  and  $C = \infty$ . For this reason, we present two examples (1) and (2) for each of these two situations:

(1) Let

$$a_n = \frac{n+1}{n}, \quad c_n = \frac{1}{n^2(n+1)}, \quad n \geq 1,$$

implying

$$A_n = n+1, \quad C_n = \frac{n}{n+1}, \quad B_n = \frac{n}{(n+1)^2}, \quad n \geq 1.$$

In this case, according to Theorem 4,

$$P(Z_n > 0) \sim n^{-\frac{1}{\theta}}, \quad E(Z_n | Z_n > 0) \rightarrow 1,$$

and

$$E(s^{Z_n} | Z_n > 0) \rightarrow s, \quad 0 \leq s \leq 1.$$

(2) Let

$$a_n = \frac{n+1}{n}, \quad A_n = n+1, \quad n \geq 1,$$

and (5.2) hold for some  $\sigma > 0$ . Then

$$C_n = \sigma n, \quad B_n = \frac{\sigma n}{n+1}, \quad n \geq 1.$$

In this case, according to Theorem 4,

$$P(Z_n > 0) \sim (1 + \sigma)^{-\frac{1}{\theta}} n^{-\frac{1}{\theta}}, \quad E(Z_n | Z_n > 0) \rightarrow (1 + \sigma)^{\frac{1}{\theta}},$$

and

$$E(s^{Z_n} | Z_n > 0) \rightarrow 1 - ((1 + \sigma)(1 - s)^{-\theta} + \sigma + \sigma^2)^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

**Example 5.** Suppose  $\theta \in (0, 1]$  and  $r = 1$ . Let

$$a_n = \begin{cases} n & \text{for } n = 2^k - 1, k \geq 1, \\ \frac{1}{n-1} & \text{for } n = 2^k, k \geq 1, \\ 1 & \text{otherwise,} \end{cases} \quad A_n = \begin{cases} n & \text{for } n = 2^k - 1, k \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Taking

$$c_n = \begin{cases} 1 & \text{for } n = 2^k, k > 1, \\ \frac{1}{n^2} & \text{otherwise,} \end{cases}$$

we get

$$C_n = \sum_{k: 2 \leq 2^k \leq n} (2^k - 1 - 2^{-2^k}) + \sum_{k=1}^n k^{-2}, \quad n \geq 1,$$

implying  $C_{k_n} \sim 2^{n+1}$ , provided  $2^n - 1 \leq k_n < 2^{n+1} - 1$ . Thus, by Theorem 5, for  $k_n = 2^n$ ,  $\lambda_n = \lambda(2n)^{-\frac{1}{\theta}}$ ,

$$P(Z_{k_n} > 0) \sim (2k_n)^{-\frac{1}{\theta}}, \quad E(e^{-\lambda_{k_n} Z_{k_n}} | Z_{k_n} > 0) \rightarrow 1 - (1 + \lambda^{-\theta})^{-\frac{1}{\theta}}, \quad \lambda \geq 0,$$

and on the other hand, for  $k_n = 2^n - 1$ ,

$$P(Z_{k_n} > 0) \sim (3k_n)^{-\frac{1}{\theta}}, \quad E(s^{Z_{k_n}} | Z_{k_n} > 0) \rightarrow 1 - (3(1-s)^{-\theta} + 6)^{-\frac{1}{\theta}}, \quad 0 \leq s \leq 1.$$

**Example 6.** Suppose  $\theta = 0$ ,  $r = 1$ , and assume  $c_n = 1 - e^{-n^\sigma}$ ,  $-\infty < \sigma < \infty$ ,  $n \geq 1$ , yielding

$$D_n = \exp\left(-\sum_{i=1}^n i^\sigma (A_{i-1} - A_i)\right), \quad n \geq 1.$$

Notice that (5.3) implies  $A = 0$  and

$$D_n = \exp\left(-\sum_{i=1}^n \frac{i^{\sigma-1}}{i+1}\right), \quad n \geq 1,$$

on the other hand, (5.4) implies  $A = \frac{1}{3}$  and

$$D_n = \exp\left(-\sum_{i=1}^n \frac{2i^{\sigma-1}}{3(i+1)}\right), \quad n \geq 1.$$

(i) If (5.3) holds and  $\sigma \geq 1$ , then

$$A_n \sim n^{-1}, \quad D_n = \exp\left(-\sum_{i=1}^n \frac{i^{\sigma-1}}{i+1}\right) \rightarrow 0,$$

so that the conditions of Theorem 6 (i) are satisfied.

(ii) If (5.3) holds and  $\sigma < 1$ , then

$$A_n \sim n^{-1}, \quad D = \exp\left(-\sum_{i=1}^{\infty} \frac{i^{\sigma-1}}{i+1}\right),$$

so that the conditions of Theorem 6 (ii) are satisfied.

(iii) If (5.4) holds and  $\sigma \geq 1$ , then

$$A = \frac{1}{3}, \quad D_n = \exp\left(-\sum_{i=1}^n \frac{2i^{\sigma-1}}{3(i+1)}\right) \rightarrow 0,$$

so that the conditions of Theorem 6 (iii) are satisfied.

(iv) If (5.4) holds and  $\sigma < 1$ , then

$$A = \frac{1}{3}, \quad D = \exp\left(-\sum_{i=1}^{\infty} \frac{2i^{\sigma-1}}{3(i+1)}\right),$$

so that the conditions of Theorem 6 (iv) are satisfied.

**Example 7.** Suppose  $\theta \in (0, 1]$ ,  $r > 1$  assuming (5.1) with  $r^{-\theta} \leq \sigma \leq (r-1)^{-\theta}$ .

(i) If (5.3), then the conditions of Theorem 7 (i) hold with  $A_n \sim n^{-1}$  and  $C = \sigma$ .

(ii) If (5.4), then the conditions of Theorem 7 (ii) hold with  $A = \frac{1}{3}$  and  $C = \frac{2\sigma}{3}$ .

**Example 8.** Suppose  $\theta \in (-1, 0)$ ,  $r > 1$  assuming (5.1) with  $r-1 \leq \sigma^\alpha \leq r$ , where  $\alpha = -\frac{1}{\theta}$ .

(i) If (5.3), then the conditions of Theorem 8 (i) hold with  $A_n \sim n^{-1}$  and  $C = \sigma$ .

(ii) If (5.4), then the conditions of Theorem 8 (ii) hold with  $A = \frac{1}{3}$  and  $C = \frac{2\sigma}{3}$ .

**Example 9.** Suppose  $\theta = 0$  and  $r > 1$  and assume

$$c_n = \sigma, \quad 0 \leq \sigma \leq 1, \quad n \geq 1,$$

which implies

$$D_n = (r - \sigma)^{1-A_n}, \quad n \geq 1.$$

(i) If (5.3), then by Theorem 9 (i), we get in particular,

$$P(\tau > n) \sim \gamma n^{-1}, \quad \gamma = (r - \sigma) \ln \frac{r}{r-1}.$$

(ii) If (5.3), then by Theorem 9 (ii), we get in particular,

$$q = r - r^{\frac{1}{3}}(r - \sigma)^{\frac{2}{3}}, \quad q_\Delta = 1 - r + (r-1)^{\frac{1}{3}}(r - \sigma)^{\frac{2}{3}}, \quad Q = 1 - (r^{\frac{1}{3}} - (r-1)^{\frac{1}{3}})(r - \sigma)^{\frac{2}{3}}.$$

**Example 10.** Suppose  $\theta \in (-1, 0)$ ,  $r = 1$ . Put  $\alpha = -\frac{1}{\theta}$  and assume (5.1) with  $0 < \sigma \leq 1$ .

(i) If (5.3), then by Theorem 10 (i), we get in particular,  $q_\Delta = \sigma^\alpha$  and

$$P(\tau > n) \sim \alpha \sigma^{\alpha-1} n^{-1}.$$

(ii) If (5.4), then by Theorem 10 (ii), we get in particular,  $Q = 1 - \left(\frac{1}{3} + \frac{2\sigma}{3}\right)^\alpha + \frac{2\sigma^\alpha}{3}$ .

## 6 Proofs

In this section we sketch the proofs of lemmas and theorems of this paper. The corollaries to Theorems 7–9 are easily obtained from the corresponding theorems.

### Proof of Lemma 1

Relations (2.1) and (2.2) imply respectively

$$(r - f_k \circ f_{k+1}(s))^{-\theta} = a_k(r - f_{k+1}(s))^{-\theta} + c_k = a_k a_{k+1}(r - s)^{-\theta} + c_k + a_k c_{k+1},$$

and

$$r - f_k \circ f_{k+1}(s) = (r - c_k)^{1-a_k}(r - f_{k+1}(s))^{a_k} = (r - c_k)^{1-a_k}(r - c_{k+1})^{(1-a_{k+1})a_k}(r - s)^{a_k a_{k+1}},$$

entailing the main claims of Lemma 1. Parts (a)–(f) follow from the respective restrictions (a)–(f) on  $(a_n, c_n)$  stated in Definition 1.

### Proof of Lemma 2

(a) In the case  $\theta \in (0, 1]$ ,  $r = 1$ , the claim follows from the existence of  $\lim C_n$  and  $\lim(A_n + C_n)$ , which in turn, follows from monotonicity of the two sequences. To see that  $A_n + C_n \leq A_{n+1} + C_{n+1}$ , it suffices to observe that

$$A_n - A_{n+1} = A_n(1 - a_{n+1}) \leq A_n c_{n+1} = C_{n+1} - C_n.$$

The second part of Lemma 2 is a direct implication of the definition of  $C_n$ .

(b)–(f) The rest of the stated results follows immediately from the restrictions (b)–(f) imposed on  $(a_n, c_n)$  in Definition 1.

### Church–Lindvall Condition for the $\text{GW}^\theta$ -Process

In [8] it was shown for the GW-processes in a varying environment that the almost surely convergence  $Z_n \xrightarrow{\text{a.s.}} Z_\infty$  holds with  $P(0 < Z_\infty < \infty) > 0$  if and only if the following condition holds:

$$\sum_{n \geq 1} (1 - p_n(1)) < \infty. \quad (6.1)$$

Relation (6.1) is equivalent to

$$\prod_{n \geq n_0} p_n(1) > 0 \quad (6.2)$$

for some  $n_0 \geq 1$ . For the  $\text{GW}^\theta$ -process, the equality  $p_n(1) = f'_n(0)$  implies

$$p_n(1) = a_n(a_n + c_n r^\theta)^{-\frac{1}{\theta}-1} \quad (6.3)$$

for  $\theta \neq 0$ , and for  $\theta = 0$ ,

$$p_n(1) = a_n(1 - c_n r^{-1})^{1-a_n}. \quad (6.4)$$

**Lemma 3.** *In the case  $\theta \in (0, 1]$  and  $r = 1$ , relation (6.1) holds if and only if*

$$A_n \rightarrow A \in (0, \infty) \quad (6.5)$$

and

$$\sum_{n \geq 1} c_n < \infty. \quad (6.6)$$

*Proof.* In view of (6.3), we have

$$\prod_{i=1}^n p_i(1) = A_n G_n^{-\frac{1}{\theta}-1}, \quad G_n := \prod_{i=1}^n (a_i + c_i).$$

Since  $a_n + c_n \geq 1$ , we have

$$\lim G_n = G \in [1, \infty].$$

If  $G = \infty$ , then (6.2) is not valid, implying that (6.1) is equivalent to (6.5) plus  $G < \infty$ . It remains to verify that under (6.5), the inequality  $G < \infty$  is equivalent to (6.6). Suppose (6.5) holds, and observe that in this case,  $G < \infty$  is equivalent to

$$\prod_{n \geq 1} \left(1 + \frac{c_n}{a_n}\right) < \infty,$$

which is true if and only if

$$\sum_{n \geq 1} \frac{c_n}{a_n} < \infty.$$

Since under (6.5),  $a_n \rightarrow 1$ , the latter condition is equivalent to (6.6).  $\square$

**Lemma 4.** *In the case  $\theta = 0$  and  $r = 1$ , relation (6.1) holds if and only if  $A \in (0, 1)$  and  $D \in (0, 1)$ .*

*Proof.* In view of (6.4), we have

$$\prod_{n \geq 1} p_n(1) = A \prod_{n \geq 1} (1 - c_n)^{1-a_n}.$$

It remains to observe that given  $A \in (0, 1)$  the relation  $D \in (0, 1)$  is equivalent to

$$\prod_{n \geq 1} (1 - c_n)^{1-a_n} > 0. \quad \square$$

**Lemma 5.** *Assume that  $\theta \neq 0$  and  $r > 1$ , and consider  $\{\tilde{Z}_n\}$ , a GW-process in a varying environment with the proper probability generating functions*

$$\tilde{f}_n(s) = \frac{f_n(s)}{f_n(1)} = \frac{r - (a_n(r-s)^{-\theta} + c_n)^{-\frac{1}{\theta}}}{r - (a_n(r-1)^{-\theta} + c_n)^{-\frac{1}{\theta}}}.$$

*Relation (3.1) implies*

$$\sum_{n=1}^{\infty} (1 - \tilde{p}_n(1)) < \infty.$$

*Proof.* Assume  $\theta \in (0, 1]$  and  $r > 1$  together with (3.1). Then  $A_n \rightarrow A \in (0, 1)$ ,  $a_n \rightarrow 1$ , and  $c_n \rightarrow 0$ . We have

$$\tilde{p}_n(1) = \tilde{f}'_n(0) = a_n h_n^{-\frac{1}{\theta}-1} k_n^{-1},$$

where

$$h_n = a_n + c_n r^\theta, \quad k_n = r - (a_n(r-1)^{-\theta} + c_n)^{-\frac{1}{\theta}}$$

are such that  $h_n \geq 1$  and  $k_n \in (0, 1]$ . The statement follows from the representation

$$\prod_{n \geq 1} p_n(1) = A H^{-\frac{1}{\theta}-1} K^{-1},$$

where  $H = \prod_{n \geq 1} h_n$  and  $K = \prod_{n \geq 1} k_n$ . It is easy to show that (3.1) and  $(1 - a_n)r^{-\theta} \leq c_n \leq (1 - a_n)(r-1)^{-\theta}$  yield

$$\sum_{n \geq 1} (h_n - 1) \leq r^\theta \sum_{n \geq 1} c_n \leq r^\theta (r-1)^{-\theta} \sum_{n \geq 1} (1 - a_n) < \infty,$$

implying  $H \in [1, \infty)$ . On the other hand,  $K \in (0, 1]$ , since

$$\sum_{n \geq 1} (1 - k_n) < \infty,$$

which follows from

$$1 - k_n \leq (r-1)(a_n + c_n(r-1)^{\theta})^{-\frac{1}{\theta}} - 1 \leq r(1 - (a_n + c_n(r-1)^{\theta})^{\frac{1}{\theta}}) \leq r\theta^{-1}(1 - a_n).$$

In the other case, when (3.1) holds together with  $\theta \in (-1, 0)$  and  $r > 1$ , the lemma is proven similarly.  $\square$

## Proof of Theorems 1–5

The proofs of these theorems are done using the usual for these kind of results arguments applied to the explicit expressions available for  $F_n(s)$ . In particular, the following standard formula is a starting point for computing the conditional limit distributions:

$$E(s^{Z_n} | Z_n > 0) = \frac{E(s^{Z_n}) - P(Z_n = 0)}{P(Z_n > 0)} = 1 - \frac{1 - F_n(s)}{1 - F_n(0)}. \quad (6.7)$$

Thus in the case  $\theta \in (0, 1]$  and  $r > 1$ , Lemma 1 and (6.7) imply

$$E(s^{Z_n} | Z_n > 0) = 1 - \frac{((1-s)^{-\theta} + B_n)^{-\frac{1}{\theta}}}{(1+B_n)^{-\frac{1}{\theta}}} \rightarrow 1 - \frac{((1-s)^{-\theta} + B)^{-\frac{1}{\theta}}}{(1+B)^{-\frac{1}{\theta}}},$$

proving the main statement of Theorem 4. The almost sure convergence stated in Theorem 2 follows from Lemma 3 and the earlier cited criterium of [8].

## Proof of Theorem 6

Suppose  $\theta = 0$ ,  $r = 1$ , in which case  $A \in [0, 1)$  and  $D \in [0, 1]$ .

(i) Suppose  $A = D = 0$ . In this case  $q = 1 - D = 1$ , and by (6.7) and Lemma 1,

$$E(s^{Z_n} | Z_n > 0) = 1 - (1-s)^{A_n}.$$

Putting here  $s_n = \exp(-\lambda e^{-\frac{x}{A_n}})$ , we get as  $n \rightarrow \infty$ ,

$$E(s_n^{Z_n} | Z_n > 0) = 1 - (1 - \exp(-\lambda e^{-\frac{x}{A_n}}))^{A_n} = 1 - \exp(A_n \ln(\lambda e^{-\frac{x}{A_n}} (1 + o(1)))) \rightarrow 1 - e^{-x}.$$

This implies a convergence in distribution

$$(Z_n e^{-\frac{x}{A_n}} | Z_n > 0) \xrightarrow{d} W(x),$$

where the limit  $W(x)$  has a degenerate distribution with

$$P(W(x) \leq w) = (1 - e^{-x}) \mathbf{1}_{\{0 \leq w < \infty\}}.$$

In other words,

$$P(Z_n \leq w e^{\frac{x}{A_n}} | Z_n > 0) \rightarrow (1 - e^{-x}) \mathbf{1}_{\{0 \leq w < \infty\}}.$$

After taking the logarithm of  $Z_n$ , we arrive at the statement of Theorem 6 (i).

(ii) Statement (ii) follows from Lemma 1 and relation (6.7) in a similar way as statement (i).

(iii) If  $A \in (0, 1)$  and  $D = 0$ , then  $q = 1$  and by relation (6.7) and Lemma 1,

$$E(s^{Z_n} | Z_n > 0) = 1 - (1-s)^{A_n} \rightarrow 1 - (1-s)^A.$$

(iv) Let  $A > 0$  and  $D > 0$ . Since  $q = 1 - D$ , similarly to part (iii), we obtain

$$E(s^{Z_n}) \rightarrow 1 - (1-s)^A D.$$

By Lemma 4, the convergence in distribution  $Z_n \xrightarrow{d} Z_\infty$  can be upgraded to the almost surely convergence  $Z_n \xrightarrow{\text{a.s.}} Z_\infty$ .

## Proof of Theorems 7 and 8

In this section we prove only Theorem 7. Theorem 8 is proven similarly.

By Lemma 1,

$$F_n(0) = r - (A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}}, \quad F_n(1) = r - (A_n (r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}.$$

It follows that

$$\begin{aligned} P(\tau > n) &= (A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n (r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}, \\ E(Z_n | \tau > n) &= \frac{F'_n(1)}{F_n(1) - F_n(0)} = \frac{\theta^{-1} A_n (A_n + C_n (r-1)^\theta)^{-\frac{1}{\theta}-1}}{(A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n (r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}}, \\ E(s^{Z_n} | \tau > n) &= \frac{F_n(s) - F_n(0)}{F_n(1) - F_n(0)} = \frac{(A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n (r-s)^{-\theta} + C_n)^{-\frac{1}{\theta}}}{(A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n (r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}}. \end{aligned}$$

(i) Assume that  $A = 0$ . Then the sequence of positive numbers

$$V_n = A_n^{-1}(C - C_n) = c_{n+1} + c_{n+2}a_{n+1} + c_{n+3}a_{n+2}a_{n+1} + \dots$$

satisfies

$$r^{-\theta} \leq \liminf V_n \leq \limsup V_n \leq (r-1)^{-\theta}.$$

For a given  $x \in (0, \infty)$ , put

$$W_n(x) = A_n^{-1}(C^{-\frac{1}{\theta}} - (A_n x + C_n)^{-\frac{1}{\theta}}).$$

Since

$$W_n(x) = A_n^{-1}(C^{-\frac{1}{\theta}} - (A_n(x - V_n) + C)^{-\frac{1}{\theta}}) = \theta^{-1} C^{-\frac{1}{\theta}-1}(x - V_n + o(1)),$$

the representation

$$A_n^{-1}P(\tau > n) = W_n((r-1)^{-\theta}) - W_n(r^{-\theta})$$

yields the first asymptotic result stated in part (i) of Theorem 7. The other two asymptotic results follow from the representations

$$\begin{aligned} E(Z_n | \tau > n) &= \frac{\theta^{-1}(A_n + C_n(r-1)^\theta)^{-\frac{1}{\theta}-1}}{W_n((r-1)^{-\theta}) - W_n(r^{-\theta})}, \\ E(s^{Z_n} | \tau > n) &= \frac{W_n((r-s)^{-\theta}) - W_n(r^{-\theta})}{W_n((r-1)^{-\theta}) - W_n(r^{-\theta})}. \end{aligned}$$

(ii) The second claim follows from the equality

$$E(s^{Z_n}; \tau_\Delta > n) = r - (A_n(r-s)^{-\theta} + C_n)^{-\frac{1}{\theta}}.$$

## Proof of Theorem 9

If  $\theta = 0$  and  $r > 1$ , then by Lemma 1

$$P(Z_n = 0) = r - r^{A_n} D_n, \quad P(Z_n \neq \Delta) = r - (r-1)^{A_n} D_n.$$

It follows that

$$q = r - r^A D, \quad q_\Delta = 1 - r + (r-1)^A D, \quad Q = 1 - (r^A - (r-1)^A) D.$$

(i) If  $A = 0$ , then clearly

$$q = r - D, \quad q_\Delta = 1 - r + D, \quad Q = 1,$$

and

$$P(\tau > n) = (r^{A_n} - (r-1)^{A_n}) D_n \sim (\ln r - \ln(r-1)) A_n D_n.$$

Furthermore,

$$\begin{aligned} E(Z_n | \tau > n) &= \frac{F'_n(1)}{F_n(1) - F_n(0)} = \frac{A_n (r-1)^{A_n-1}}{r^{A_n} - (r-1)^{A_n}} \rightarrow \frac{(r-1)^{-1}}{\ln r - \ln(r-1)}, \\ P(s^{Z_n} | \tau > n) &= \frac{F_n(s) - F_n(0)}{F_n(1) - F_n(0)} = \frac{r^{A_n} - (r-s)^{A_n}}{r^{A_n} - (r-1)^{A_n}} \rightarrow \frac{\ln r - \ln(r-s)}{\ln r - \ln(r-1)}. \end{aligned}$$

(ii) In the case  $A > 0$ , the main claim is obtained as

$$E(s^{Z_n}; \tau_\Delta > n) = r - (r-s)^{A_n} D_n \rightarrow r - (r-s)^A D.$$

## Proof of Theorem 10

If  $\theta \in (-1, 0)$  and  $r = 1$ , then by Lemmas 1 and 2,

$$F_n(0) = 1 - (A_n + C_n)^\alpha, \quad F_n(1) = 1 - C_n^\alpha$$

and

$$q = 1 - (A + C)^\alpha, \quad q_\Delta = C^\alpha,$$

where  $\alpha = -\frac{1}{\theta}$  and  $0 < C \leq 1 - A$ .

(i) Suppose  $A = 0$ . Then the sequence of positive numbers  $V_n = A_n^{-1}(C - C_n)$  satisfies

$$0 \leq \liminf V_n \leq \limsup V_n \leq 1.$$

For a given  $x \in (0, \infty)$ , put

$$W_n(x) = A_n^{-1}((A_n x + C_n)^\alpha - C^\alpha).$$

Since

$$W_n(x) = \alpha C^{\alpha-1}(x - V_n + o(1)),$$

the representation

$$A_n^{-1}P(\tau > n) = W_n(1) - W_n(0)$$

yields the first asymptotic result stated in part (i) of Theorem 10. The other asymptotic result follows from the representation

$$E(s^{Z_n} | \tau > n) = \frac{W_n(1) - W_n((1-s)^{\frac{1}{\alpha}})}{W_n(1) - W_n(0)}.$$

(ii) Claim (ii) is derived as

$$P(s^{Z_n}; \tau > n) = 1 - (A_n(1-s)^{\frac{1}{\alpha}} + C_n)^\alpha \rightarrow 1 - (A(1-s)^{\frac{1}{\alpha}} + C)^\alpha.$$

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