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## THE ISOSPECTRAL PROBLEM FOR FLAT TORI FROM THREE PERSPECTIVES

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ABSTRACT. Flat tori are among the only types of Riemannian manifolds for which the Laplace eigenvalues can be explicitly computed. In 1964, Milnor used a construction of Witt to find an example of isospectral nonisometric Riemannian manifolds, a striking and concise result that occupied one page in the *Proceedings of the National Academy of Science* of the USA. Milnor’s example is a pair of 16-dimensional flat tori, whose set of Laplace eigenvalues are identical, in spite of the fact that these tori are not isometric. A natural question is, What is the *lowest* dimension in which such isospectral nonisometric pairs exist? This isospectral question for flat tori can be equivalently formulated in analytic, geometric, and number theoretic language. We explore this question from all three perspectives and describe its resolution by Schieman in the 1990s. Moreover, we share a number of open problems.

### 1. INTRODUCTION

The Laplace eigenvalue problem is broadly appealing because it connects physics, number theory, analysis, and geometry. At the same time, it is a challenging and frustrating problem because in general, one cannot solve it analytically. Wielding heavy tools from functional analysis, one can prove that solutions exist, but this is not nearly as satisfying as being able to write down a solution in closed form. There is, however, a notable exception: flat tori. Although there is no smooth isometric embedding of a flat  $n$ -dimensional torus into  $(n+1)$ -dimensional Euclidean space, there is a  $C^1$  embedding discovered by Nash [39] and Kuiper [36]. This embedding remained mysterious, eluding visualization until the 2012 papers by Borrelli, Jabrane, Lazarus, and Thibert [3], [4]. It is apparent from this visualization in Figure 1 that the embedding is not smooth, because the surface of the torus exhibits a fractal behavior in the normal direction.

A flat torus is the quotient of  $\mathbb{R}^n$  by a full-rank lattice with Riemannian metric induced by the standard Euclidean metric on  $\mathbb{R}^n$ . It is a smooth and compact Riemannian manifold, whose Riemannian curvature tensor is identically zero. For the sake of completeness and inclusivity, we recall the following.

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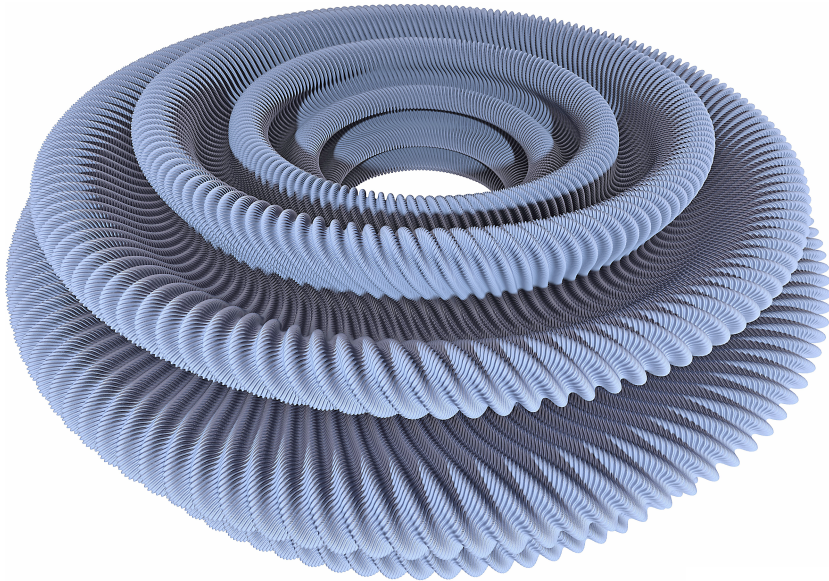


FIGURE 1. A  $C^1$  isometric embedding of a two-dimensional flat torus into three-dimensional Euclidean space has a rough, corrugated looking surface, as depicted in [3]. The above image is from V. Borelli, S. Jabrane, F. Lazarus, D. Rohmer and B. Thivert, *Hevea Project*, licensed under Creative Commons Attribution–ShareAlike 2.0 Generic.

**Definition 1.1.** An  $n$ -dimensional (full-rank) *lattice*  $\Gamma \subset \mathbb{R}^n$  is a set which can be expressed as  $\Gamma := A\mathbb{Z}^n$  for an invertible  $n \times n$  matrix  $A$  with real coefficients. The matrix  $A$  is called a *basis matrix* of  $\Gamma$ . The lattice defines the *flat torus*  $\mathbb{T}_\Gamma = \mathbb{R}^n / \Gamma$ , with Riemannian metric induced by the Euclidean metric on  $\mathbb{R}^n$ .

The Laplace eigenvalue problem in this context is to find all functions defined on  $\mathbb{R}^n$  for which there exists  $\lambda \in \mathbb{C}$  such that

$$(1.1) \quad \begin{aligned} \Delta f(x) &= \lambda f(x), & f(x + \ell) &= f(x) \text{ and} \\ \nabla f(x + \ell) &= \nabla f(x) & \forall \ell \in \Gamma \text{ and } x \in \mathbb{R}^n. \end{aligned}$$

Our sign convention for the Euclidean Laplace operator is

$$\Delta = - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

If  $f$  is a solution to the Laplace eigenvalue problem, it is an *eigenfunction*. With a bit of functional analysis [12], one can prove that the corresponding eigenvalue is nonnegative. The simplest case is a one-dimensional lattice, which consists of all integer multiples of a number  $\ell$ . The Laplace eigenvalue problem is then to find all functions which satisfy

$$-f''(x) = \lambda f(x), \quad f(x + k\ell) = f(x), \quad f'(x + k\ell) = f'(x) \quad \forall k \in \mathbb{Z}.$$

Using calculus, the functions

$$f_n(x) := e^{2\pi i n x / \ell}, \quad n \in \mathbb{Z},$$

are eigenfunctions, with corresponding eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{\ell^2}.$$

To prove that the functions  $\{f_n\}_{n \in \mathbb{Z}}$  are *all* of the eigenfunctions up to scaling, it suffices to prove that they constitute an orthogonal basis for the Hilbert space  $\mathcal{L}^2$ , as demonstrated in [18]. The set of all of its Laplace eigenvalues with multiplicities is known as its *spectrum*.

In physics, the Laplace eigenvalue problem is a crucial step in solving the wave equation, as solutions can be expressed in terms of the Laplace eigenfunctions and eigenvalues. The Laplace eigenvalues are in bijection with the resonant frequencies of solutions to the wave equation. For this reason, Kac [29] would say that *spectral invariants*, which are all quantities that are entirely determined by the spectrum, are *audible*. We therefore ask, *Can one hear the shape of a flat torus?*

**1.1. Three perspectives.** We are interested in the question, If two flat tori are isospectral, then are they isometric? This has an equivalent formulation in both number theoretic terms as well as purely geometric terms, as summarized in Table 1. This observation is crucial to obtain a thorough investigation.

TABLE 1

Analysis	<i>Does the spectrum of the Laplace operator determine the geometry of flat tori?</i>
Number Theory	<i>Are positive-definite quadratic forms determined by their values over the integers counting multiplicity?</i>
Geometry	<i>Do the lengths of points of a lattice with multiplicity determine the lattice itself up to congruency?</i>

The first perspective gives the subject physical motivation. The second perspective is number theoretical, thereby opening the possibility to employ powerful techniques from analytic number theory. The third perspective is a more intuitive and purely geometric question about lattices. It is unfortunate that the mathematical language is quite different when the problem is investigated from these different viewpoints in the sense that disparate fields do not always cross-reference each other. Consequently, we will take this opportunity to connect some of the different terminologies. We aim to provide interested readers with a more thorough understanding of the question by studying it from all three perspectives.

**1.2. Organization.** In Section 2 we collect the essential ingredients required to investigate flat tori and their spectra. We conclude that section by combining some of these ingredients to prove that isospectral rectangular flat tori are isometric, as are isospectral Euclidean boxes. In Section 3 we introduce useful techniques, or “key kitchenware”, for investigating the spectrum and geometry of flat tori. We explore in Section 4 famous examples of nonisometric isospectral flat tori. Section 5 is dedicated to popularizing the fundamental yet not widely known theorem of Schiemann. We conclude in Section 6 with a collection of conjectures and open problems. This includes a discussion of the open question, How *many* flat tori can be mutually isospectral and nonisometric in a given dimension?

## 2. ESSENTIAL INGREDIENTS AND A CASE STUDY

We begin with the key ingredients in the mathematics of flat tori and their spectra.

**2.1. Lattices, congruency, and isometry.** The general linear group is the set of all invertible  $n \times n$  matrices with real coefficients, denoted  $\mathrm{GL}_n(\mathbb{R})$ . The unimodular group  $\mathrm{GL}_n(\mathbb{Z}) \subset \mathrm{GL}_n(\mathbb{R})$  consists of those matrices  $M \in \mathrm{GL}_n(\mathbb{R})$  such that both  $M$  and  $M^{-1}$  have integer coefficients; necessarily all elements of  $\mathrm{GL}_n(\mathbb{Z})$  have determinant equal to  $\pm 1$ . The orthogonal group  $\mathrm{O}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$  consists of those matrices whose inverse matrix and transpose matrix are equal. The following lemma summarizes basic facts about the orthogonal group. Its proof is straightforward and therefore omitted.

**Lemma 2.1.** *A matrix is an element of  $\mathrm{O}_n(\mathbb{R})$  if and only if its column vectors form an orthonormal basis of  $\mathbb{R}^n$ . Further,  $A \in \mathrm{O}_n(\mathbb{R})$  if and only if  $Ax \cdot Ay = x \cdot y$  for any  $x, y \in \mathbb{R}^n$ . A matrix  $C \in \mathrm{O}_n(\mathbb{R})$  if and only if the linear transformation  $C : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by  $Cv = Cv$  maps any orthonormal basis of  $\mathbb{R}^n$  to another orthonormal basis of  $\mathbb{R}^n$ .*

**Definition 2.2** (Lattices of arbitrary rank, trivial lattices, sublattices, and integer lattices). A set  $v_1, \dots, v_k \in \mathbb{R}^n$  of linearly independent vectors define the  $k$ -rank lattice, also called a  $k$ -dimensional lattice:

$$\Gamma := v_1\mathbb{Z} + \dots + v_k\mathbb{Z} = \left\{ \sum_{j=1}^k z_j v_j : z_j \in \mathbb{Z} \forall j \right\}.$$

The matrix whose column vectors are equal to  $v_1, \dots, v_k$  is a *basis matrix*. Full-rank lattices are  $n$ -rank lattices in  $\mathbb{R}^n$ . A *trivial lattice* is a set whose only element is 0. A *sublattice*  $\Lambda$  of a lattice  $\Gamma$  is a lattice such that  $\Lambda \subseteq \Gamma$ . An *integer lattice* is a sublattice of  $\mathbb{Z}^n$ .

We next collect basic facts about lattices.

**Proposition 2.3.** *Let  $\Lambda$  and  $\Gamma$  be full-rank lattices and let  $A_\Lambda, A_\Gamma$  be corresponding bases.*

- (1)  $\Lambda$  is a sublattice of  $\Gamma$  if and only if  $A_\Lambda = A_\Gamma V$  for some  $V \in \mathbb{Z}^{n \times n}$ .
- (2) If  $\Lambda \subseteq \Gamma$ , then  $\det(A_\Lambda)/\det(A_\Gamma) \in \mathbb{Z} \setminus \{0\}$  and

$$(\det(A_\Lambda)/\det(A_\Gamma))\Gamma \subseteq \Lambda.$$

- (3) Two matrices  $A_1$  and  $A_2$  in  $\mathrm{GL}_n(\mathbb{R})$  are both bases for the same lattice if and only if there is a matrix  $B \in \mathrm{GL}_n(\mathbb{Z})$  such that  $A_2 = A_1 B$ .
- (4) If  $\Lambda \subseteq \Gamma$ , then the index of  $\Lambda$  in  $\Gamma$  as a subgroup, denoted  $[\Gamma : \Lambda]$ , is equal to  $|\det(A_\Lambda)/\det(A_\Gamma)|$ .
- (5) A nontrivial additive subgroup  $\Gamma \subset \mathbb{R}^n$  is discrete if and only if it is a  $k$ -rank lattice for some  $1 \leq k \leq n$ .

*Proof.* Statements (1), (2), and (3) are direct observations. A proof of (4) can be found in [17, §1.2.4], and (5) is contained in [41, p. 24].  $\square$

**Definition 2.4** (Congruent lattices and isometric flat tori). Let  $\Gamma_1 \subset \mathbb{R}^n$  and  $\Gamma_2 \subset \mathbb{R}^m$  be lattices. If  $n = m$ , then  $\Gamma_1$  and  $\Gamma_2$  are *congruent* if there is a  $C \in \mathrm{O}_n(\mathbb{R})$  such that  $\Gamma_2 = C\Gamma_1$ . If  $n > m$ , then  $\Gamma_1$  is congruent to  $\Gamma_2$  if  $\Gamma_2 \times \{0\} = C\Gamma_1$  for

an orthogonal matrix  $C \in O_n(\mathbb{R})$  and a trivial lattice in  $\mathbb{R}^{n-m}$ . Two flat tori are *isometric* if they are isometric as Riemannian manifolds.

We use the same notation  $\cong$  for both isometry of flat tori as well as congruency of lattices as justified by the following theorem which shows that flat tori are isometric if and only if the lattices that define them are congruent.

**Theorem 2.5** ([1, p. 5]). *Two flat tori are isometric in the Riemannian sense if and only if their associated lattices are congruent.*

Necessary and sufficient conditions for full-rank lattices to be congruent are given in the following lemma.

**Lemma 2.6.** *Let  $\Gamma_1 = A_1\mathbb{Z}^n$  and  $\Gamma_2 = A_2\mathbb{Z}^n$  be two full-rank lattices. Then*

$$\Gamma_1 \cong \Gamma_2 \iff CA_1 = A_2B$$

for some  $B \in GL_n(\mathbb{Z})$ ,  $C \in O_n(\mathbb{R})$ .

**2.2. The spectrum of a flat torus.** The spectrum of a flat torus,  $\mathbb{R}^n/\Gamma$ , is in bijection with the lengths of the vectors in the dual lattice of  $\Gamma$ .

**Definition 2.7** (Dual lattice). For a full-rank lattice  $\Gamma \subset \mathbb{R}^n$ , its *dual lattice* is defined to be

$$\Gamma^* := \{\ell \in \mathbb{R}^n : \ell \cdot \gamma \in \mathbb{Z} \quad \forall \gamma \in \Gamma\}.$$

It is straightforward to show that the dual lattice of a full-rank lattice  $\Gamma$  is itself a lattice, and there is a natural bijection between  $\Gamma^*$  and  $\text{Hom}(\Gamma, \mathbb{Z})$  justifying the name dual lattice. If a full-rank lattice  $\Gamma$  has basis matrix  $A$ , then  $(A^{-1})^T$  is a basis matrix for the dual lattice.

**Theorem 2.8** (The Laplace spectrum of a flat torus). *The eigenvalues of a flat torus  $\mathbb{R}^n/\Gamma$  are precisely  $4\pi^2\|\ell\|^2$  such that  $\ell$  is an element of the dual lattice,  $\Gamma^*$ . The multiplicity of such an eigenvalue is the number of distinct elements of  $\Gamma^*$  that have the same length as  $\ell$ . The eigenspace is spanned by the functions*

$$\{u_\ell(x) = e^{2\pi i x \cdot \ell}\}_{\ell \in \Gamma^*}.$$

*The collection of eigenvalues, counted with multiplicity, is the spectrum of the flat torus.*

*Proof.* For any  $\ell$  in the dual lattice, the function  $u_\ell(x) = e^{2\pi i x \cdot \ell}$  satisfies both the Laplace eigenvalue equation on  $\mathbb{R}^n$  as well as  $u_\ell(x + \gamma) = u_\ell(x)$ , and  $\nabla u_\ell(x + \gamma) = \nabla u_\ell(x)$  for all  $\gamma \in \Gamma$ . Moreover, one can demonstrate that the functions  $u_\ell$  are an orthogonal basis for the Hilbert space  $\mathcal{L}^2$  on the torus, for example by showing they are the solutions of a regular Sturm–Liouville problem [18].  $\square$

**Definition 2.9** (Isospectrality). Two flat tori  $\mathbb{R}^n/\Gamma$  and  $\mathbb{R}^n/\Lambda$  are *isospectral* if they have the same Laplace spectrum, in which case we say that the lattices  $\Gamma$  and  $\Lambda$  are also isospectral.

One can readily check that if two flat tori are isometric, then they are isospectral. Is the converse true? We explore this question in Section 5.

2.2.1. *Poisson's summation formula.* Poisson's summation formula is a powerful tool because it equates a purely analytical object with a purely geometric one. Two important geometric ingredients in the formula are the volume of the flat torus and its length spectrum.

**Definition 2.10** (Volume). For a flat torus  $\mathbb{T}_\Gamma = \mathbb{R}^n/\Gamma$ , with full-rank lattice  $\Gamma = A\mathbb{Z}^n$ , the *volume* of  $\mathbb{T}_\Gamma$  (with respect to the flat Riemannian metric induced by the Euclidean metric on  $\mathbb{R}^n$ ) is equal to

$$\text{vol}(\mathbb{T}_\Gamma) := |\det(A)|.$$

It is straightforward to show that the volume is independent of the choice of basis matrix. If  $\Gamma$  is an integer lattice, then  $\text{vol}(\Gamma)\mathbb{Z}^n$  is a sublattice of  $\Gamma$ .

**Definition 2.11** (The length spectrum). For a flat torus  $\mathbb{T}_\Gamma = \mathbb{R}^n/\Gamma$ , with lattice  $\Gamma = A\mathbb{Z}^n$ , the *length spectrum* of  $\mathbb{T}_\Gamma$  (with respect to the flat Riemannian metric induced by the Euclidean metric on  $\mathbb{R}^n$ ) is equal to the collection of lengths of closed geodesics, counted with multiplicity, and denoted by  $\mathcal{L}_\Gamma$ . This length spectrum is also equal to the collection of lengths of lattice vectors,  $\|\gamma\|$  for  $\gamma \in \Gamma$ , counted with multiplicity, which is how we define the length spectrum of the lattice  $\Gamma$ .<sup>1</sup>

The Poisson summation formula [8, p. 125] equates a sum over the Laplace spectrum with a sum over the length spectrum, thereby relating these two spectra.

**Theorem 2.12** (Poisson summation formula). *For a full-rank lattice  $\Gamma \subset \mathbb{R}^n$  the following series converge for any  $t \in (0, \infty)$  and satisfy*

$$(2.1) \quad \sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 \|\gamma^*\|^2 t} = \frac{\text{vol}(\Gamma)}{(4\pi t)^{n/2}} \sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2/4t},$$

$$(2.2) \quad \sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2/4t} = \frac{(4\pi t)^{n/2}}{\text{vol}(\Gamma)} \sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 \|\gamma^*\|^2 t}.$$

The first series in Poisson's summation formula is a spectral invariant known as the *heat trace*.

**Definition 2.13** (Heat trace). The *heat trace* of a flat torus  $\mathbb{T}_\Gamma$  (and lattice  $\Gamma$ ) is defined for  $t > 0$  by

$$\sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 \|\gamma^*\|^2 t} = \sum_{k \geq 0} e^{-\lambda_k t},$$

where the eigenvalues  $\lambda_k$  are ordered as  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \uparrow \infty$  and counting multiplicities.

A closely related spectral invariant is the *theta series*.

**Definition 2.14** (Theta series). Let  $\Gamma$  be a lattice. Then we define the *theta series* of the lattice (and flat torus) as

$$\theta_\Gamma(z) := \sum_{\gamma \in \Gamma} e^{i\pi z \|\gamma\|^2}, \quad z \in \mathbb{C} \text{ with } \text{Im } z > 0.$$

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<sup>1</sup>We note that there is a related question of *length-equivalence* of lattices [43], in which one considers the set of lengths of lattice vectors, ignoring their multiplicity.

By the Poisson summation formula, the lattices  $\Gamma, \Lambda$  are isospectral if and only if their dual lattices  $\Gamma^*, \Lambda^*$  are isospectral. The Poisson summation formula shows that the dimension, volume, and length spectrum are all spectral invariants. Moreover, it establishes the equivalence of the analytical and geometrical formulations in Section 1.1.

**Corollary 2.15.** *The following are equivalent:*

- (1) *The flat tori  $\mathbb{T}_\Gamma$  and  $\mathbb{T}_\Lambda$  are isospectral.*
- (2) *The flat tori  $\mathbb{T}_\Gamma$  and  $\mathbb{T}_\Lambda$  have identical heat traces.*
- (3) *The flat tori  $\mathbb{T}_\Gamma$  and  $\mathbb{T}_\Lambda$  have identical theta series.*
- (4) *The flat tori  $\mathbb{T}_\Gamma$  and  $\mathbb{T}_\Lambda$  have identical length spectra.*

*If any of the above hold, then the flat tori have identical dimension and volume.*

*Proof.* The implication (1)  $\implies$  (2) follows immediately from the definition. To prove the converse, proceed inductively to prove that the heat trace determines all of the eigenvalues and their multiplicities by analyzing the asymptotic behavior of the heat trace as  $t \nearrow \infty$ . The equivalence of (1) and (4) is an immediate consequence of the Poisson summation formula. The heat trace is obtained by evaluating the theta series of the dual lattice at  $z = 4\pi it$  for  $t > 0$ . Consequently, identical theta series imply that the dual lattices have identical heat traces and are therefore isospectral. Isospectrality of lattices is equivalent to isospectrality of their dual lattices, showing that (3) is equivalent to (1). We have therefore established that (1) is equivalent to (2), (4), and (3). To conclude that isospectral tori have identical dimension and volume, investigate the asymptotic behavior for  $t$  approaching 0 and for  $t$  approaching infinity in Poisson's formula.  $\square$

**2.3. Quadratic forms equate the number theoretic formulation.** We have seen that the analytical and geometrical formulations in Section 1.1 are equivalent with the help of Poisson's powerful summation formula. To equate these formulations in number theoretic language, we collect several facts about quadratic forms. In Section 2.3.1, we will use these facts to associate a quadratic form to a flat torus and identify its representation numbers with the spectrum. Quadratic forms will also be essential in Section 5.

**Definition 2.16.** A *quadratic form*,  $q$ , of  $n$  variables is a homogeneous polynomial of degree 2. If  $q(x) \geq 0$  for all nonzero  $x \in \mathbb{R}^n$ , it is *positive semidefinite*, and if the inequality is strict, then the form is *positive definite*. We may equivalently refer to quadratic forms of  $n$  variables as  $n$ -dimensional quadratic forms.

There is a well-known natural bijection between positive-definite quadratic forms and symmetric matrices. For any quadratic form  $q$  of  $n$  variables, there is a unique symmetric  $n \times n$  matrix  $Q$ , known as the *associated matrix*, such that

$$q(x) = x^T Q x \quad \forall x \in \mathbb{R}^n.$$

The matrix  $Q$  is positive (semi-)definite if and only if the quadratic form  $q$  is. Although it is a slight abuse of notation, we hope the reader will pardon our identification of quadratic forms with their associated matrices in the following.

**Definition 2.17.** The notation  $\mathcal{S}_{>0}^n$  (resp.,  $\mathcal{S}_{\geq 0}^n$ ) is the set of  $n \times n$  positive definite (resp., semidefinite) matrices and is also identified with the set of positive-definite (resp., semidefinite) quadratic forms of  $n$  variables. A quadratic form is *rational* if the entries of its associated matrix are all rational. A quadratic form is *even* if

the entries of its associated matrix are integers, and the diagonal consists of even numbers. If a quadratic form  $q$  is positive definite, then its *dual form*,  $q^*$ , is defined by

$$q^*(x) := x^T Q^{-1} x, \quad \forall x \in \mathbb{R}^n.$$

The matrix  $Q$  associated with a positive-definite quadratic form admits a *Cholesky factorization*.

**Theorem 2.18** ([51, Theorem 11.2]). *Assume that  $Q$  is a symmetric  $n \times n$  real matrix such that  $x^T Q x > 0$  for all nonzero  $x \in \mathbb{R}^n$ . Then there is an invertible matrix  $A$  with  $Q = A^T A$ . This is known as a Cholesky factorization. The matrix  $A$  is unique up to left-multiplication with an orthogonal matrix.*

*Proof.* A symmetric matrix  $Q$  can be diagonalized as  $D = U^T Q U$ , with  $U^T = U^{-1}$  an orthogonal matrix. Since  $Q$  is positive definite, each diagonal entry of  $D$  is positive, and hence there is a well-defined diagonal matrix  $S$  with positive diagonal entries such that  $D = S^2 = S S^T$ . Define  $A = (U S U)^T$ . For the uniqueness, observe that if  $B^T B = Q$ , then  $(A^T)^{-1} B^T = (B A^{-1})^T = A B^{-1} = (B A^{-1})^{-1}$ , so  $B = C A$  for the orthogonal matrix  $C = B A^{-1}$ .  $\square$

Motivated by the Cholesky factorization, we define *underlying lattices* of a quadratic form.

**Definition 2.19** (Underlying lattice). For a positive-definite  $n \times n$  matrix  $Q$  with Cholesky factorization  $Q = A^T A$ , we say that  $A \mathbb{Z}^n$  is an *underlying lattice* of both  $Q$  and the associated quadratic form.

Cholesky factorization is unique up to left-multiplication with an orthogonal matrix; therefore, all underlying lattices of a given positive-definite quadratic form are congruent.

**Definition 2.20** (Integral equivalence). Two quadratic forms  $q$  and  $p$  on  $\mathbb{R}^n$  are *integrally equivalent* if their associated matrices  $Q$  and  $P$  satisfy  $B^T Q B = P$ , for a unimodular matrix  $B$ .

The following proposition collects several useful facts about quadratic forms. It can be proven using the spectral theorem for symmetric matrices.

**Proposition 2.21.** *Let  $Q$  be a real symmetric  $n \times n$  matrix. Let  $\lambda_{\min}$  be its smallest eigenvalue and  $\lambda_{\max}$  its biggest.*

- (1)  $x^T Q x > 0$  for all  $x \in \mathbb{Z}^n \setminus \{0\}$  if and only if  $x^T Q x > 0$  for all  $x \in \mathbb{Q}^n \setminus \{0\}$ .
- (2)  $x^T Q x > 0$  for all  $x \in \mathbb{Z}^n \setminus \{0\}$  implies  $Q \in \mathcal{S}_{>0}^n$ .
- (3)  $Q \in \mathcal{S}_{\geq 0}^n$  if and only if  $Q$  has only nonnegative real eigenvalues.
- (4)  $Q \in \mathcal{S}_{>0}^n$  if and only if  $Q$  has only positive real eigenvalues.
- (5)  $\lambda_{\min} \|x\|^2 \leq x^T Q x \leq \lambda_{\max} \|x\|^2$  for any  $x \in \mathbb{R}^n$ .
- (6)  $Q \in \mathcal{S}_{>0}^n$  implies  $Q = E^T E$  for some  $n \times n$  real matrix  $E$ .
- (7)  $Q \in \mathcal{S}_{>0}^n$  if and only if  $Q \in \mathcal{S}_{\geq 0}^n$  and  $Q$  is of full rank.
- (8) Viewed as a quadratic form, the image of  $Q \in \mathcal{S}_{>0}^n$  over  $\mathbb{Z}^n$  is discrete, and all multiplicities are finite.

**2.3.1. Representation numbers of quadratic forms.** The connection between the spectra of flat tori and quadratic forms is obtained using the *representation numbers* of quadratic forms. The representation numbers are the image of  $\mathbb{Z}^n$  under the quadratic form, taking into account multiplicities.

**Definition 2.22** (Representation numbers). If  $q$  is an  $n$ -dimensional positive-definite quadratic form, its *representation numbers* are defined for  $t \in \mathbb{R}_{\geq 0}$  as follows:

$$\mathcal{R}(q, t) := \#\{x \in \mathbb{Z}^n : q(x) = t\}.$$

We may also consider a subset  $X \subset \mathbb{Z}^n$  and define

$$\mathcal{R}_X(q, t) := \#\{x \in X : q(x) = t\}.$$

A convenient reduction for verifying whether two quadratic forms have the same representation numbers is obtained by defining

$$(2.3) \quad \mathbb{Z}_*^n := \{x \in \mathbb{Z}^n \setminus \{0\} : \gcd(x_1, \dots, x_n) = 1,$$

and the last nonzero coordinate is positive\}.

Then, one can show that for two quadratic forms  $q_1$  and  $q_2$ , for any  $t \geq 0$ ,

$$\mathcal{R}(q_1, t) = \mathcal{R}(q_2, t) \iff \mathcal{R}_X(q_1, t) = \mathcal{R}_X(q_2, t) \text{ for } X = \mathbb{Z}_*^n.$$

A positive-definite quadratic form  $q$  has a collection of underlying lattices, all of which are congruent. Consequently, the associated flat tori are all isometric and therefore also isospectral. On the one hand, for a full-rank lattice  $A\mathbb{Z}^n$ , for any unimodular matrix  $G \in \text{GL}_n(\mathbb{Z})$ ,  $AG\mathbb{Z}^n$  and  $A\mathbb{Z}^n$  are the same lattice. The quadratic form with matrix  $(AG)^T(AG)$  is not necessarily the same as the quadratic form with matrix  $A^T A$ . These two quadratic forms are, however, integrally equivalent, and any two  $n$ -dimensional positive-definite quadratic forms that are integrally equivalent have identical representation numbers for all  $t \geq 0$ .

For a flat torus  $\mathbb{R}^n/\Gamma$  with  $\Gamma = A\mathbb{Z}^n$  we associate the equivalence class of quadratic forms that are integrally equivalent to  $A^T A$ . There is a natural bijection between the length spectrum and the representation numbers of this equivalence class, taking for some  $x \in \mathbb{Z}^n$ ,  $\|Ax\|$  to  $q(x) = x^T A^T A x = \|Ax\|^2$ . It then follows from Corollary 2.15 that two flat tori are isospectral if and only if their representation numbers associated in this way are identical. They are isometric if and only if their equivalence classes of quadratic forms are in fact identical. We therefore define isospectrality for flat tori, lattices, and quadratic forms.

**Definition 2.23** (Isospectrality of lattices and quadratic forms). Two lattices  $\Gamma_i \subset \mathbb{R}^n$ ,  $i = 1, 2$ , are isospectral if and only if the flat tori  $\mathbb{R}^n/\Gamma_i$  have identical Laplace spectra, or equivalently, they have identical length spectra. Two quadratic forms are isospectral if and only if they have identical representation numbers.

The precise number theoretic formulation in Section 1.1 is then, Is a quadratic form uniquely determined by its representation numbers, up to integral equivalence? We will see that the answer to the question depends on the dimension.

**2.4. Constructing and deconstructing lattices with implications for isospectrality.** One can construct higher-dimensional flat tori by taking products of lower-dimensional ones, or equivalently, one can build a full-rank lattice by summing lower-rank lattices. This technique has been historically important, which will be apparent in Section 4. We will also use it to give an elegant proof of the lower bound for the number of isospectral but nonisometric flat tori in each dimension in Section 6.1. If a flat torus has been built as a product of lower-dimensional flat tori, it is *reducible*, and if not, it is *irreducible*.<sup>2</sup> We use the same terminology for

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<sup>2</sup>Irreducible may also be termed *indecomposable*.

the lattice that defines the flat torus. Reducibility of lattices have connections to root systems and Dynkin diagrams that are studied in the theory of Lie groups; see for example [23, p. 217]. A key ingredient in the definition of reducibility is the Minkowski sum.

**Definition 2.24** (Minkowski sum). Let  $A$  and  $B$  be two nonempty sets in  $\mathbb{R}^n$ . Their *Minkowski sum*, denoted  $A + B$  is

$$A + B := \{a + b \in \mathbb{R}^n : a \in A, b \in B\}.$$

The product

$$A \cdot B := \{a \cdot b : a \in A, b \in B\},$$

where  $a \cdot b$  is the scalar product in  $\mathbb{R}^n$ . If  $A \cdot B = \{0\}$ ,  $A + B = A \oplus B$  is a *direct sum*.

**Definition 2.25** (Reducible and irreducible lattices and flat tori). Let  $\Gamma \subseteq \mathbb{R}^n$  be a nontrivial lattice. Then  $\Gamma$  is reducible if  $\Gamma = \Gamma_1 \oplus \Gamma_2$  for two nontrivial lattices  $\Gamma_i$ ,  $i = 1, 2$ . Otherwise  $\Gamma$  is irreducible. The associated flat torus  $\mathbb{R}^n/\Gamma$  is reducible or irreducible if  $\Gamma$  is reducible or irreducible, respectively.

By the definition of irreducibility, a full-rank lattice  $\Gamma \subset \mathbb{R}^n$  can be decomposed as a sum of irreducible sublattices  $\Gamma_i$ , so that

$$\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k,$$

for some  $1 \leq k \leq n$ . This is known as the *irreducible decomposition*, and is unique up to reordering. Kneser investigated a more general setup in [33]. The following lemma shows that an equivalent way to define reducibility and irreducibility is through products.

**Lemma 2.26.** *A lattice  $\Gamma \neq \{0\}$  is reducible if and only if it is congruent to a lattice of the form  $\Gamma_1 \times \Gamma_2$ , where  $\Gamma_1, \Gamma_2$  are of dimensions at least 1. Conversely, if there are no such  $\Gamma_1$  and  $\Gamma_2$ , then  $\Gamma$  is irreducible.*

*Proof.* Without loss of generality, we may assume that  $\Gamma$  is a full-rank lattice in  $\mathbb{R}^n$ . Then,  $\Gamma$  is reducible if and only if there are two nontrivial sublattices  $G_1$  and  $G_2$  such that  $\Gamma = G_1 \oplus G_2$ . These induce an orthogonal decomposition of  $\mathbb{R}^n$  into two subspaces of dimensions  $k_1$  and  $k_2$ , the ranks of  $G_1$  and  $G_2$ , respectively, with  $k_1 + k_2 = n$ . There is an isometry, denoted  $\Phi_i$ , between each of these subspaces and  $\mathbb{R}^{k_i}$ , respectively, for  $i = 1, 2$ . Consequently,  $G_i \cong \Phi_i(G_i) =: \Gamma_i \subset \mathbb{R}^{k_i}$ , for each  $i = 1, 2$ , and  $\Gamma = G_1 \oplus G_2 \cong \Phi_1(G_1) \times \Phi_2(G_2) = \Gamma_1 \times \Gamma_2$ .  $\square$

We note that a lattice is irreducible if and only if its dual lattice is irreducible. The following lemma shows that rearranging the constituents in a product of lattices results in a congruent lattice.

**Lemma 2.27.** *The product of lattices  $\Gamma_1 \times \cdots \times \Gamma_m$  is congruent to the product  $\Gamma_{\sigma(1)} \times \cdots \times \Gamma_{\sigma(m)}$  for any permutation  $\sigma \in S_m$ .*

*Proof.* We outline the key ideas. Let  $A = [a_j]$  be an  $n \times n$  matrix. Right-multiplication by elements of  $\mathrm{GL}_n(\mathbb{Z})$  can reorder the columns of  $A$  in any desired way. Left multiplication by elements of  $O_n(\mathbb{R})$  can reorder of the rows of  $A$  in any desired way.  $\square$

**Proposition 2.28.** *Assume that a lattice  $\Gamma$  in  $\mathbb{R}^n$  can be decomposed into an orthogonal sum of sublattices*

$$\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k.$$

*Then  $\Lambda \cong \Gamma$  if and only if  $\Lambda$  is a direct sum of  $k$  sublattices  $\Lambda_i \cong \Gamma_i$ .*

*Proof.* Without loss of generality we assume that  $\Lambda \subset \mathbb{R}^n$ , and that  $\Gamma$  is full rank. If  $\Lambda \cong \Gamma$ , there is  $C \in O_n(\mathbb{R})$  such that  $\Lambda = C\Gamma = C(\Gamma_1 \oplus \cdots \oplus \Gamma_k)$ . Since  $C$  preserves orthogonality,  $\Lambda = C\Gamma_1 \oplus \cdots \oplus C\Gamma_k$ . Thus, defining  $\Lambda_i := C\Gamma_i \cong \Gamma_i$  completes the proof in this direction. For the other direction, assume that  $\Lambda_i \cong \Gamma_i$  for each  $i = 1, \dots, k$ , with  $\Lambda := \Lambda_1 \oplus \cdots \oplus \Lambda_k$ . Then, there exist orthogonal transformations  $C_i \in O_n(\mathbb{R})$  such that  $C_i\Gamma_i = \Lambda_i$ . By orthogonality,  $\mathbb{R}^n$  admits an orthogonal decomposition into  $k$  subspaces, the  $i$ th subspace containing  $\Gamma_i$ . Let  $\Pi_i$  be an orthogonal projection onto the  $i$ th subspace. We therefore define

$$C := \sum_{i=1}^k C_i \circ \Pi_i \in O_n(\mathbb{R}), \quad C\Gamma = \Lambda_1 \oplus \cdots \oplus \Lambda_k = \Lambda \implies \Gamma \cong \Lambda. \quad \square$$

An immediate consequence is

**Corollary 2.29.** *Two products of irreducible lattices are congruent,*

$$\Gamma_1 \times \cdots \times \Gamma_k \cong \Lambda_1 \times \cdots \times \Lambda_{k'},$$

*if and only if  $k = k'$ , and up to reordering  $\Gamma_i \cong \Lambda_i$ .*

**Proposition 2.30.** *Two lattices  $\Gamma, \Lambda$  are congruent if and only if  $\Gamma^n, \Lambda^n$  are congruent for some  $n \in \mathbb{Z}_{\geq 2}$ .*

*Proof.* If  $\Gamma$  and  $\Lambda$  are congruent, we leave it to the reader to show that  $\Gamma^n$  and  $\Lambda^n$  are congruent. On the other hand we may, up to congruence, consider irreducible decompositions

$$\Gamma_1^n \times \cdots \times \Gamma_k^n \quad \text{and} \quad \Lambda_1^n \times \cdots \times \Lambda_{k'}^n.$$

By Corollary 2.29,  $nk = nk'$ , and therefore  $k = k'$ . By possibly reordering and renaming, without loss of generality,  $\Lambda_i \cong \Gamma_i$  for each  $i = 1, \dots, k$ .  $\square$

The following theorem is an immediate consequence of the definitions and preceding results together with Witt's cancellation theorem. Witt's theorem [62] shows that if the same term appears in a product, one can *cancel* that term.

**Theorem 2.31.** *Fix two congruent lattices  $\Lambda$  and  $\Lambda'$ . Two lattices  $\Gamma$  and  $\Gamma'$  are congruent if and only if  $\Gamma \times \Lambda$  is congruent to  $\Gamma' \times \Lambda'$ .*

With help from the theta series, we obtain a similar cancellation result for isospectrality.

**Lemma 2.32.** *If  $\Gamma_1$  and  $\Gamma_2$  are lattices and  $\Gamma = \Gamma_1 \times \Gamma_2$ , then their theta series satisfy  $\theta_\Gamma = \theta_{\Gamma_1}\theta_{\Gamma_2}$ . As a consequence, for an arbitrary lattice  $\Lambda$ ,  $\Lambda \times \Gamma_1$  is isospectral to  $\Lambda \times \Gamma_2$  if and only if  $\Gamma_1$  and  $\Gamma_2$  are isospectral.*

*Proof.* The first statement is left to the reader; it can be demonstrated using the Pythagorean theorem. Consider  $n$ -dimensional lattices  $\Gamma_1, \Gamma_2$  and an  $m$ -dimensional lattice  $\Gamma$ . By Corollary 2.15  $\Gamma_1 \times \Gamma$  and  $\Gamma_2 \times \Gamma$  are isospectral if and only if  $\theta_{\Gamma_1 \times \Gamma} = \theta_{\Gamma_2 \times \Gamma}$ . This is equivalent to  $\theta_{\Gamma_1}\theta_\Gamma = \theta_{\Gamma_2}\theta_\Gamma$ . By Definition 2.14, evaluating the theta series at  $i\mathbb{R}_{>0}$ , they are positive, so we may divide obtaining

$\theta_{\Gamma_1}(iy) = \theta_{\Gamma_2}(iy)$  holds for all  $y > 0$ . By the identity theorem, since theta series are holomorphic in the upper half-plane,  $\theta_{\Gamma_1}$  and  $\theta_{\Gamma_2}$  are identical. By Corollary 2.15, this is true if and only if  $\Gamma_1$  and  $\Gamma_2$  are isospectral.  $\square$

The next lemma can be proven by combining the ingredients we have collected thus far to show that once we have isospectral nonisometric flat tori in dimension  $n$ , then we have them in all higher dimensions.

**Lemma 2.33.** *If there exist  $k$  mutually isospectral and pairwise nonisometric flat tori in dimension  $n$ , then there exist  $k$  mutually isospectral and pairwise nonisometric flat tori in all higher dimensions.*

**2.5. One can hear the shapes of rectangular flat tori and Euclidean boxes.** A *rectangular flat torus* can be completely reduced to a product of intervals.

**Definition 2.34** (Rectangular lattices). A *rectangular lattice*  $\Gamma$  is a lattice that has a diagonal basis matrix. The associated flat torus  $\mathbb{R}^n/\Gamma$  is a *rectangular flat torus*.

If an  $n$ -rank lattice  $\Gamma$  has a diagonal basis matrix, then there are scalars  $\{c_j\}_{j=1}^n$  such that a basis for the lattice is  $\{c_j e_j\}_{j=1}^n$ , where  $e_j$  are the standard orthonormal basis vectors of  $\mathbb{R}^n$ . Consequently, the rectangular lattice  $\Gamma \cong \Gamma_1 \times \cdots \times \Gamma_n$ , for the one-dimensional lattices  $\Gamma_j = \mathbb{Z}c_j \cong \mathbb{Z}c_j e_j$ .

**Theorem 2.35.** *If two rectangular flat tori are isospectral, then they are isometric.*

*Proof.* The proof is by induction on the dimension. If two rectangular flat tori are isospectral, then they are the same dimension, so they are both defined by full-rank lattices in  $\mathbb{R}^n$  that have diagonal basis matrices. The case  $n = 1$  follows immediately from the equality of the first positive eigenvalue that completely determines a one-dimensional flat torus. So we now assume the theorem has been proven for dimensions up to some  $n \geq 1$ . Assume that two  $(n + 1)$ -dimensional rectangular flat tori are isospectral. The rectangular tori are each defined by the products of the one-dimensional lattices

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_{n+1} \quad \text{and} \quad \Lambda = \Lambda_1 \times \cdots \times \Lambda_{n+1},$$

with each  $\Gamma_j = \mathbb{Z}c_j \cong \mathbb{Z}c_j e_j$ , for the standard unit vector  $e_j$  and for some nonzero  $c_j$ . The length of the shortest nonzero vector in  $\Gamma$  is therefore the minimal  $|c_j|$ . The length of the shortest nonzero vectors of  $\Gamma$  and  $\Lambda$  are the same by Corollary 2.15. By Lemma 2.27 we can without loss of generality rearrange these products to assume that  $\Gamma_1 = \mathbb{Z}c_1 e_1$  and  $\Lambda_1 = \mathbb{Z}c_1 e_1$ . Consequently, these are congruent and isospectral. By Lemma 2.32, we therefore have that  $\Gamma_2 \times \cdots \times \Gamma_{n+1}$  and  $\Lambda_2 \times \cdots \times \Lambda_{n+1}$  are isospectral, and they are  $n$ -dimensional. Consequently, by the induction assumption, they are congruent. We therefore have by Theorem 2.31 that  $\Gamma$  and  $\Lambda$  are also congruent, and consequently the flat tori they define are isometric.  $\square$

Is there a rectangular flat torus that is isospectral to a nonrectangular flat torus? The answer will be revealed in Section 3.3.1.

**2.5.1. One can hear the shape of a Euclidean box.** Although we would expect the following result to be known, we are unaware of a reference in the literature and therefore include it here. A *Euclidean box* is a bounded domain in  $\mathbb{R}^n$  that is the Cartesian product of  $n$  bounded intervals.

**Theorem 2.36.** *Assume that two Euclidean boxes are isospectral with respect to the Dirichlet boundary condition or the Neumann boundary condition. Then the two boxes are isometric.*

*Proof.* If two boxes are isospectral, then the same boundary condition must be taken on both boxes. The Dirichlet boundary condition requires eigenfunctions to vanish on the boundary, whereas the Neumann boundary condition requires eigenfunctions to have a vanishing normal derivative on the smooth components of the boundary. In the Neumann case, 0 is an eigenvalue, whereas in the Dirichlet case, the spectrum is strictly positive. Although the proof is similar to the one for rectangular flat tori, the flavor is a bit different. For flat tori, one has the Poisson summation formula, which we do not have for Euclidean boxes. We replace this ingredient with the explicit calculation of the eigenvalues. We will prove the theorem for the Dirichlet boundary condition because the proof for the Neumann condition is completely analogous.

In one dimension, the spectrum consists of  $n^2\pi^2/\ell^2$  for  $n \in \mathbb{N}$ , with  $\ell$  the length of the one-dimensional box (interval). Thus, if one-dimensional boxes are isospectral, then they are isometric. Now assume the statement is true for all dimensions from 1 to  $n$  for some  $n \geq 1$ . By Weyl's law [60], if the eigenvalues are listed in nondecreasing order, then the  $k$ th eigenvalue grows with  $k \rightarrow \infty$  on the order of  $k^{2/n}$  for an  $n$ -dimensional box. As a consequence, if two boxes are isospectral, then they are the same dimension. Consider now an  $(n+1)$ -dimensional box. The first two smallest eigenvalues are

$$\lambda_1 = \sum_{k=1}^{n+1} \frac{\pi^2}{\ell_k^2}, \quad \lambda_2 = \frac{4\pi^2}{\ell_{\max}^2} + \sum_{\ell_k \neq \ell_{\max}} \frac{\pi^2}{\ell_k^2}.$$

Here,  $\ell_{\max}$  is the length of the longest side of the box. If two boxes are isospectral, they have the same first two eigenvalues, as well as the same difference between these first eigenvalues, that is  $3\pi^2/\ell_{\max}^2$ . Consequently,  $\ell_{\max}$  is the same for both boxes. The two boxes are therefore each respectively isometric to

$$B \times [0, \ell_{\max}], \quad B' \times [0, \ell_{\max}].$$

Here,  $B$  and  $B'$  are boxes of dimension  $n$ . For simplicity, set  $\ell := \ell_{\max}$ . Since the eigenvalues of  $B \times [0, \ell]$  are equal to the sum of the eigenvalues of  $B$  and the eigenvalues of  $[0, \ell]$ , the heat trace satisfies  $H_{B \times [0, \ell]}(t) = H_B(t)H_{[0, \ell]}(t)$  and, similarly,  $H_{B' \times [0, \ell]}(t) = H_{B'}(t)H_{[0, \ell]}(t)$ . The two heat traces are equal by isospectrality, and therefore  $H_B = H_{B'}$ , from which it follows that  $B$  and  $B'$  are isospectral. Since  $B$  and  $B'$  are dimension  $n$ , by induction they are also isometric. By Theorem 2.31, the two boxes are respectively isometric to  $B \times [0, \ell] \cong B' \times [0, \ell]$ .  $\square$

### 3. KEY KITCHENWARE FOR COMPUTATION AND CONSTRUCTION

In the preceding section we introduced the essential ingredients for understanding the geometry and spectra of flat tori. Here we present useful tools for manipulating and combining these ingredients. In this section, all lattices are full rank.

**3.1. Congruency tests.** Given two lattices  $\Gamma_1 = A_1\mathbb{Z}^n$  and  $\Gamma_2 = A_2\mathbb{Z}^n$  with two explicit bases, we can easily check if they are the same lattice, since this is the case if and only if  $A_1^{-1}A_2$  is a unimodular matrix. Checking whether or not the two lattices are congruent is not as simple. The lattices are congruent if and only if there is an

orthogonal matrix  $C$  and a unimodular matrix  $B$  such that  $CA_1B = A_2$ . There are infinitely many orthogonal matrices and infinitely many unimodular matrices, so checking congruency is a seemingly infinite task. One way to conclude that lattices are not congruent is furnished by Proposition 2.28, from which we immediately obtain

**Corollary 3.1.** *If  $\Gamma$  is reducible and  $\Lambda$  is not, then  $\Gamma$  is not congruent to  $\Lambda$ .*

A second method uses Corollary 2.15 together with the fact that orthogonal transformations preserve the scalar product.

**Corollary 3.2.** *Let  $\Gamma_1, \Gamma_2$  be two lattices. For  $s > 0$ , let  $S_i(s)$  be the sets of all vectors in  $\Gamma_i$  of length  $s$ . If there exists an  $s > 0$  such that the number of elements in  $S_i(s)$  are not equal, then  $\Gamma_i$  are not congruent. Let  $P_i(s) = \{a \cdot b : a, b \in S_i(s)\}$ . If there exists an  $s > 0$  such that  $P_i(s)$  are not equal, then  $\Gamma_i$  are not congruent.*

A third method uses the equivalence classes of quadratic forms we associate to lattices, with the help of the following corollary that is a consequence of Proposition 2.21.

**Corollary 3.3.** *Let  $Q_1$  and  $Q_2$  be positive-definite  $n \times n$  matrices. Let  $\lambda_{\min}$  be the smallest eigenvalue of  $Q_1$ . If  $B = [b_j]$  is a matrix with column vectors  $b_j$ , and  $B^T Q_1 B = Q_2$ , then*

$$b_j^T Q_1 b_j = (Q_2)_{jj}, \quad j = 1, \dots, n.$$

*Moreover,  $\|b_j\|^2 \leq (Q_2)_{jj} / \lambda_{\min}$  for each  $j = 1, \dots, n$ .*

To lattices  $A_i \mathbb{Z}^n$  for  $i = 1, 2$  we associate the class of quadratic forms that are integrally equivalent to  $Q_i = A_i^T A_i$  for each  $i$ . The lattices are congruent if and only if these equivalence classes are identical. In turn, these equivalence classes are identical if and only if there is a unimodular matrix  $B$  such that  $B^T Q_1 B = Q_2$ . Since the entries of unimodular matrices are integers, there are only finitely many unimodular matrices whose column vectors satisfy the conditions of Corollary 3.3. Consequently, one can write a computer program to check congruency based on Corollary 3.3. A greater challenge is to investigate infinite families of quadratic forms. In 2011, Cervino and Hein [25] developed a systematic method to analyze isometry and nonisometry of infinite families of pairs of quadratic forms using modular forms.

**3.2. Modular forms in connection to isospectrality.** Verifying with complete confidence that two flat tori are isospectral requires checking that all of their eigenvalues are identical counting multiplicity, yet again an apparently infinite task. It turns out that the theory of *modular forms* gives a convenient criterion for determining precisely when flat tori are isospectral. This is one of several reasons why the theory and language of modular forms appear in many of the articles related to the spectral theory of flat tori.

To state the definition, recall that  $\mathrm{SL}_2(\mathbb{Z})$  is the *special linear group* of  $2 \times 2$  integer matrices with determinant 1. The *congruence groups* are defined as

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Let  $\mathbb{H}$  denote the complex upper half-plane.

**Definition 3.4** (Dirichlet characters and modular forms). A Dirichlet character mod  $N$  is a function  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$  such that  $\chi(mn) = \chi(m)\chi(n)$ ,  $\chi(m+N) = \chi(m)$ ,  $\chi(a) = 0$  if  $\gcd(a, N) > 1$ , and  $\chi(a) \neq 0$  if  $\gcd(a, N) = 1$ . A Dirichlet character defines a character on the congruence group  $\Gamma_0(N)$  via

$$\chi(\gamma) := \chi(d), \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N).$$

Given a congruence group  $\Gamma_0(N)$ , a *modular form* of weight  $k$  and character  $\chi$  mod  $N$  is a function  $f : \mathbb{H} \rightarrow \mathbb{C}$  such that:

- (1)  $f$  is holomorphic.
- (2) For any  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$ ,  $f(\frac{az+b}{cz+d}) = \chi(d)(cz+d)^k f(z)$ .
- (3) For any  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ ,  $(cz+d)^{-k} f(\frac{az+b}{cz+d})$  is bounded as  $\mathrm{Im}(z) \rightarrow \infty$ .

The set of such forms is denoted  $M_k(\Gamma_0(N), \chi)$ , following the notation of [35, p. 127].

In fact, it is enough to check conditions (2) and (3) for the generators of  $\Gamma_0(N)$  and  $\mathrm{SL}_2(\mathbb{Z})$ , respectively. One set of generators for  $\mathrm{SL}_2(\mathbb{Z})$  consists of the matrices  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix}$ .

Let us now restrict our inquiry to *rational* quadratic forms (resp., rational flat tori and rational lattices). Up to a constant, any rational quadratic form is an even quadratic form, as in Definition 2.17. In the sense of Proposition 6.5, this restriction does not limit the investigation of the relationship between isospectrality and congruence because this can be reduced to only considering rational quadratic forms. One connection between modular forms and the spectrum of flat tori is through the following theorem which shows that the theta series of an even-dimensional rational lattice is a modular form.

**Theorem 3.5** ([16, p. 295] or [38, Corollary 4.9.5(iii)]). *Let  $Q$  be an even positive-definite quadratic form of  $2k$  variables, and let  $N_Q$  be the smallest positive integer such that  $N_Q Q^{-1}$  is even. If  $\Gamma$  is an underlying lattice, then  $\theta_\Gamma(z) \in M_k(\Gamma_0(N_Q), \chi)$ . The character  $\chi$  mod  $N_Q$  is defined for*

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N_Q),$$

$$\chi(\gamma) := \begin{cases} \mathrm{sgn}(d)^k, & (-1)^k \det(Q) \not\equiv 0 \pmod{|d|} \text{ and is a quadratic residue mod } |d|, \\ -\mathrm{sgn}(d)^k, & (-1)^k \det(Q) \text{ is a nonquadratic residue mod } |d|, \\ 0, & (-1)^k \det(Q) \equiv 0 \pmod{|d|}. \end{cases}$$

One can use the dual quadratic form of  $q$  to show that  $N_Q$  is a spectral invariant of even positive-definite quadratic forms and their underlying lattices. Modular forms in the same space  $M_k(\Gamma_0(N_Q), \chi)$  are determined by finitely many coefficients in their Fourier expansions. Moreover, this number of coefficients is estimated from above by the *Sturm bound* [53].

**Theorem 3.6** (Hecke's identity theorem for modular forms [24, p. 811]). *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \in M_k(\Gamma_0(N), \chi), \quad \mu_0(N) := N \prod_{p|N, \text{ prime}} \left(1 + \frac{1}{p}\right).$$

Assume that  $\chi$  is real valued. Then the first  $\left\lceil \frac{\mu_0(N)k}{12} + 1 \right\rceil$  coefficients  $a_n$  completely determine  $f$ . Equivalently,

$$a_n = 0 \quad \forall n \text{ with } 0 \leq n \leq \frac{\mu_0(N)k}{12} + 1 \implies f = 0.$$

We first apply the identity theorem to modular forms in even dimensions.

**Corollary 3.7** (Isospectrality certificate). *Let  $P$  and  $Q$  be two even positive-definite quadratic forms of  $2k$  variables. They are isospectral if and only if  $\det(P) = \det(Q)$ ,  $N_P = N_Q$ , and their multiplicities over the integers of values less than or equal to  $\frac{\mu_0(N_P)k}{12} + 1$  coincide.*

*Proof.* It is straightforward to verify that if  $P$  and  $Q$  are isospectral, then the statements of the corollary hold. For the converse, note that for any underlying lattice of  $Q$ , denoted  $A\mathbb{Z}^n$ , the theta series of this lattice is

$$\sum_{\gamma \in A\mathbb{Z}^n} e^{i\pi z \|\gamma\|^2} = \sum_{x \in \mathbb{Z}^n} e^{i\pi z (Ax)^T (Ax)} = \sum_{x \in \mathbb{Z}^n} e^{i\pi z x^T Q x}.$$

Consequently, the theta series is identical for all underlying lattices, and we can therefore define the theta series associated to  $P$  and  $Q$ ,

$$\theta_P(z) := \sum_{x \in \mathbb{Z}^n} e^{i\pi z x^T P x}, \quad \theta_Q(z) := \sum_{x \in \mathbb{Z}^n} e^{i\pi z x^T Q x}.$$

If  $N_P = N_Q$ , and  $\det(P) = \det(Q)$ , then their corresponding theta series  $\theta_P, \theta_Q$  lie in  $M_k(\Gamma_0(N_P), \chi)$  for some  $\chi$  determined by  $\det(P) = \det(Q)$ , by Theorem 3.5. The quadratic forms are isospectral if and only if their theta series are identical. So, consider  $f(z) := \theta_P(z) - \theta_Q(z)$ . We have

$$\begin{aligned} M_k(\Gamma_0(N_P), \chi) \ni f(z) &= \sum_{x \in \mathbb{Z}^n} e^{\pi i z x^T P x} - \sum_{x \in \mathbb{Z}^n} e^{\pi i z x^T Q x} \\ &= \sum_{n \in \mathbb{N}} (m_P(n) - m_Q(n)) e^{\pi i z n}, \end{aligned}$$

where  $m_P(n), m_Q(n)$  denote the multiplicities of the value  $n$  of  $P$  and  $Q$ , respectively, as quadratic forms over  $\mathbb{Z}^n$ . By Theorem 3.6, if  $m_P(n) - m_Q(n) = 0$  for all  $0 \leq n \leq \mu_0(N_P)k/12 + 1$ , then  $f(z) = 0$  for all  $z$ , and  $\theta_P = \theta_Q$ .  $\square$

To extend Corollary 3.7 to odd dimensions, note that if  $P, Q$  are positive-definite  $k \times k$  matrices, they define isospectral quadratic forms if and only if the  $(k+1)$ -dimensional forms defined by

$$P' = \begin{bmatrix} 2 & 0 \\ 0 & P \end{bmatrix} \quad \text{and} \quad Q' = \begin{bmatrix} 2 & 0 \\ 0 & Q \end{bmatrix}$$

are isospectral. Moreover,  $P'$  and  $Q'$  are integrally equivalent if and only if  $P$  and  $Q$  are as well.

**3.3. Building lattices from linear codes.** Codes are used in numerous everyday circumstances, including data compression, cryptography, error detection and correction, data transmission, and data storage. Linear codes are useful for studying lattices because they allow one to translate questions for lattices, infinite discrete groups, into questions for finite groups, known as linear codes. For a general treatment we refer to [15] and [40].

**Definition 3.8.** A linear  $q$ -nary code  $C$  of length  $n$  is a  $\mathbb{Z}$ -linear subspace (and a subgroup) of the module  $(\mathbb{Z}/q\mathbb{Z})^n$ , where  $\mathbb{Z}/q\mathbb{Z}$  is the ring of integers modulo  $q$ . Its elements are *codewords*. The linear space  $(\mathbb{Z}/q\mathbb{Z})^n$  is equipped with the inner product  $x \cdot y := \sum_{i=1}^n x_i y_i \pmod{q}$ .

In the literature,  $q$  is often assumed to be prime, but we do not need this assumption for our purposes. To construct a lattice from a linear code, consider the projection

$$\pi : \mathbb{Z}^n \rightarrow (\mathbb{Z}/q\mathbb{Z})^n \quad \text{by} \quad z \mapsto z \pmod{q},$$

where  $\text{mod } q$  acts coordinate-wise. Importantly,  $\pi$  is a group homomorphism.

Let  $C$  be a linear code in  $(\mathbb{Z}/q\mathbb{Z})^n$ . The preimage under  $\pi_q$ ,

$$\pi_q^{-1}(C) := \{\ell \in \mathbb{Z}^n : \ell \pmod{q} \in C\},$$

is a full-rank lattice; see [30, Prop. 16.2]. The codewords  $c_i \in C$  partition  $\pi_q^{-1}(C)$  into the subsets  $\pi_q^{-1}(c_i)$ . So, if  $C_1, C_2 \subseteq (\mathbb{Z}/q\mathbb{Z})^n$  are different codes, then  $\pi_q^{-1}(C_1) \neq \pi_q^{-1}(C_2)$ . If  $c_1, \dots, c_k$  are generators of  $C$ , then viewing them as elements of  $\{0, 1, \dots, q-1\}^n \subseteq \mathbb{Z}^n$  gives

$$(3.1) \quad \pi_q^{-1}(C) = [c_1 \ \cdots \ c_k \ qI] \mathbb{Z}^{k+n} \subseteq \mathbb{Z}^n,$$

with  $I$  the  $n \times n$  identity matrix. In [10], building a lattice in this way is called *construction A*. In the following theorem, we prove that *all* integer lattices can be obtained by construction A, indicating the usefulness of this approach. As a consequence of this characterization, there are only finitely many distinct integer lattices of a given determinant  $q$ , because the number of linear codes in  $(\mathbb{Z}/q\mathbb{Z})^n$  is finite.

**Theorem 3.9.** *Any integer lattice  $L$  is the preimage of the  $\text{vol}(L)$ -nary code  $\pi_{\text{vol}(L)}(L)$ .*

*Proof.* Assume that  $L$  is an integer lattice. For any  $q$ , define

$$L' := \pi_q^{-1}(\pi_q(L)) = \{x \in \mathbb{Z}^n : \pi_q(x) \in \pi_q(L)\}.$$

For  $x \in \mathbb{Z}^n$ ,  $\pi_q(x) \in \pi_q(L)$  if and only if there is some  $\gamma \in L$  such that  $\pi_q(x) = \pi_q(\gamma)$ . This is equivalent to  $\pi_q(x - \gamma) = 0$ , which holds if and only if  $x = \gamma + qz$  for some  $z \in \mathbb{Z}^n$ . This proves  $L' = L + q\mathbb{Z}^n$ . For  $q = \text{vol}(\Gamma)$ ,  $q\mathbb{Z}^n \subseteq L$  by Proposition 2.3, and therefore  $L' = L$ .  $\square$

**Corollary 3.10.** *Let  $C \subseteq (\mathbb{Z}/q\mathbb{Z})^n$  be a linear code, and let  $L = AZ^n$  be a lattice, where  $A = [a_j]$ . If the codewords  $\pi_q(a_j)$  generate  $C$ , then  $L \subseteq \pi_q^{-1}(C)$ . If in addition,  $q\mathbb{Z}^n \subseteq L$ , then  $L = \pi_q^{-1}(C)$ .*

*Proof.* If  $\pi_q(a_j)$  generate  $C$ , then  $\pi_q(L) = C$ , so  $L \subseteq \pi_q^{-1}(\pi_q(L)) = \pi_q^{-1}(C)$ . If  $q\mathbb{Z}^n \subseteq L$ , then the inclusion becomes an equality.  $\square$

We next relate linear codes to isospectrality. For this we need

**Definition 3.11.** Let  $C_1, C_2 \subseteq (\mathbb{Z}/q\mathbb{Z})^n$  be two linear codes of equal cardinality, and list their respective elements as  $c_1^{(i)}, \dots, c_k^{(i)}$  for  $i = 1, 2$ . The codes have the same *weight distribution* if for each pair  $(c_j^{(1)}, c_j^{(2)})$ , there is a permutation  $\sigma \in S_n$  such that  $(c_j^{(2)})_k = (c_j^{(1)})_{\sigma(k)}$  for each coordinate  $k$ . The codes constitute an *absolute*

pairing if for each  $(c_j^{(1)}, c_j^{(2)})$  we have  $(c_j^{(2)})_k = \pm(c_j^{(1)})_k$  in  $\mathbb{Z}/q\mathbb{Z}$  for each  $k$ . Both relations are equivalence relations.

**Proposition 3.12.** *Let  $C_1, C_2$  be  $q$ -nary linear codes, and let  $L_i = \pi_q^{-1}(C_i)$ . If the weight distributions of  $C_1$  and  $C_2$  are the same, then  $L_1$  and  $L_2$  are isospectral. The converse does not hold.*

*Proof.* We prove the first part. Let  $c_1^{(i)}, \dots, c_m^{(i)}$  be lists of codewords as in the definition of equal weight distribution. We give length-preserving bijections from  $\pi_q^{-1}(c_j^{(1)})$  to  $\pi_q^{-1}(c_j^{(2)})$  for each  $j$  which is sufficient since the inverse images of the codewords partition  $\pi_q^{-1}(C_i)$ . Let  $\sigma$  be the permutation corresponding to the pair  $(c_j^{(1)}, c_j^{(2)})$ . It can be realized as a permutation matrix  $\Sigma$ , which is orthogonal. Then  $x \mapsto \Sigma x$  is length-preserving and by construction a bijection from  $\pi_q^{-1}(c_j^{(1)})$  to  $\pi_q^{-1}(c_j^{(2)})$ . Consequently,  $L_i$  have identical length spectra and are therefore isospectral.  $\square$

**Theorem 3.13.** *Let  $C_1, C_2 \subseteq (\mathbb{Z}/q\mathbb{Z})^n$  be linear codes with*

$$L_i = A_i \mathbb{Z}^n = \pi_q^{-1}(C_i).$$

*There is a bijection  $\phi : L_1 \rightarrow L_2$  preserving the absolute value of each coordinate if and only if  $C_1, C_2$  make an absolute pairing. If either is true, then  $DL_1$  and  $DL_2$  are isospectral for any diagonal invertible matrix  $D$ .*

*Proof of Theorem 3.13.*  $\Leftarrow$ ) Let  $c_1^{(i)}, \dots, c_m^{(i)}$  be lists of codewords as in the definition of an absolute pairing, and view them as elements of  $\{0, 1, \dots, q-1\}^n \subseteq \mathbb{Z}^n$ . We again give a length-preserving bijection  $\phi : \pi_q^{-1}(c_j^{(1)}) \rightarrow \pi_q^{-1}(c_j^{(2)})$  for a fixed  $j$ . Write  $x \in \pi_q^{-1}(c_j^{(1)})$  uniquely as  $c_j^{(1)} + qt$ , where  $t \in \mathbb{Z}^n$ . Let  $\phi$  act coordinate-wise bijectively as follows:

$$x_k = (c_j^{(1)})_k + qt_k \mapsto \begin{cases} (c_j^{(2)})_k + qt_k, & \text{if } (c_j^{(2)})_k = (c_j^{(1)})_k, \\ (c_j^{(2)})_k - q(t_k + 1), & \text{otherwise if } (c_j^{(2)})_k = q - (c_j^{(1)})_k. \end{cases}$$

This map preserves the absolute values of the coordinates. We leave the direction  $\Rightarrow$  of the proof to the reader. Isomorphic codes (in the sense of groups) do not in general correspond to either isospectral or congruent lattices. For the last part, consider  $\phi : L_1 \rightarrow L_2$ , a bijection between lattices and a diagonal matrix  $D$  as in the statement above. By assumption, if  $x \in L_1$ , then  $|x_i| = |\phi(x)_i|$  for each coordinate. Now define  $\varphi : DL_1 \rightarrow DL_2$  as  $\varphi(Dx) := D\phi(x)$  for  $x \in L_1$ . We have for  $x \in L_1$ ,  $|(Dx)_i| = |d_i x_i| = |d_i \phi(x)_i| = |(D\phi(x))_i| = |\varphi(Dx)_i|$ . Therefore  $\varphi$  is again a bijection that preserves the absolute values of coordinates, from whence it follows that  $DL_1$  and  $DL_2$  are isospectral.  $\square$

**3.3.1. Can one hear cubicity?** In Section 2.5 we proved that if a pair of rectangular flat tori are isospectral, then they are isometric. Suppose we only know a priori that one flat torus in the pair is rectangular, then surely the other must be as well? As an application of linear codes, we show that the answer depends on the dimension. Following Conway [9, pp. 40–42], we define *cubic lattices* as those that are congruent up to scaling to  $\mathbb{Z}^n$  and say that *cubicity* is the property of being cubic.

**Proposition 3.14.** *The two lattices*

$$\Lambda = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \mathbb{Z}^6 \quad \text{and} \quad \Omega = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{bmatrix} \mathbb{Z}^6$$

are isospectral and noncongruent. In particular,  $\Lambda$  is cubic, but  $\Omega$  is not. Cubicity is audible in dimensions 5 and lower; it is not audible in dimensions 6 and higher.

*Proof.* We prove that the pair is isospectral and noncongruent. As a consequence, Lemma 2.33 shows that cubicity is not audible for  $n > 6$ . The fact that cubicity is audible when  $n < 6$  is shown in [9, p. 60]. Consider the following two binary linear codes of length 6. The rows are the codewords, and the codewords of the same row differ by permutation, showing that the codes are of the same weight distribution:

$$C_1 : \begin{array}{c} 000000 \\ 110000 \\ 001100 \\ 000011 \\ 111100 \\ 110011 \\ 001111 \\ 111111 \end{array} \quad \text{and} \quad C_2 : \begin{array}{c} 000000 \\ 101000 \\ 001010 \\ 100010 \\ 010111 \\ 110101 \\ 011101 \\ 111111 \end{array}.$$

We have that, modulo 2, the vectors of the given bases of  $\Lambda$ ,  $\Omega$  generate the codes  $C_1$ ,  $C_2$ , respectively. It is straightforward to check that  $2\mathbb{Z}^6 \subseteq \Lambda$ ,  $\Omega$ . So, the preimages of the codes are equal to  $\Lambda$ ,  $\Omega$ , respectively, by Corollary 3.10. They are isospectral by Proposition 3.12 and nonisometric since the vectors of length  $\sqrt{2}$  in  $\Lambda$  that are nonparallel are orthogonal, while this is not the case for  $\Omega$ .  $\square$

#### 4. DUETS OF ISOSPECTRAL NONISOMETRIC FLAT TORI

In 1964, a paper of one single page was published in *Proceedings of the National Academy of Science* that is famous to this day [37], “Eigenvalues of the Laplace operator on certain manifolds”. In it Milnor described an example of two 16-dimensional flat tori that are isospectral but not isometric. One can imagine this as a *duet*, a pair of perfectly attuned yet differently shaped flat tori that would resonate with identical frequencies. Milnor’s paper inspired Kac’s acclaimed work [29] entitled “Can one hear the shape of a drum?” The resolution of Kac’s question by Gordon, Webb, and Wolpert [20] was based in part on adapting the *Sunada method* [54] from four dimensions to two. Sunada humbly described this method [54, p. 169] as “a *geometric* analogue of a routine method in number theory.” This simple and elegant method constructs not only isospectral Riemannian manifolds but also families of isospectral abelian varieties.

**4.1. Milnor’s duet.** Milnor’s paper [37] referred to a construction of two lattices by Witt [61] that begins with the root lattice  $D_n$ , also called the *checkerboard lattice*,

$$D_n := \left\{ z = (z_1, \dots, z_n) \in \mathbb{Z}^n : \sum_{i=1}^n z_i \in 2\mathbb{Z} \right\}.$$

A basis for  $D_n$  is given by the vectors  $\{e_1 + e_2, e_{j-1} - e_j\}_{j=2}^n$ . The root lattice of the  $E_n$  root system, also denoted by  $E_n$ , for  $n$  divisible by 4 is

$$E_n := \left\{ x \in \mathbb{Z}^n \cup \left( \frac{1}{2} \mathbf{1} + \mathbb{Z}^n \right) : \sum x_i \in 2\mathbb{Z} \right\}, \quad \mathbf{1} := \sum_{j=1}^n e_j.$$

We note that  $E_8$  is sometimes known as the Gosset lattice after [21].

When  $n$  is divisible by 4, a basis for  $E_n$  is given by  $\{e_1 + e_2, e_{j-1} - e_j, \frac{1}{2}\mathbb{1}\}_{j=2}^{n-1}$ . The classical theory of root lattices tells us that  $D_n$  with  $n > 2$  and  $E_{4n}$  are irreducible lattices; see [15, §1.4] and [11, §4.7–4.8.1]. We give an alternative method for checking irreducibility that may be of independent interest as we have not seen this elsewhere in the literature. This method can be applied to prove irreducibility of  $D_n$  and  $E_{4n}$  but may also be more broadly applicable.

**Lemma 4.1.** *Let  $A = [a_j]$  be a basis of a lattice,  $\Gamma \subseteq \mathbb{R}^n$  with irreducible components  $\Gamma_i$ . For a scalar  $s \neq 0$ , and a vector  $v \in \mathbb{R}^n$ , the lattice*

$$\Lambda := \begin{bmatrix} A & v \\ 0 & s \end{bmatrix} \mathbb{Z}^{n+1}$$

is irreducible if:

- (a) each  $(a_j, 0)$  is of shortest nonzero length in  $\Lambda$ ;
- (b)  $(v + \gamma) \cdot \Gamma_i \neq \{0\}$  for each  $\gamma \in \Gamma$  and  $i$ .

*Proof.* By contradiction, assume that we can write  $\Lambda = \Lambda_1 \oplus \Lambda_2$ , where  $\Lambda_i$  are nontrivial and  $\Lambda_1$  is irreducible. Without loss of generality, since  $s \neq 0$ , we may assume that there is some  $\omega \in \Lambda_1$  with  $\omega = (x, t)$  for some  $t \neq 0$  (for simplicity we use row vector notation in this proof). Then,  $\{\omega, (a_j, 0)\}_{j=1}^n$  are all contained in  $\Lambda$ , and they are all linearly independent. Each  $(a_j, 0)$  lies in precisely one of  $\Lambda_i$ , by virtue of (a) and the Pythagorean theorem. Consequently, the number of  $(a_j, 0)$  that lie in  $\Lambda_2$  is equal to the rank of  $\Lambda_2$ . In particular, the last coordinate of every element of  $\Lambda_2$  vanishes. Therefore, the projection of  $\Lambda_2$  onto the first  $n$  coordinates is an orthogonal sum of some  $\Gamma_i$ . We fix an element  $\Gamma_i$  in this sum. Since  $(v, s) \in \Lambda$ ,  $(v, s)$  is equal to a sum of an element of  $\Lambda_1$  and an element of  $\Lambda_2$ . We can therefore, without loss of generality, choose  $\omega = (x, s) \in \Lambda_1$  such that  $\omega = (v, s) + (\gamma, 0)$  with  $(\gamma, 0) \in \Lambda_2$ , and  $\gamma \in \Gamma$ . Since  $\omega \in \Lambda_1$ ,  $(v + \gamma) \cdot \Gamma_i$  vanishes, a contradiction to (b).  $\square$

Since  $E_{16}$  is irreducible, it is not congruent to  $E_8 \times E_8$ . Milnor's result is completed by showing that they are isospectral. These lattices are even and unimodular, and consequently their theta series are modular forms for  $\mathrm{PSL}_2(\mathbb{Z})$ . In 16 dimensions there is only one such form (up to multiplication by scalars), so since these lattices have identical volume, they also have identical theta series and are therefore isospectral.

**4.2. The race to find duets.** As we have seen, one-dimensional isospectral flat tori are always isometric, so a natural question is, What is the lowest dimension in which there are isospectral nonisometric flat tori? Following Milnor, the search for isospectral and nonisometric duets of flat tori became a race towards the lowest possible dimension. Kneser found a 12-dimensional example in 1967 [34]. Ten years later, Kitaoka [32] reduced this to eight. In 1986, Conway and Sloane [10] found five- and six-dimensional examples. In 1990, Schiemann [47] constructed a four-dimensional example. Independently, and using a different approach, Shiota [52] found another example one year later in 1991. The same year, Earnest and Nipp [14] contributed one more pair.

Kneser's 12-dimensional pair is  $D_{12}$  and  $E_8 \times D_4$  [34]. Kitaoka also made use of  $D_4$  in his construction [32]. We refer interested readers to the literature for the

above-mentioned constructions and recall here Schiemann's four-dimensional pair. Consider the positive-definite matrices

$$\begin{bmatrix} 4 & 2 & 0 & 1 \\ 2 & 8 & 3 & 1 \\ 0 & 3 & 10 & 5 \\ 1 & 1 & 5 & 10 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 0 & 1 & 1 \\ 0 & 8 & 1 & -4 \\ 1 & 1 & 8 & 2 \\ 1 & -4 & 2 & 10 \end{bmatrix}.$$

Schiemann proved that the quadratic forms they define have identical representation numbers using Corollary 3.7. He showed that these forms are not integrally equivalent using the theory of Minkowski reduction; see Section 5.2. This can also be seen by Corollary 3.3. These two quadratic forms were systematically found in the sense that they are the integral forms that satisfy these conditions and have the smallest determinant.

**4.3. Conway and Sloane's isospectral family.** One way to obtain a family of infinitely many pairs of isospectral nonisometric four-dimensional flat tori is to start with Schiemann's pair, denoted by  $(S_1, S_2)$ , and take the one-parameter family  $(cS_1, cS_2)_{c>0}$ . We say that such a family is obtained by scaling. Can one obtain an infinite family of isospectral nonisometric pairs that are not simply obtained by scaling? Conway and Sloane were the first to present an infinite family of four-dimensional pairs:

**Theorem 4.2** (Conway and Sloane [10] and Cervino and Hein [25]). *Let  $a, b, c, d > 0$ . Consider the two matrices*

$$A_{\pm} = \frac{1}{\sqrt{12}} \begin{bmatrix} \sqrt{a} & 0 & 0 & 0 \\ 0 & \sqrt{b} & 0 & 0 \\ 0 & 0 & \sqrt{c} & 0 \\ 0 & 0 & 0 & \sqrt{d} \end{bmatrix} \underbrace{\begin{bmatrix} \pm 3 & 1 & 1 & 1 \\ -1 & \pm 3 & -1 & 1 \\ -1 & 1 & \pm 3 & -1 \\ -1 & -1 & 1 & \pm 3 \end{bmatrix}}_{=:T_{\pm}}.$$

*There is a bijection  $T_+\mathbb{Z}^4 \rightarrow T_-\mathbb{Z}^4$  preserving absolute values of the coordinates, and therefore the lattices  $A_+\mathbb{Z}^4, A_-\mathbb{Z}^4$  are isospectral for any  $a, b, c, d > 0$ . They are nonisometric if and only if  $a, b, c$ , and  $d$  are all distinct.*

Both  $A_{\pm}$  have determinant  $\sqrt{abcd}$ . The observant reader may see a connection to Definition 3.13. Indeed,  $T_{\pm}\mathbb{Z}^4$  correspond to linear codes that constitute an absolute pairing. However, since  $\det(T_{\pm}) = 144$ , this is not easy to check directly. This theorem encompasses, up integral equivalence, Schiemann's pair  $(S_1, S_2)$  of quadratic forms by letting  $a = 1$ ,  $b = 7$ ,  $c = 13$ ,  $d = 19$ , which can be seen using Corollary 3.3. Conway and Sloane were only able to verify nonisometry for integers  $a < b < c < d$  with  $abcd < 10,000$  [10]; for other values of  $a, b, c, d$  the nonisometry was a conjecture at that time. Their proof of isospectrality [10] is short and simple, but the proof of the nonisometry conjecture resisted solution for nearly 20 years. Its proof was completed by Cervino and Hein [25] in 2011 and is significantly more complicated than the proof of isospectrality. Conway, Sloane, Cervino, and Hein therefore together obtained the following corollary.

**Corollary 4.3.** *There are infinitely many pairs of isospectral nonisometric four-dimensional flat tori, that are not simply related by scaling.*

## 5. THE RACE ENDS TO THE SOUND OF SCHIEMANN'S SYMPHONY

The race to find isospectral nonisometric flat tori in successively lower dimensions finally ended in 1994, over three decades after it began, when Alexander Schiemann determined the final answer to the three equivalent questions in Section 1.1. The aim of this section is to describe *Schiemann's symphony*, the elaborate algorithm that gave a final answer to this question and that was performed with a comprehensive computer search [48], [49].

**5.1. Schiemann's theorem.** To quantify the isospectral question for flat tori, we introduce the *choir numbers*.

**Definition 5.1.** In each dimension  $n \in \mathbb{N}$ , we define the *choir number*  $b_n$  to be the maximal number  $k$  such that the three following equivalent conditions hold:<sup>3</sup> there is a sequence of  $k$   $n$ -dimensional mutually isospectral

- (1) flat tori that are nonisometric;
- (2) positive-definite quadratic forms that are nonintegrally equivalent;
- (3) lattices that are noncongruent.

Since isospectral flat tori would sound identical, the choir numbers describe how many geometrically distinct flat tori can resonate in perfect unison. In 1990–1994, Schiemann proved the next remarkable theorem [47], [48].

**Theorem 5.2** (Schiemann's theorem). *The choir numbers are equal to 1 for  $n = 1, 2, 3$  and greater than 1 otherwise.*

*Proof that  $b_2 = 1$ .* In dimension 2, a full-rank lattice always has a basis consisting of a shortest nonzero vector, and a shortest vector that is linearly independent of the first one. Interestingly, it is *not* always possible to find such a basis for higher-dimensional lattices; see Section 5.2. Let  $A, A'$  be bases of two isospectral lattices  $\Gamma, \Gamma'$  in  $\mathbb{R}^2$  with the aforementioned property. By left-multiplication with an orthogonal matrix, we may assume that

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} a' & c' \\ 0 & b' \end{bmatrix}.$$

Here,  $a, a', b, b'$  are positive numbers. By Corollary 2.15,  $a = a'$  and  $b = b'$ . If  $|c| < |c'|$ , then the closed ball  $\overline{D(0, \|(c, b)\|)}$  contains more points from  $\Gamma$  than from  $\Gamma'$ , implying that  $\Gamma, \Gamma'$  are not isospectral. If  $c = -c'$ , then by letting  $B = C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ , we see that  $CA' = AB$  and  $\Gamma$  is congruent to  $\Gamma'$ .  $\square$

The proof that  $b_1 = 1$  was shown in the introduction, and the proof that  $b_2 = 1$  above is also quite short. In contrast, proving  $b_3 = 1$  is so difficult that, to the best of our knowledge, it has never been achieved without the help of a computer. This proof in full detail is only available in German [47]. Consequently, we take this opportunity to present the main ideas, general structure, and strategy of the proof. This strategy is independently interesting, and we further suspect that Schiemann's methods can be generalized to higher dimensions in order to determine, say  $b_4$ , which is unknown. We describe the proof of Theorem 5.2 as *Schiemann's symphony*. This piece is performed using positive-definite quadratic forms. Up to integral equivalence, these forms can be geometrically represented as a polyhedral cone, which is a convenient structure for our computer algorithms. The proof is

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<sup>3</sup>We use the musical symbol “b” (flat), since we are working with flat tori.

rather technical, so we employ musical analogies as a mnemonic technique to keep track of the different elements in the proof and the roles they play.

**5.2. Minkowski reduction.** A *representative* of a quadratic form  $Q$  is any element of the equivalence class of integrally equivalent quadratic forms containing  $Q$ . There are infinitely many representatives in each integral equivalence class of positive-definite quadratic forms. A *Minkowski reduced form* is a particularly natural representative which has been of historical interest as described in Schürman's survey [50].

**Definition 5.3** (Minkowski reduced forms and bases). A positive-definite quadratic form  $q$  is *Minkowski reduced* if for all  $k = 1, \dots, n$  and for all  $x \in \mathbb{Z}^n$  with  $\gcd(x_k, \dots, x_n) = 1$ , we have  $q(x) \geq q_{kk}$ . Moreover, a lattice basis  $A = [a_j] \in \mathrm{GL}_n(\mathbb{R})$  is *Minkowski reduced* if for each  $j$ ,

$$a_j \in A\mathbb{Z}^n \setminus \{0, a_1, \dots, a_{j-1}\}$$

is a shortest choice of vector such that  $a_1, \dots, a_j$  is part of some basis of  $\Gamma$ .

We summarize the key properties of Minkowski reduced forms and bases in the following proposition.

**Proposition 5.4.**

- (1) A quadratic form  $q$  is positive definite and Minkowski reduced if and only if  $q_{11} > 0$  and  $q(x) \geq q_{kk}$  for all  $k = 1, \dots, n$  as long as  $x \in \mathbb{Z}^n$  with  $\gcd(x_k, \dots, x_n) = 1$ .
- (2)  $A$  is a Minkowski reduced lattice basis if and only if  $A^T A$  is a Minkowski reduced positive-definite form.
- (3) Each positive-definite quadratic form has a positive, finite number of Minkowski reduced representatives.

*Proof.* The first statement follows from Definition 5.3. The second statement is demonstrated in [6, Cor. 4, p. 14]. The third statement is proven in [6, pp. 27–28], alternatively [57, Section 4.4.2].  $\square$

Is it possible to find an even more intuitive reduction? For example, is there always, given a lattice  $\Gamma$ , a basis matrix  $A = [a_j]$  such that each

$$a_j \in \Gamma \setminus \mathrm{Span}\{0, a_1, \dots, a_{j-1}\}$$

is *any* shortest choice of vector? In four dimensions, the following example constructed by van der Waerden [58, p. 286] shows that this is not always possible. Consider the basis matrix

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

of the lattice  $\Gamma = A_4\mathbb{Z}^4$ , with the column vector notation  $A_4 = [a_j]$ . We see that  $e_4 = -a_1 - a_2 - a_3 + 2a_4$  is of the same length as  $a_4$  and is linearly independent of  $a_1, a_2, a_3$ ; however, the vectors  $a_1, a_2, a_3, e_4$  do not make a basis for  $\Gamma$ . In general for  $n \geq 5$ , let  $A_n = [e_1, \dots, e_{n-1}, \frac{1}{2}\mathbb{1}]$ , where  $\mathbb{1} = e_1 + \dots + e_n$ . Then  $e_n \in A_n\mathbb{Z}^n$  is shorter than  $\frac{1}{2}\mathbb{1}$ , but  $e_1, \dots, e_n$  is not a basis for  $A_n\mathbb{Z}^n$ .

**Theorem 5.5** ([58, p. 278]). *As long as  $n \leq 3$ , we can find a basis matrix  $A = [a_j]$  for any  $n$ -dimensional lattice  $\Gamma$  such that  $a_1$  is a shortest nonzero vector of  $\Gamma$ , and each  $a_i$  is any shortest choice such that  $a_1, \dots, a_i$  are linearly independent. If  $n = 4$ , then we have the equivalent statement if we replace “any” with “some”.*

**Definition 5.6** (Minkowski domain). We define  $\mathcal{M}_n$  to be the set of  $n$ -dimensional symmetric positive-definite quadratic forms that are Minkowski reduced.

5.2.1. *Polyhedral cones.* To give a geometric description of the Minkowski domain we define *polyhedral cones*. Properties of these cones are used throughout Schiemann’s symphony. First note that we can naturally embed  $n$ -dimensional symmetric quadratic forms  $q(x) = x^T Q x$ , where  $Q = (q_{ij})_{ij}$  in  $\mathbb{R}^{n(n+1)/2}$ . There are many ways to perform such an embedding. In three dimensions perhaps the most common embedding is

$$q \mapsto (q_{11}, q_{22}, q_{33}, q_{12}, q_{13}, q_{23}).$$

**Definition 5.7** (Polyhedral cone). Let  $A, B$  be (possibly empty) finite sets of nonzero  $n$ -dimensional vectors. Then a set of the form

$$P(A, B) := \{x \in \mathbb{R}^n : a \cdot x \geq 0, \quad b \cdot x > 0 \text{ for each } a \in A, \text{ and } b \in B\}$$

is a *polyhedral cone*. For  $a \in A$ , a set of the form  $\{x \in \mathbb{R}^n : a \cdot x = 0\}$  is a *supporting hyperplane*, similarly for  $b \in B$ . The dimension of  $P(A, B)$  is the dimension of the smallest vector space containing it. A polyhedral cone is *pointed* if it does not contain any lines.

It follows from the definition that the closure of a nonempty polyhedral cone

$$\overline{P(A, B)} = P(A \cup B, \emptyset).$$

Rational polyhedral cones (i.e., the elements of  $A$  and  $B$  are vectors with rational entries) are amenable to computers because they are easily stored and a computer can do exact calculations. Polyhedral cones have facets, faces, edges, and vertices.

**Definition 5.8** (Facets,  $j$ -faces, edges, and vertices). Let  $P$  be a  $k$ -dimensional polyhedral cone with closure  $\overline{P}$ . A *facet* of  $P$  is a  $(k - 1)$ -dimensional intersection of  $\overline{P}$  with a collection of its supporting hyperplanes. Similarly, a  *$j$ -face* is a  $j$ -dimensional intersection. A 1-face is an *edge*, and a 0-face is a *vertex*.

The following proposition collects properties of polyhedral cones. The proof is omitted as these properties can be readily checked using Definition 5.7; see also [28, Theorem 3.10].

**Proposition 5.9.** *Let  $P(A, B) \subset \mathbb{R}^n$  and  $P(C, D) \subset \mathbb{R}^m$  be polyhedral cones.*

- (1) *The Cartesian product is also a polyhedral cone, specifically*

$$P(A, B) \times P(C, D) = P(A \times \{0\} \cup \{0\} \times C, B \times \{0\} \cup \{0\} \times D).$$

- (2) *If  $n = m$ , then  $P(A, B) \cap P(C, D) = P(A \cup C, B \cup D)$ .*  
 (3) *If  $U$  is a closed set in  $\mathbb{R}^n$  and  $P(A, B) \neq \emptyset$ , then  $P(A, B) \subseteq U$  if and only if  $P(A \cup B, \emptyset) \subseteq U$ .*  
 (4) *If the polyhedral cone  $P(A, B)$  is pointed, then it has finitely many edges that are sets of the form  $k_i \mathbb{R}_{\geq 0}$ , for  $k_i \in \mathbb{R}^n$ , and*

$$(5.1) \quad \overline{P(A, B)} = \sum_{i=1}^r k_i \mathbb{R}_{\geq 0}.$$

*Consequently, the dimension of  $P(A, B)$  is the dimension of the span of  $\{k_i\}_{i=1}^r$ .*

- (5) *If  $k_i \mathbb{R}_{\geq 0}$  are the edges of  $P(A, \emptyset)$  and  $k'_j \mathbb{R}_{\geq 0}$  are those of  $P(C, \emptyset)$ , then the edges of  $P(A, \emptyset) \times P(C, \emptyset)$  are  $(k_i, 0) \mathbb{R}_{\geq 0}$  and  $(0, k'_j) \mathbb{R}_{\geq 0}$ .*

If no confusion shall arise, we may simply write the vectors  $k_i$  to denote the edges of a pointed polyhedral cone. To prove that every element of a pointed polyhedral cone has a certain property, it is often enough to check the property for its edges. For example, if  $P(A, B)$  is a pointed polyhedral cone and  $C$  is a convex set, then  $P(A, B) \subseteq C$  if and only if the edges  $k_i$  of  $P(A, B)$  lie in  $C$ .

5.2.2. *Minkowski's domain.* The next lemma describes  $\mathcal{M}_n$  for  $n \leq 4$  as a pointed polyhedral cone. We refer to a proof that is quite elegant but technical [5, pp. 257–258].

**Lemma 5.10.** *For any  $n \leq 4$ , a quadratic form  $q$  is a Minkowski reduced positive-definite form if and only if the following hold:*

- (1)  $0 < q_{11} \leq q_{22} \leq \dots \leq q_{nn}$ ;  
 (2)  $q(x) \geq q_{kk}$  for  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  with coordinates  $x_i \in \{-1, 0, 1\}$  for all  $i = 1, \dots, n$  and satisfying  $x_k = 1$  and  $x_j = 0$  for  $j > k$ .

Based on the preceding lemma, we give a precise description of  $\mathcal{M}_3$  as a polyhedral cone.

**Theorem 5.11** ( $\mathcal{M}_3$  as a polyhedral cone). *The set of symmetric positive-definite Minkowski reduced forms  $q(x) = x^T Qx$  in three dimensions is precisely those which*

satisfy

$$\begin{aligned}
& 0 < q_{11}, \\
& 0 \leq q_{22} - q_{11}, \\
& 0 \leq q_{33} - q_{22}, \\
& 0 \leq q_{11} - 2q_{12}, \\
& 0 \leq q_{11} + 2q_{12}, \\
(5.2) \quad & 0 \leq q_{11} + q_{22} + 2q_{12} - 2q_{13} - 2q_{23}, \\
& 0 \leq q_{11} + q_{22} - 2q_{12} - 2q_{13} + 2q_{23}, \\
& 0 \leq q_{11} + q_{22} - 2q_{12} + 2q_{13} - 2q_{23}, \\
& 0 \leq q_{11} + q_{22} + 2q_{12} + 2q_{13} + 2q_{23}, \\
& 0 \leq q_{11} - 2q_{13}, \\
& 0 \leq q_{11} + 2q_{13}, \\
& 0 \leq q_{22} - 2q_{23}, \\
& 0 \leq q_{22} + 2q_{23}.
\end{aligned}$$

*Proof.* This follows from Lemma 5.10.  $\square$

The Minkowski domain is always a pointed polyhedral cone. Unfortunately, in higher dimensions the number of inequalities *explode*; see Tammela's list [50, p. 20], [56].

**Theorem 5.12** ( $\mathcal{M}_n$  is a polyhedral cone [5, pp. 256–257]). *Only finitely many of the conditions for a quadratic form  $q$  to be in  $\mathcal{M}_n$  according to Proposition 5.4 are nonredundant. These conditions define  $\mathcal{M}_n$  as a pointed polyhedral cone.*

Although the set of Minkowski reduced forms may contain more than one representative for each equivalence class, the representatives are *almost* unique in the following sense.

**Theorem 5.13.** *Assume that  $q$  is in the interior of  $\mathcal{M}_n$  and has associated matrix  $Q$ . Then for any unimodular  $B$ , the quadratic form with associated matrix  $B^T Q B$  is in  $\mathcal{M}_n$  if and only if  $B$  is diagonal.*

*Proof.* Write  $B = [b_j]$ , where  $b_j$  are column vectors. Since the determinant of a unimodular matrix is  $\pm 1$ , the entries of the first column,  $b_1$ , satisfy  $\gcd((b_1)_1, \dots, (b_n)_1) = 1$ . Then,  $q(b_1) = q(Be_1) > q_{11}$  unless  $b_1 = \pm e_1$ . However, we know that  $q(b_1) = q_{11}$ , because by Minkowski reduction  $q(Be_1)$  is a smallest nonzero value. This means that the lower right  $(n-1) \times (n-1)$  matrix of  $B$ , say  $B'$ , has determinant  $\pm 1$  and is also unimodular. For this reason, the second vector  $b_2$  has  $\gcd((b_2)_2, \dots, (b_n)_2) = 1$ . Therefore  $q(b_2) > q_{22}$  unless  $b_2 = \pm e_2$ . The equality  $q(b_2) = q_{22}$  follows from the second formulation of Minkowski reduction in terms of lattices; write  $Q = A^T A$ ,  $A = [a_j]$  and note that  $AB = [\pm a_1 \quad Ab_2 \quad \cdots \quad Ab_n]$ . Since  $a_1, Ab_2$  are part of a basis of  $A\mathbb{Z}^n$ , the length of  $Ab_2$  must be equal to the length of  $a_2$ . We can repeat this process until each  $b_i = \pm e_i$ .  $\square$

**Corollary 5.14.** *The set  $\mathcal{M}_n^+ := \mathcal{M}_n \cap \{q : q_{1j} \geq 0 \text{ for each } 1 \leq j \leq n\}$  contains a representative of each positive-definite quadratic form. In the interior, the representatives are unique.*

This corollary is a direct consequence of Theorem 5.13.

5.2.3. *Successive minima.* To understand one of the most striking results about Minkowski reduction, we require *successive minima*.

**Definition 5.15.** Assume that  $Q$  is a positive-definite symmetric  $n \times n$  matrix. The  $i$ th successive minimum of  $Q$  is

$$\lambda_i(Q) := \min \left\{ q \left( x^{(i)} \right) : x^{(1)}, \dots, x^{(i)} \in \mathbb{Z}^n \text{ are linearly independent,} \right. \\ \left. \text{and } q \left( x^{(j)} \right) \leq q \left( x^{(j+1)} \right) \text{ for each } 1 \leq j \leq i-1 \right\}.$$

**Theorem 5.16** (van der Waerden, [58, Satz 7]). *For a Minkowski reduced positive-definite form  $q$ , we have*

$$q_{ii} \leq \Delta_i \lambda_i(Q) \quad \text{with} \quad \Delta_i := \max \left\{ 1, \left( \frac{5}{4} \right)^{i-4} \right\}.$$

It is conjectured in [50] that there is an even tighter bound:  $q_{ii} \leq i \lambda_i(Q)/4$  for  $i > 5$ . By Theorem 5.16, if two Minkowski reduced forms  $q^{(1)}, q^{(2)}$  are isospectral, then  $q_{22}^{(1)} = q_{22}^{(2)}$ . There is no guarantee however that  $q_{33}^{(1)} = q_{33}^{(2)}$ , as we observed in Section 4.

5.3. **Schiemann's reduction.** In his thesis [48], Schiemann introduced the following reduction, which he called the *Vorzeichennormalform*. We call it *Schiemann reduction*.

**Definition 5.17** (Schiemann reduction). A ternary positive-definite form  $f$  is said to be *Schiemann reduced* if the following hold:

- 1a)  $f$  is Minkowski reduced,
- 1b)  $f_{12} \geq 0$ , and  $f_{13} \geq 0$ ,
- 1c)  $2f_{23} > -f_{22}$ ;

and the following facet conditions hold:

- 2a)  $f_{12} = 0 \implies f_{23} \geq 0$ ,
- 2b)  $f_{13} = 0 \implies f_{23} \geq 0$ ;
- 3a)  $f_{11} = f_{22} \implies |f_{23}| \leq f_{13}$ ,
- 3b)  $f_{22} = f_{33} \implies f_{13} \leq f_{12}$ ;
- 4a)  $f_{11} + f_{22} - 2f_{12} - 2f_{13} + 2f_{23} = 0 \implies f_{11} - 2f_{13} - f_{12} \leq 0$ ,
- 4b)  $2f_{12} = f_{11} \implies f_{13} \leq 2f_{23}$ ,
- 4c)  $2f_{13} = f_{11} \implies f_{12} \leq 2f_{23}$ ,
- 4d)  $2f_{23} = f_{22} \implies f_{12} \leq 2f_{13}$ .

*Schiemann's domain*, denoted  $\mathcal{C}$ , is the set of Schiemann reduced forms embedded in  $\mathbb{R}^6$ . Schiemann's pointed polyhedral cone, denoted  $\mathcal{C}_p$  is the set of elements of  $\mathbb{R}^6$  that satisfy conditions 1a), 1b), and 1c).

Schiemann's pointed polyhedral cone,  $\mathcal{C}_p$  is a nonclosed pointed polyhedral cone that properly contains  $\mathcal{C}$ . It has several facets. The facet conditions 2), 3), and 4) determine which forms of  $\mathcal{C}_p$  are contained in  $\mathcal{C}$ . Unfortunately, the facet conditions cannot be expressed according to the definition of polyhedral cone, which is why we cannot express  $\mathcal{C}$  itself as a pointed polyhedral cone. The more well-known set of Eisenstein reduced forms (see [44, pp. 142–143]) contains precisely one representative of each positive-definite ternary quadratic form, and in fact, Schiemann used this theory to prove

**Theorem 5.18.** *Any positive-definite quadratic form in three variables has a unique representative that is Schiemann reduced.*

The proof is not very interesting, and it is long and technical [48]. What is interesting is that the Schiemann reduction of a positive-definite quadratic form is *unique*.

**Lemma 5.19.** *The closure of  $\mathcal{C}$  is a pointed polyhedral cone equal to the closure of  $M_3^+$ . Alternatively, it is given by the system*

$$(5.3) \quad \begin{cases} 0 \leq q_{11} \leq q_{22} \leq q_{33}, \\ 0 \leq 2q_{12} \leq q_{11} \text{ and } 0 \leq 2q_{13} \leq q_{11}, \\ -q_{22} \leq 2q_{23} \leq q_{22}, \\ 0 \leq q_{11} + q_{22} - 2q_{12} - 2q_{13} + 2q_{23}. \end{cases}$$

One can calculate the edges of  $\overline{\mathcal{C}}$ , for instance, via a package like `Polymake` [19] or by applying Section 5.5.1. The set of edges is denoted by

$$M = M_1 \cup M_2 \cup M_3,$$

$$(5.4) \quad M_1 := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad M_2 := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix} \right\}, \quad \text{and}$$

$$(5.5) \quad M_3 := \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & \pm 1 \\ 1 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \right\}.$$

Note that any element of  $\overline{\mathcal{C}}$  can be expressed as an  $\mathbb{R}_{\geq 0}$ -linear combination of elements in  $M$ . By definition of  $\mathcal{C}$ , we can for three sets of vectors  $\mathfrak{A}$ ,  $\mathfrak{B}$ , and  $\mathfrak{C}$  write

$$\mathcal{C} = \text{P}(\mathfrak{A}, \mathfrak{B}) \cap \{f \in \mathbb{R}^6 : \forall (c, d) \in \mathfrak{C} : (c \cdot f = 0 \Rightarrow d \cdot f \geq 0)\}.$$

The sets  $\mathfrak{A}$ ,  $\mathfrak{B}$  consist of vectors in  $\mathbb{R}^6$  that correspond to the conditions 1a), 1b), and 1c) in the definition of Schiemann reduced forms. The set of vectors in  $\mathfrak{C} \subseteq \mathbb{R}^6 \times \mathbb{R}^6$  produce the facet conditions. The elements of these three sets can be written out explicitly using Theorem 5.11 and Definition 5.17. There are Schiemann reduced forms that have representatives contained in  $\overline{\mathcal{C}} \setminus \mathcal{C}$ . The following algorithm provides a means to remove those superfluous representatives. To state the algorithm, for  $v \in \mathbb{R}^n$ , we define the sets

$$(5.6) \quad v^{\geq 0} := \{x \in \mathbb{R}^n : v \cdot x \geq 0\}, \quad v^\perp := \{x \in \mathbb{R}^n : v \cdot x = 0\}.$$

The superfluous forms are removed by checking if the first or second factor lies in a facet of  $\mathcal{C}$  which has facet conditions, and then removing forms that do not satisfy the facet conditions.

**Lemma 5.20.** *Assume that the polyhedral cone  $T \subseteq \overline{\mathcal{C}} \times \overline{\mathcal{C}}$ . We define a sequence of polyhedral cones  $T_i$  for  $i \in \mathbb{N}_0$  by*

$$T_0 := \left( \text{P}(\mathfrak{A}, \mathfrak{B}) \times \text{P}(\mathfrak{A}, \mathfrak{B}) \right) \cap T,$$

$$T_i := T_{i-1} \cap \bigcap_{(c,d) \in \mathfrak{C} : T_{i-1} \subseteq (c,0)^\perp} (d,0)^\geq \bigcap_{(c,d) \in \mathfrak{C} : T_{i-1} \subseteq (0,c)^\perp} (0,d)^\geq.$$

*The sequence  $T_i$  becomes stationary at some  $i_0$ . We define  $T_{\mathcal{C} \times \mathcal{C}} := T_{i_0}$ . Further,  $T \cap (\mathcal{C} \times \mathcal{C}) = \overline{T_{\mathcal{C} \times \mathcal{C}}}$  and  $T \cap (\mathcal{C} \times \mathcal{C}) \subseteq T_{\mathcal{C} \times \mathcal{C}} \subseteq T$ .*

*Proof.* Clearly, we have  $T_{i+1} \subseteq T_i$  for each  $i$ . Let  $C_1^i$  (resp.,  $C_2^i$ ) be the set of  $c$  belonging to a pair  $(c, d) \in \mathfrak{C}$ , such that  $T_i \subseteq (c, 0)^\perp$  (resp.,  $T_i \subseteq (0, c)^\perp$ ). Since  $T_i$  monotonically decreases,  $C_1^i$  and  $C_2^i$  monotonically increase. However  $\mathfrak{C}$  is finite, meaning that  $C_1^i$  and  $C_2^i$  converge and become stationary. It follows that  $T_i$  must become stationary.

We also show  $T \cap (\mathcal{C} \times \mathcal{C}) \subseteq T_{\mathcal{C} \times \mathcal{C}}$ . Take any  $(f, g) \in T \cap (\mathcal{C} \times \mathcal{C})$ . It is clear that  $(f, g) \in T_0$ . Now say  $(f, g) \in T_i$ . When calculating  $T_{i+1}$ , note that  $T_i \subseteq (c, 0)^\perp$  only if  $f \cdot c = 0$  at which point we already know  $f \cdot d \geq 0$  for  $(c, d) \in \mathfrak{C}$ , since  $(f, g) \in \mathcal{C} \times \mathcal{C}$ . We have the analogous situation for  $T_i \subseteq (0, c)^\perp$ . In either case, this implies  $(f, g) \in T_{i+1}$ .  $\square$

Note that we always have  $(T_{\mathcal{C} \times \mathcal{C}})_{\mathcal{C} \times \mathcal{C}} = T_{\mathcal{C} \times \mathcal{C}}$ .

**5.4. The minimal sets.** To identify quadratic forms that have identical representation numbers, Schiemann defined minimal sets which induce a transitive relation. Our treatment here is a slight generalization of Schiemann's.

**Definition 5.21** (Minimal set). Let  $\mathcal{Q}$  be a finite set of symmetric quadratic forms in  $n$  dimensions. First define the transitive relation

$$x \preceq_{\mathcal{Q}} y \Leftrightarrow q(x) \leq q(y) \text{ for each } q \in \mathcal{Q}.$$

The set of minimal vectors with respect to a set  $X \subseteq \mathbb{Z}^n$  is defined

$$\text{MIN}_{\mathcal{Q}}(X) := \{x \in X : y \not\preceq_{\mathcal{Q}} x \text{ for all } y \in X \setminus \{x\}\}.$$

In case  $n = 3$ , and  $\mathcal{Q}$  is the set  $M$  of edges of  $\overline{\mathcal{C}}$ , listed in (5.4) and (5.5), we simply write  $\text{MIN}$  and  $\preceq$ .

Note that  $\text{MIN}_{\mathcal{Q}}(X)$  consists of all  $x \in X$  for which no other element in  $X$  is smaller with respect to  $\preceq_{\mathcal{Q}}$ . As an example, note that

$$\text{MIN}(\mathbb{Z}_*^3) = \{(1, 0, 0)\}.$$

**Definition 5.22.** For a set  $\mathcal{Q}$  of  $n$ -dimensional quadratic forms,  $q_1, \dots, q_N$ , recall its positive hull,

$$\text{pos}(\mathcal{Q}) := \left\{ \sum_{i=1}^N \lambda_i q_i : \lambda_i \geq 0, \quad q_i \in \mathcal{Q} \right\}.$$

The set  $\mathcal{Q}$  is *regular* if

- (i) all its elements are rational and positive semidefinite,
- (ii) the positive hull of  $\mathcal{Q}$  contains the identity matrix, and
- (iii) the embedding of  $\mathcal{Q}$  into  $\mathbb{R}^{n(n+1)/2}$  spans  $\mathbb{R}^{n(n+1)/2}$ .

One can check that the edges of  $\overline{\mathcal{C}}$  form a regular set. Next we characterize  $\text{MIN}_{\mathcal{Q}}$  and describe its most important properties.

**Lemma 5.23.** *We have  $x \preceq_{\mathcal{Q}} y$  if and only if  $q(x) \leq q(y)$  for each  $q \in \text{pos}(\mathcal{Q})$ . Assume that  $\mathcal{Q}$  is a regular set of quadratic forms, and that  $\text{MIN}_{\mathcal{Q}}(X) \neq \emptyset$ . The minimal set  $\text{MIN}_{\mathcal{Q}}(X)$  is the unique smallest set in  $X$  such that for any  $z \in X$  there is a  $y \in \text{MIN}_{\mathcal{Q}}(X)$  with  $y \preceq_{\mathcal{Q}} z$ .*

*Proof.* The first statement can be verified from the definitions. To prove the second statement, we proceed by contradiction. Assume that for some  $z \in X$ , each  $y \in \text{MIN}_{\mathcal{Q}}(X)$  has  $y \not\preceq_{\mathcal{Q}} z$ . Then  $z \notin \text{MIN}_{\mathcal{Q}}(X)$ , meaning that there is an element  $z^{(1)} \in X \setminus \{z\}$  with  $z^{(1)} \preceq_{\mathcal{Q}} z$ . If  $z_1 \in \text{MIN}_{\mathcal{Q}}(X)$ , then we arrive at a contradiction.

Otherwise, we construct a sequence  $z^{(i)}$  inductively by letting  $z^{(i+1)}$  be an element in  $X \setminus \{z, z^{(1)}, \dots, z^{(i)}\}$  such that  $z^{(i+1)} \preceq_{\mathcal{Q}} z^{(i)}$ . By transitivity of  $\preceq_{\mathcal{Q}}$ , we would arrive at a contradiction if at any point  $z^{(i)} \in \text{MIN}_{\mathcal{Q}}(X)$ . Since  $I \in \text{pos}(\mathcal{Q})$ , by the first statement of the lemma, we know that  $\|z^{(i)}\| \leq \|z^{(i-1)}\| \leq \dots \leq \|z\|$ . Each  $z^{(i)} \in \mathbb{Z}^n$  is distinct, therefore this sequence of norms must become stationary. This will eventually contradict  $z^{(i+1)} \in X \setminus \{z, z^{(1)}, \dots, z^{(i)}\}$ , because there are only finitely many vectors in  $\mathbb{Z}^n$  of any given fixed length. To see that  $\text{MIN}_{\mathcal{Q}}(X)$  is the unique smallest set with this property, note that if  $z \in \text{MIN}_{\mathcal{Q}}(X)$ , then the only element  $y \in X$  with  $y \preceq_{\mathcal{Q}} z$  is  $z$  itself.  $\square$

**Proposition 5.24.** *Let  $\mathcal{Q}$  be a regular set. We have the following:*

- (i)  $x \preceq_{\mathcal{Q}} y$  implies  $\|x\| \leq \|y\|$ .
- (ii) If  $x \preceq_{\mathcal{Q}} y \preceq_{\mathcal{Q}} x$ , then  $x = \pm y$ .
- (iii)  $\text{MIN}_{\mathcal{Q}}(X)$  is finite.
- (iv) If  $0 \notin X \neq \emptyset$  and for any shortest vector  $u$  of  $X$  we have  $-u \notin X$ , then  $\text{MIN}_{\mathcal{Q}}(X) \neq \emptyset$ .
- (v)  $\emptyset \neq X \subseteq \mathbb{Z}_*^n$  implies  $\text{MIN}_{\mathcal{Q}}(X) \neq \emptyset$ .

*Proof.*

- (i) Follows from Lemma 5.23 and the fact that  $I \in \text{pos}(\mathcal{Q})$ .
- (ii) We have  $q(x) = q(y)$  for all  $q \in \mathcal{Q}$ . This can be written

$$0 = q(x) - q(y) = \sum_i q_{ii}(x_i^2 - y_i^2) + \sum_{i < j} 2q_{ij}(x_i x_j - y_i y_j) = 0.$$

The fact that  $\mathcal{Q}$  spans  $\mathbb{R}^{n(n+1)/2}$  implies that  $x_i x_j = y_i y_j$  for each  $i, j$ . Then  $y_i = \pm x_i$  for each  $i$ . In fact, it must be either  $y_i = x_i$  for all  $i$  or  $y_i = -x_i$  for all  $i$ . To see this, assume  $x_1, x_2 \neq 0$ , and  $y_1 = x_1, y_2 = -x_2$ . This would contradict  $x_1 x_2 = y_1 y_2$ .

(iii) If  $\text{MIN}_{\mathcal{Q}}(X)$  were infinite, then there would be a sequence  $x^{(i)} \in \text{MIN}_{\mathcal{Q}}(X)$  of distinct elements such that  $x^{(1)} \not\preceq_{\mathcal{Q}} x^{(2)} \not\preceq_{\mathcal{Q}} x^{(3)} \dots$ . We can find a subsequence  $x^{(i_j)}$  such that there is a  $q \in \mathcal{Q}$  with  $q(x^{(i_1)}) > q(x^{(i_2)}) > q(x^{(i_3)}) > \dots$ . The elements of  $\mathcal{Q}$  are semipositive and rational. Consequently, for each  $q \in \mathcal{Q}$ ,  $q(\mathbb{Z}^n)$  is a discrete set. So the sequence must become stationary, a contradiction.

(iv) Denote the distinct vectors in  $X$  that share the shortest length in  $X$  by  $u^{(1)}, \dots, u^{(k)} \in X$ . Assume by contradiction that no  $u^{(i)}$  is in  $\text{MIN}_{\mathcal{Q}}(X)$ . Then for each  $u^{(i)}$  there is an  $x \in X \setminus \{u^{(i)}\}$  such that  $x \preceq_{\mathcal{Q}} u^{(i)}$ . By statement (i),  $x = u^{(j)}$  for some  $j \neq i$ . If  $k = 1$ , we directly have a contradiction. If  $k > 1$ , then we could find two vectors  $u^{(i)} \neq u^{(j)}$  such that  $u^{(j)} \preceq u^{(i)} \preceq u^{(j)}$  as a consequence of the set of  $u^{(i)}$  being finite. By (ii) this implies  $u^{(j)} = \pm u^{(i)}$  which is a contradiction to the assumption in (iv).

- (v) Follows directly from (iv) and the definition of  $\mathbb{Z}_*^n$ .  $\square$

5.4.1. *Calculating  $\text{MIN}_{\mathcal{Q}}(X)$ .* It helps to know that any minimal set is finite given a regular set  $\mathcal{Q}$ , but how do we find its elements? The following lemma was used by Schiemann to reduce the calculation of  $\text{MIN}$  under certain conditions.

**Lemma 5.25.** *Let  $\emptyset \neq Y \subseteq X \subseteq \mathbb{Z}_*^n$ , and let  $\mathcal{Q}$  be regular. If*

$$W \supseteq \{x \in X : x \not\preceq_{\mathcal{Q}} y \quad \forall y \in Y\} \cup Y,$$

*then  $\text{MIN}_{\mathcal{Q}}(X) = \text{MIN}_{\mathcal{Q}}(X \cap W)$ .*

*Proof.* We first show  $\text{MIN}_{\mathcal{Q}}(X) \subseteq \text{MIN}_{\mathcal{Q}}(X \cap W)$ , and it is not hard to see that it suffices to prove  $\text{MIN}_{\mathcal{Q}}(X) \subseteq X \cap W$ . Take  $z \in \text{MIN}_{\mathcal{Q}}(X) \subseteq X$ . If  $z \not\preceq_{\mathcal{Q}} y$  for all  $y \in Y$ , then by the assumption on  $W$ ,  $z \in W$ . If for some  $y \in Y$  we have  $z \succeq_{\mathcal{Q}} y$ , then by construction of  $\text{MIN}_{\mathcal{Q}}(X)$ ,  $y = z \in Y$ , meaning that also in this case we have  $z \in W$ .

Secondly we prove  $\text{MIN}_{\mathcal{Q}}(X) \supseteq \text{MIN}_{\mathcal{Q}}(X \cap W)$ . Let  $z \in \text{MIN}_{\mathcal{Q}}(X \cap W)$  and take any  $x \in X$  such that  $x \preceq_{\mathcal{Q}} z$ . We show that  $x$  is necessarily equal to  $z$ , since then  $z \in \text{MIN}_{\mathcal{Q}}(X)$  by definition. If  $x \in X \cap W$ , then since  $z \in \text{MIN}_{\mathcal{Q}}(X \cap W)$ , we have  $z = x$ . If  $x \in X \setminus W$ , then it follows that there is a  $y \in Y$  such that  $y \preceq_{\mathcal{Q}} x$ . By transitivity,  $y \preceq_{\mathcal{Q}} z$ . Now since  $Y \subseteq X \cap W$ , this implies  $y = z$ . As a consequence,  $z \preceq_{\mathcal{Q}} x \preceq_{\mathcal{Q}} z$ , which by Proposition 5.24 means  $z = x$ .  $\square$

To determine  $\text{MIN}_{\mathcal{Q}}(X)$ , we want to find appropriate, finite  $Y$  and  $W$  as above. We present a new result for this purpose. First, let  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  denote the projection onto the last  $n - 1$  coordinates.

**Theorem 5.26.** *Assume that  $\mathcal{E}$  is a regular set of  $n$ -dimensional quadratic forms. Let  $\mathcal{E}'$  be the set of noninvertible matrices of  $\mathcal{E}$ , with the first row and column deleted. Let  $\mathcal{E}^\times$  be the invertible matrices of  $\mathcal{E}$ . If  $\mathcal{E}'$  is regular, then for  $a \in \mathbb{Z}$  and  $X \subseteq \mathbb{Z}^n$  such that  $\text{MIN}_{\mathcal{E}'}(\pi(X)) \neq \emptyset$ , we define  $Y(a) := \{(a, z) : z \in \text{MIN}_{\mathcal{E}'}(\pi(X))\}$ . Then,*

$$\begin{aligned} W(a) &:= \{x \in X : t \not\preceq_{\mathcal{E}} x \text{ for each } t \in Y(a)\} \cup Y(a) \\ &\subseteq \{x \in \mathbb{Z}^n : \lambda_{\min} \|x\|^2 < \lambda_{\max} \max_{t \in Y(a)} \{\|t\|^2\}\} \cup Y(a), \end{aligned}$$

where  $\lambda_{\min}$ ,  $\lambda_{\max}$  are the smallest and largest eigenvalues, respectively, among elements of  $\mathcal{E}^\times$ . In particular,  $W(a)$  is a finite set.

*Proof.* Take any  $x \in W(a) \setminus Y(a)$  and note that if  $f \in \mathcal{E}$  is noninvertible, and  $f'$  is  $f$  with first row and column removed, then  $f(x) = f'(\pi(x))$ . By assumption and Proposition 5.23 there is an element  $z$  of  $\text{MIN}_{\mathcal{E}'}(\pi(X))$  such that  $z \preceq_{\mathcal{E}'} \pi(x)$ . Therefore, if  $(a, z) \not\preceq_{\mathcal{E}} x$  we would require  $g(a, z) > g(x)$  for some  $g \in \mathcal{E}^\times$ . It follows that  $\lambda_{\min} \|x\|^2 < g(a, z) \leq \max\{g(t)\} \leq \lambda_{\max} \max\{\|t\|^2\}$ , where the maximum is taken over  $t \in Y(a)$ . This proves the inclusion, since  $\lambda_{\min} > 0$ ,  $\lambda_{\max}/\lambda_{\min} \geq 1$ , and we end by noting that both  $\mathcal{E}^\times$  and  $Y(a)$  are finite.  $\square$

If  $\mathcal{E} = M$  is the set of edges of  $\overline{\mathcal{C}}$ , then the conditions of the theorem are satisfied. To apply the theorem in Schiemann's symphony, let

$$(5.7) \quad \begin{aligned} X_0 &:= \mathbb{Z}_*^3 \setminus e_1\mathbb{Z}, & X_1 &:= \mathbb{Z}_*^3 \setminus (e_1\mathbb{Z} + e_2\mathbb{Z}), \\ X_2 &:= \mathbb{Z}_*^3 \setminus ((e_1\mathbb{Z} + e_2\mathbb{Z}) \cup (e_1\mathbb{Z} + e_3\mathbb{Z})). \end{aligned}$$

In Theorem 5.26, the corresponding

$$(5.8) \quad Y_0(a) = \{(a, 1, 0)\}, \quad Y_1(a) = \{(a, 0, 1)\}, \quad Y_2(a) = \{(a, 1, 1), (a, -1, 1)\},$$

$$(5.9) \quad W_i(a) \subseteq \left\{x : \|x\| < 2\sqrt{2(a^2 + 2)}\right\}.$$

Schiemann showed similar, sharper bounds through less general means. We could find sharper bounds by determining  $\max\{g(t)\}$  as in the proof above, instead of estimating using  $\lambda_{\max}$ . In Schiemann's symphony, we calculate the minimal sets of  $X_i \setminus Z$  for finite sets  $Z$ . Since  $Z$  is finite, we can always find an appropriate  $a$  such that  $Y_i(a) \subseteq X \setminus Z$ , and in this way apply Lemma 5.25.

Although we work in three dimensions here, the statements can be generalized to any dimension without problems, essentially by replacing  $\overline{\mathcal{C}}$  by the closure of  $\mathcal{M}_n^+$ .

**Definition 5.27.** Let  $x^{(1)}, \dots, x^{(k)} \in X \subseteq \mathbb{Z}_*^3$  be distinct elements. We define

$$\begin{aligned} \mathcal{S} \left( X, \{x^{(i)}\}_{i=1}^k \right) \\ := \left\{ f \in \overline{\mathcal{C}} : f \left( x^{(j)} \right) = \min \left\{ f \left( X \setminus \left\{ x^{(1)}, \dots, x^{(j-1)} \right\} \right) \right\} \quad \forall j = 1, \dots, k \right\}. \end{aligned}$$

We set  $\mathcal{S}(X, \emptyset) := \overline{\mathcal{C}}$ .

Above,  $\mathcal{S} \left( X, \{x^{(i)}\}_{i=1}^k \right)$  is the set of quadratic forms in  $\overline{\mathcal{C}}$  for which  $x^{(1)}, \dots, x^{(k)}$  yield the successively smallest values of the forms evaluated at elements of  $X$ . To show the connection to the representation numbers, let  $\{x^{(i)}, x\}_{i=1}^k$  denote the sequence  $x^{(1)}, \dots, x^{(k)}, x$  in this order. If  $f \in \mathcal{S}(X, \{x^{(i)}, x\}_{i=1}^k)$ , then

$$\begin{aligned} (5.10) \quad f(x) &= \min f \left( X \setminus \left\{ x^{(1)}, \dots, x^{(k)} \right\} \right) \\ &= \min \left\{ t_0 \in \mathbb{R}_{\geq 0} : \sum_{0 \leq t \leq t_0} \mathcal{R}_X(f, t) \geq k + 1 \right\}. \end{aligned}$$

The sets  $\mathcal{S} \left( X, \{x^{(i)}\}_{i=1}^k \right)$  are pointed polyhedral cones contained in the closure of Schiemann's domain.

**Lemma 5.28.** *Let  $X \subseteq \mathbb{Z}_*^3$  and  $x^{(1)}, \dots, x^{(k)} \in X$  be distinct. Assume that  $\text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k) \neq \emptyset$ . Then  $\mathcal{S}(X, \{x^{(i)}\}_{i=1}^k)$  is a pointed polyhedral cone, specifically*

$$\begin{aligned} \mathcal{S}(X, \{x^{(i)}\}_{i=1}^k) \\ = \left\{ f \in \overline{\mathcal{C}} : f(x^{(1)}) \leq \dots \leq f(x^{(k)}) \text{ and } f(x^{(k)}) \leq f(x) \quad \forall x \in \text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k) \right\}. \end{aligned}$$

To see how the sets  $\mathcal{S} \left( X, \{x^{(i)}\}_{i=1}^k \right)$  are realized as polyhedral cones in a computer, we proceed as follows. For fixed  $x, y \in \mathbb{R}^n$  and any quadratic form  $f$ , we have

$$f(y) \geq f(x) \iff \sum_i f_{ii}(y_i^2 - x_i^2) + \sum_{i < j} f_{ij}(2y_i y_j - 2x_i x_j) \geq 0.$$

The right-hand side is a linear inequality for the values of  $f_{ij}$ . Then, given  $x^{(i)}$  and  $\text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k)$ , which we calculated in the last section, we note that  $\text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k)$  is a finite set, and it is therefore possible to explicitly write out all the linear inequalities defining the set  $\mathcal{S} \left( X, \{x^{(i)}\}_{i=1}^k \right)$ .

**Lemma 5.29.** *Let  $X \subseteq \mathbb{Z}_*^3$  and  $x^{(1)}, \dots, x^{(k)} \in X$  be distinct. If  $X \setminus \{x^{(i)}\}_{i=1}^k \neq \emptyset$ , then*

$$\mathcal{S}(X, \{x^{(i)}\}_{i=1}^k) = \bigcup_{y \in \text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k)} \mathcal{S}(X, \{x^{(i)}, y\}_{i=1}^k).$$

*Proof.*

“ $\supseteq$ ”: This inclusion is immediate since  $\mathcal{S}(X, \{x^{(i)}, y\}_{i=1}^k) \subseteq \mathcal{S}(X, \{x^{(i)}\}_{i=1}^k)$  by definition.

“ $\subseteq$ ”: Let  $f \in \mathcal{S}(X, \{x^{(i)}\}_{i=1}^k)$  and define

$$Y_f := \left\{ y \in X \setminus \{x^{(i)}\}_{i=1}^k : f(y) = \min f(X \setminus \{x^{(i)}\}_{i=1}^k) \right\}.$$

The set  $Y_f$  is nonempty by the discreteness of  $f(X \setminus \{x^{(i)}\}_{i=1}^k)$ . By Proposition 5.24(v),  $\text{MIN}(Y_f) \neq \emptyset$ . Fix some  $y \in \text{MIN}(Y_f)$ . We proceed to show that  $y \in \text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k)$ . In that case, we are done since by definition we have  $f \in \mathcal{S}(X, \{x^{(i)}, y\}_{i=1}^k)$ . We consider the following decomposition,

$$X \setminus \{x^{(i)}\}_{i=1}^k = Y_f \cup \left( (X \setminus \{x^{(i)}\}_{i=1}^k) \setminus Y_f \right).$$

To see that  $y \in \text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k)$ , it suffices to show that  $x \not\leq y$  for any  $x \in X \setminus \{x^{(i)}, y\}_{i=1}^k$ . Since  $y \in \text{MIN}(Y_f)$ , we know that  $x \not\leq y$  for  $x \in Y_f \setminus \{y\}$ . Now consider  $x \in (X \setminus \{x^{(i)}\}_{i=1}^k) \setminus Y_f$ , and note that  $x \neq y$ . If  $(X \setminus \{x^{(i)}\}_{i=1}^k) \setminus Y_f = \emptyset$ , then we are done. Otherwise, for such an  $x$ ,  $f(x) > \min f(X \setminus \{x^{(i)}\}_{i=1}^k)$ , since  $x \notin Y_f$ . This means  $f(x) > f(y)$ , because  $y \in Y_f$  which implies  $x \not\leq y$ , completing the proof.  $\square$

**5.5. Schiemann’s symphony.** We are almost ready to perform Schiemann’s symphony, the algorithm that proves  $b_3 = 1$ ; three-dimensional flat tori are determined by their spectra. For this we define the *duets*, that are pairs of isospectral Schiemann reduced forms

$$\mathcal{D} := \{(f, g) \in \mathcal{C} \times \mathcal{C} : f \text{ and } g \text{ are isospectral}\}.$$

The goal of Schiemann’s symphony is to prove that all duets are in fact *solos*, meaning the two forms are identical. We therefore aim to prove that  $\mathcal{D} \subset \mathcal{S}$ , where

$$\mathcal{S} := \{(f, f) : f \in \mathbb{R}^6\}.$$

**Definition 5.30** (Coverings and refinements). For  $A \subseteq \mathbb{R}^n$ ,  $P$  is a *covering* of  $A$  if

$$A \subseteq \bigcup_{U \in P} U.$$

If  $P'$  is also a covering of  $A$ , it is a *refinement* if each element of  $P'$  is contained in some  $U \in P$ .

To prove that  $\mathcal{D} \subset \mathcal{S}$ , from which it immediately follows that  $b_3 = 1$ , we use a computer algorithm to calculate polyhedral cones that cover  $\mathcal{D}$  and iteratively refine these coverings to prove that eventually the coverings are all contained in  $\mathcal{S}$ .

**Definition 5.31** (In tune). A polyhedral cone  $T$  is *in tune with respect to*  $\Lambda(T)$ ,  $k = k(T)$  and sequences  $x^{(1)}, \dots, x^{(k)}, y^{(1)}, \dots, y^{(k)}$  if the following properties hold.

**P1.**  $T \subseteq \overline{\mathcal{C}} \times \overline{\mathcal{C}}$  is a polyhedral cone and  $T_{\mathcal{C} \times \mathcal{C}} = T$  as in Lemma 5.20.

**P2.**  $\Lambda := \Lambda(T)$  is the largest of the three nested subsets of  $\mathbb{R}^3$ ,

$$e_1\mathbb{Z} \subseteq e_1\mathbb{Z} + e_2\mathbb{Z} \subseteq (e_1\mathbb{Z} + e_2\mathbb{Z}) \cup (e_1\mathbb{Z} + e_3\mathbb{Z})$$

such that  $f|_{\Lambda(T)} = g|_{\Lambda(T)}$  for all  $(f, g) \in T$ .

**P3.** We have  $x^{(i)}, y^{(i)} \in \mathbb{Z}_*^3 \setminus \Lambda$  and

$$T \subseteq \mathcal{S}(\mathbb{Z}_*^n \setminus \Lambda, \{x^{(i)}\}_{i=1}^k) \times \mathcal{S}(\mathbb{Z}_*^n \setminus \Lambda, \{y^{(i)}\}_{i=1}^k),$$

$$T \subseteq \{(f, g) \in \overline{\mathcal{C}} \times \overline{\mathcal{C}} : f(x^{(i)}) = g(y^{(i)}) \quad \forall i = 1, \dots, k\}.$$

A covering  $\mathcal{T}$  of  $\mathcal{D}$  is *in tune* if each  $T \in \mathcal{T}$  is in tune.

Aiming to define a sequence of coverings of  $\mathcal{D}$ , we start with

$$(5.11) \quad \mathcal{T}_0 := \{(f, g) \in \mathcal{C}_p \times \mathcal{C}_p : f_{11} = g_{11}\}_{\mathcal{C} \times \mathcal{C}}.$$

Denote by  $T$  the single element of  $\mathcal{T}_0$ . Observe that  $\mathcal{D} \subseteq T$ , by Minkowski reduction and Corollary 2.15. The set  $T$  satisfies properties P1, P2, and P3 by setting  $k = 1$ ,  $x^{(1)} = y^{(1)} = e_1$ , and  $\Lambda = e_1\mathbb{Z}$ . For an in tune covering  $\mathcal{T}$  of  $\mathcal{D}$ , we define a refinement as follows.

**Definition 5.32.** Let  $T$  be an in tune polyhedral cone with corresponding  $\Lambda$ ,  $k$ ,  $x^{(i)}$ , and  $y^{(i)}$  as in Definition 5.31. We define its *refinement*  $\mathcal{R}_T$  according to the following two cases.

**Case 1.**  $T \subseteq \mathcal{S}$ : Let  $\mathcal{R}_T := \{T\}$ . The set  $T \in \mathcal{R}_T$  is in tune with  $\Lambda$ ,  $k$ ,  $x^{(i)}$ ,  $y^{(i)}$ .

**Case 2.**  $T \not\subseteq \mathcal{S}$ : Write  $\mathfrak{N}(\Lambda, \{z^{(i)}\}) := \text{MIN}((\mathbb{Z}_*^n \setminus \Lambda) \setminus \{z^{(i)}\}_{i=1}^k)$  for a sequence  $z^{(i)}$ . Let for each  $x \in \mathfrak{N}(\Lambda, \{x^{(i)}\})$  and  $y \in \mathfrak{N}(\Lambda, \{y^{(i)}\})$ ,

$$S_{xy} := T \cap (\mathcal{S}(\mathbb{Z}_*^n \setminus \Lambda, \{x^{(i)}, x\}_{i=1}^k) \times \mathcal{S}(\mathbb{Z}_*^n \setminus \Lambda, \{y^{(i)}, y\}_{i=1}^k)),$$

$$T_{xy} := [S_{xy} \cap \{(f, g) \in \overline{\mathcal{C}} \times \overline{\mathcal{C}} : f(x) = g(y)\}]_{\mathcal{C} \times \mathcal{C}}.$$

We define

$$\mathcal{R}_T := \bigcup_{x \in \mathfrak{N}(\Lambda, \{x^{(i)}\}), y \in \mathfrak{N}(\Lambda, \{y^{(i)}\})} \{T_{xy}\}.$$

Each  $T_{xy}$  is in tune with variables as follows. Let  $\Lambda_{xy} = \Lambda(T_{xy})$  be maximal with  $f|_{\Lambda_{xy}} = g|_{\Lambda_{xy}}$  for all  $(f, g) \in T_{xy}$ . Let  $k(T_{xy}) = k(T) + 1$  and  $x^{(k+1)} = x, y^{(k+1)} = y$ . If  $\Lambda_{xy} \neq \Lambda$ , then let  $0 \leq r \leq k(T) + 1$  be maximal with  $\#(\{x^{(i)}\}_{i=1}^r \setminus \Lambda_{xy}) = \#(\{y^{(i)}\}_{i=1}^r \setminus \Lambda_{xy})$ , and instead set  $k(T_{xy}) = r$ . In this case,  $T_{xy}$  is in tune with  $x^{(i)}, y^{(i)}$  for  $i = 1, \dots, r$ .

Observe that  $T_{xy} \subseteq S_{xy}$ , and  $(f, g) \in T_{xy}$  implies  $(f, g) \in T$ , and  $f(x) = g(y)$ . The sets  $S_{xy}$  decompose  $T$  into possibly overlapping subsets whose union is equal to  $T$ . Passing from  $S_{xy}$  to  $T_{xy}$ , multiple representatives of the same equivalence class are removed according to Lemma 5.20. We are now ready to define the sequence  $\mathcal{T}_i$  of coverings of  $\mathcal{D}$ .

**Definition 5.33.** If  $\mathcal{T}$  is a covering of  $\mathcal{D}$ , then we define its refinement as

$$\mathcal{T}' := \bigcup_{T \in \mathcal{T}} \mathcal{R}_T.$$

With  $\mathcal{T}_0$  as in equation (5.11), we define the sequence  $\mathcal{T}_i$  by  $\mathcal{T}_{i+1} = \mathcal{T}'_i$  for each  $i \geq 0$ .

We must now argue that  $\mathcal{T}'$  is actually a refinement of a given covering  $\mathcal{T}$ . This is done via the following proposition.

**Proposition 5.34.**  $\mathcal{T}_i$  is a sequence of coverings of  $\mathcal{D}$ , and each iteration is a refinement of the previous one.

*Proof.* Assume that  $\mathcal{T} = \mathcal{T}_i$  for some  $i$  is a covering of  $\mathcal{D}$ . We show that  $\mathcal{T}' = \mathcal{T}_{i+1}$  is a refinement. Fix some arbitrary  $T \in \mathcal{T}$ . Let us first note that by Lemma 5.29,

$$\bigcup_{x \in \mathfrak{N}(\Lambda, \{x^{(i)}\}), y \in \mathfrak{N}(\Lambda, \{y^{(i)}\})} \mathcal{S}(\mathbb{Z}_*^3 \setminus \Lambda, \{x^{(i)}, x\}_{i=1}^k) \times \mathcal{S}(\mathbb{Z}_*^3 \setminus \Lambda, \{y^{(i)}, y\}_{i=1}^k)$$

is equal to  $\mathcal{S}(\mathbb{Z}_*^3 \setminus \Lambda, \{x^{(i)}\}_{i=1}^k) \times \mathcal{S}(\mathbb{Z}_*^3 \setminus \Lambda, \{y^{(i)}\}_{i=1}^k)$ . By property P3 of  $T$ , we have

$$\bigcup_{x \in \mathfrak{N}(\Lambda, \{x^{(i)}\}), y \in \mathfrak{N}(\Lambda, \{y^{(i)}\})} S_{xy} = T \cap \mathcal{S}(\mathbb{Z}_*^3 \setminus \Lambda, \{x^{(i)}\}_{i=1}^k) \times \mathcal{S}(\mathbb{Z}_*^3 \setminus \Lambda, \{y^{(i)}\}_{i=1}^k) = T.$$

Consider  $(f, g) \in S_{xy} \cap \mathcal{D}$ . We have  $\mathcal{R}(f, t) = \mathcal{R}(g, t)$  for all  $t \in \mathbb{R}_{\geq 0}$ . Since  $f|_{\Lambda} = g|_{\Lambda}$ ,  $\mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda}(f, t) = \mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda}(g, t)$  for all  $t \in \mathbb{R}_0^+$ . By equation (5.10) and the definition of  $S_{xy}$ , we have

$$\begin{aligned} f(x) &= \min \left\{ t_0 \in \mathbb{R}_0^+ : \sum_{0 \leq t \leq t_0} \mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda}(f, t) \geq k + 1 \right\} \\ &= \min \left\{ t_0 \in \mathbb{R}_0^+ : \sum_{0 \leq t \leq t_0} \mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda}(g, t) \geq k + 1 \right\} = g(y). \end{aligned}$$

This implies  $(f, g) \in T_{xy}$  by Lemma 5.20, and  $S_{xy} \cap \mathcal{D} \subseteq T_{xy}$ . Consequently,

$$T \supseteq \bigcup_{x \in \mathfrak{N}(\Lambda, \{x^{(i)}\}), y \in \mathfrak{N}(\Lambda, \{y^{(i)}\})} T_{xy} \supseteq \bigcup_{x \in \mathfrak{N}(\Lambda, \{x^{(i)}\}), y \in \mathfrak{N}(\Lambda, \{y^{(i)}\})} (S_{xy} \cap \mathcal{D}) = T \cap \mathcal{D}.$$

By the fact that  $\mathcal{T}$  is a covering of  $\mathcal{D}$ , we have

$$\bigcup_{T \in \mathcal{T}} T \supseteq \bigcup_{T \in \mathcal{T}} \left( \bigcup_{T_{xy} \in \mathcal{D}_T} T_{xy} \right) \supseteq \bigcup_{T \in \mathcal{T}} (T \cap \mathcal{D}) = \mathcal{D} \cap \bigcup_{T \in \mathcal{T}} T = \mathcal{D}. \quad \square$$

If  $\mathcal{T}_i$  becomes stationary, by Definitions 5.32 and 5.33 each  $T \in \mathcal{T}_i$  lies in  $\mathcal{S}$ , and we have  $\mathcal{D} \subseteq \mathcal{S}$ . We cannot a priori conclude that this will occur, but we can show that the pairs of the coverings become more and more isospectral in the following sense.

**Lemma 5.35.** *We have*

$$(5.12) \quad \bigcap_{i \in \mathbb{N}} \bigcup_{T \in \mathcal{T}_i} T \cap (\mathcal{C} \times \mathcal{C}) = \mathcal{D}.$$

*Sketch of the proof.* By definition,  $\mathcal{D}$  is contained in the left side of (5.12). So, assuming that  $(f, g)$  is in the left side of (5.12), we wish to show that  $(f, g) \in \mathcal{D}$ . There is a sequence of  $T_i \in \mathcal{T}_i$  such that  $(f, g) \in T_i$  and  $T_i \subseteq T_{i+1}$  for each  $i$  by construction. Since  $\Lambda(T_i)$  increases monotonically and takes three values, it becomes stationary and equals, say,  $\Lambda$ . Clearly, for  $x \in \Lambda$ , we have  $f(x) = g(x)$ . Next we construct a bijection  $\phi$  from  $\mathbb{Z}_*^3 \setminus \Lambda$  to itself, such that  $f(x) = g(\phi(x))$  for each  $x \in \mathbb{Z}_*^3 \setminus \Lambda$ . The existence of  $\phi$  proves that  $f$  and  $g$  are isospectral. There are sequences  $x^{(i)}, y^{(i)} \in \mathbb{Z}_*^3 \setminus \Lambda$  for which  $T_i$  are in tune with, meaning  $f(x^{(k)}) = \min f((\mathbb{Z}_*^3 \setminus \Lambda) \setminus \{x^{(i)}\}_{i=1}^{k-1})$  for each  $k$ , and analogously for  $g(y^{(k)})$ . Note that  $\mathbb{Z}_*^3 \setminus \Lambda = \{x^{(i)}\}_{i=1}^{\infty} = \{y^{(i)}\}_{i=1}^{\infty}$ , or we would arrive at a contradiction by Proposition 2.21(5). In addition,  $f(x^{(k)}) = g(y^{(k)})$  for each  $k$ , and so defining  $\phi(x^{(k)}) = y^{(k)}$  finalizes the proof.  $\square$

5.5.1. *Calculating edges.* Schiemann's symphony is performed by calculating  $\mathcal{T}_i$  in Definition 5.33 using a computer. An algorithm calculates the edges of polyhedral cones in order to find  $\Lambda$  in Definition 5.31 and to check the termination criterion  $T \subseteq \mathcal{S}$  of Definition 5.32. The statements here follow from the classic literature of [22, chapters 2 and 3]. For the complete proofs we refer to Schiemann's thesis [48]; here we aim to give an understanding of the general mechanism. Consider linearly independent vectors  $a_1, \dots, a_n \in \mathbb{R}^n$ . To find the edges of  $P(\{a_i\}_{i=1}^n, \emptyset)$ ,

we calculate all the kernels of the  $n$  different  $(n-1) \times n$  matrices we obtain from the set of  $a_i$  by removing one those vectors. This approach is very computationally expensive for a larger set of constraints  $a_i$ , and therefore we proceed as follows. Assume that we know the edges  $k_1, \dots, k_r$  of some polyhedral cone  $P(A, B)$ . What can we say about the edges of a polyhedral cone  $P(A \cup \{v\}, B) \subseteq \mathbb{R}^n$  for some vector  $v$ ? Since  $P(A \cup B, B) = P(A, B)$ , we can without loss of generality assume that  $B \subseteq A$ . In this case,  $\overline{P(A, B)} = P(A, \emptyset)$ .

**Theorem 5.36.** *Let  $P(A, B) \neq \emptyset$  be a pointed polyhedral cone such that  $B \subseteq A$ . Let  $k_1, \dots, k_r$  be the edges of  $P(A, B)$ . For a nonzero vector  $v$  and the set  $P(A \cup \{v\}, B)$ , we have*

- Case 1. *If  $k_i \in v^{\geq 0}$  for each  $i$ , then the edges of  $P(A \cup \{v\}, B)$  are  $k_1, \dots, k_r$ .*  
Case 2. *If  $k_i \notin v^{\geq 0}$  for some  $i$  and each  $k_j$  has  $k_j \cdot v \leq 0$ , then proceed as follows. Let  $k'_1, \dots, k'_l$  be those edges among  $k_1, \dots, k_r$  that lie in  $v^\perp$ . If there are no such  $k'_i$ , then  $P(A \cup \{v\}, B)$  is either empty or equal to  $\{0\}$ . The set  $P(A \cup \{v\}, B)$  is empty if and only if  $k := \sum k'_i$  has  $k \cdot b = 0$  for some  $b \in B$ . If it is nonempty and nonzero, then its edges are  $k'_1, \dots, k'_l$ .*  
Case 3. *If  $k_i \notin v^{\geq 0}$  for some  $i$  and some  $k_j$  has  $k_j \cdot v > 0$ , then proceed as follows. The set  $P(A \cup \{v\}, B)$  is nonempty and its edges are calculated as those edges among  $k_1, \dots, k_r$  such that  $k_i \cdot v \geq 0$  and the elements of the set  $\{F \cap v^\perp : F = k_1 \mathbb{R}_{\geq 0} + k_2 \mathbb{R}_{\geq 0} \text{ is a 2-face of } P(A, \emptyset) \text{ with } k_1 \cdot v > 0, k_2 \cdot v < 0\}$ .*

In our algorithm, we calculate a great number of edges. Theorem 5.36 reduces the computing time since it allows us to do it cumulatively, by keeping track of all the edges at all times. Next we give a computable criterion for finding the 2-faces for Case 3 above.

**Lemma 5.37.** *Let  $k_1 \neq k_2$  represent different edges of a polyhedral cone  $P(A, \emptyset)$ . Let  $\{a_1, \dots, a_r\} = \{a \in A : k_1 \subseteq a^\perp\}$  and  $\{a'_1, \dots, a'_s\} = \{a \in A : k_2 \subseteq a^\perp\}$ . Then  $k_i \mathbb{R}_0^+ + k_j \mathbb{R}_0^+$  is a 2-face of  $P_c(A, \emptyset)$  if and only if*

$$\dim \bigcap_{a \in \{a_1, \dots, a_r\} \cap \{a'_1, \dots, a'_s\}} a^\perp = 2.$$

At each step, we want the number of elements of  $A$  (and  $B$ ) to be as few as possible to speed up the algorithm. The next lemma gives a simple condition with which we can remove some of the redundant constraints.

**Lemma 5.38.** *Let  $P(A, \emptyset)$  be a pointed polyhedral cone of dimension  $d$  and with edges  $k_1, \dots, k_r$ . We have  $P(A, \emptyset) = P(A', \emptyset)$  for*

$$A' := \{c \in A : \#\{k_i : k_i \in c^\perp\} \geq d - 1\}.$$

5.5.2. *Results from the algorithm.* The finale of Schiemann's symphony is the following result.

**Theorem 5.39.** *The sequence  $\mathcal{T}_i$  becomes stationary for  $i \geq 14$ . Further, we have that each set  $T \in \mathcal{T}_{14}$  lies in  $\mathcal{S}$ .*

As we have previously noted, this completes the symphony. In other words, we have shown  $b_3 = 1$ . As documented in Table 2, with one processor it took about three hours to finish, and at least 147,442 polyhedral cones were computed. With

TABLE 2. State after the  $i$ th iteration. This table shows our computational results for Schiemann’s symphony. Here, HH:MM correspond to the time in hours and minutes after the  $i$ th iteration, and the number of cones is to the number of elements of  $\mathcal{T}_i$ .

$i$	HH:MM	# Cones	$i$	HH:MM	# Cones
1	00:00	1	8	00:59	59,970
2	00:00	4	9	01:48	34,658
3	00:00	42	10	02:22	4,452
4	00:00	500	11	02:42	1,284
5	00:02	3,311	12	02:53	702
6	00:05	11,164	13	03:00	18
7	00:28	31,334	14	03:01	0

50 processors the algorithm took 19 minutes. We wrote the code in Julia [2] with the following packages and specifications of our computer:

**CPU:** Intel<sup>®</sup> Xeon<sup>®</sup> Platinum 8180 CPU @ 2.50 Ghz;

**OS:** Fedora 32;

**packages:** Abstract Algebra v.0.9.0, Nemo v.0.17.0 and Hecke v.0.8.0.

We found that the calculation of the minimal sets was not very time consuming in comparison to calculating edges. One way to make this program faster would be to minimize the amount of edges that need to be calculated. Here, we perform the symphony at a comfortable *andante* pace. In his thesis, Schiemann performed the symphony at a quicker *vivaci* tempo by incorporating clever tricks to speed up the algorithm which we have not included here so as to keep the focus on the main arguments of the proof.

## 6. OPEN PROBLEMS AND FOOD FOR THOUGHT

The following question is, to the best of our knowledge, open.

**Question 6.1.** What are the precise values of the choir numbers  $b_n$  for  $n \geq 4$ ?

With the three equivalent but different perspectives in mind, this question is related to many open problems in number theory concerning the classification of quadratic forms as well as open problems concerning the geometry of lattices. We take this opportunity to highlight some of these interrelated problems.

**6.1. Asymptotics of the choir numbers.** The celebrated geometer Wolpert, one of the three authors who proved *one cannot hear the shape of a drum* [20], studied the moduli space of flat tori in [63]. This space consists of the set of equivalence classes of flat tori, where all members of the same equivalence class are isometric. He gave a geometric description of this moduli space and proved that it is in a certain sense an unusual phenomenon that flat tori of different shapes are isospectral; the isometry class of a *generic* flat torus is determined by its spectrum. In his article, he showed that if each flat torus in a *continuous family* (meaning a one-parameter family of flat tori that has a continuous family of basis matrices) is isospectral to any other in the family, then all are isometric. We obtain Wolpert’s result as a consequence of the following.

**Lemma 6.2.** *Consider for  $k \in \mathbb{N}$  the sequence of full-rank lattices  $\Gamma_k = A_k \mathbb{Z}^n$ . Assume that all  $\mathbb{T}_{\Gamma_k}$  are mutually isospectral and that  $A_k \rightarrow A$ . At some point the sequence  $\Gamma_k$  becomes stationary up to congruency.*

*Proof.* The positive-definite quadratic forms  $A_k^T A_k$  all have the same image over  $\mathbb{Z}^n$ . By Corollary 2.15 and continuity of the determinant,  $\det(A) = \det(A_k)$  for each  $k$ . Therefore,  $\Gamma = A\mathbb{Z}^n$  is a full-rank lattice, and its length spectrum is a discrete set. Fix any  $x \in \mathbb{Z}^n$  and consider the triangle inequality  $|\|A_k x\| - \|Ax\|| \leq \|(A_k - A)x\|$ . For sufficiently large  $k$ ,  $\|A_k x\| = \|Ax\|$  since the spectra are discrete and identical for all  $k$ , and  $A_k \rightarrow A$ . There is now a  $k$  big enough such that for all  $x = e_i + e_j$ , where  $1 \leq i, j \leq n$ , we have

$$x^T (A_k^T A_k - A^T A) x = 0.$$

Consequently,  $A_k^T A_k = A^T A$  for all  $k$  sufficiently large and thus  $A_k = C_k A$  for  $C_k \in \mathcal{O}_n(\mathbb{R})$ .  $\square$

When studying limits of lattices, Mahler’s compactness theorem is crucial, without which this survey would not be complete. According to Cassels, this theorem “may be said to have completely transformed the subject” in the context of lattice theory. See [6, pp. 136–139] for this quote and the proof of Mahler’s compactness theorem.

**Theorem 6.3** (Mahler’s compactness theorem). *Let  $\Lambda_i$  be an infinite sequence of lattices of the same dimension, satisfying the following two conditions:*

- (1) *there exists a number  $K > 0$  such that  $\text{vol}(\Lambda_i) \leq K$  for all  $i$ ;*
- (2) *there exists a number  $r > 0$  such that  $\inf_{0 \neq v \in \Lambda_i} \|v\| \geq r$  for all  $i$ .*

*Then, there is a subsequence  $\Lambda_{i_k}$  that converges to some lattice  $\Lambda$ , in the sense that there are bases of  $\Lambda_{i_k}$  converging to a basis of  $\Lambda$ .*

As an application of the results collected above, one can prove the following finiteness theorem which was first demonstrated by Kneser in an unpublished work.

**Theorem 6.4** (Finiteness theorem, [63]). *The number of nonisometric flat tori with a given Laplace spectrum is finite.*

The following proposition generalizes Wolpert’s result to tuples.

**Proposition 6.5.** *If  $b_n \geq k$ , then there exist  $k$  isospectral nonisometric even  $n$ -dimensional quadratic forms.*

*Proof.* We follow the ideas of Wolpert [63]. Consider the  $n$ -variable positive definite forms  $Q_1, \dots, Q_k$  that are isospectral and nonisometric. Then there are bijections  $\phi_i : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  such that

$$Q_1(\phi_1(z)) = \dots = Q_k(\phi_k(z))$$

for all  $z \in \mathbb{Z}^n$ . In particular, this tuple  $Q_i$  lies in the set

$$U := \bigcap_{z \in \mathbb{Z}^n} \left\{ (P_1, \dots, P_k) \in (\mathcal{S}^n)^k : P_1(\phi_1(z)) = \dots = P_k(\phi_k(z)) \right\},$$

where  $\mathcal{S}^n$  is the set of real symmetric  $n \times n$  matrices, and it is not hard to see that  $U$  is a linear space. Since it is defined by integer constraints, we can find a basis  $(f_1^{(i)}, \dots, f_k^{(i)})$  of integer matrices for  $U$ . This means that  $Q_1$  can be written as a

linear combination of the  $f_1^{(i)}$  with real coefficients  $\lambda_i$ . Since the set of positive-definite forms is open, we can approximate  $Q_1$  by a rational matrix  $\tilde{Q}_1$ , by choosing rational  $\tilde{\lambda}_i \approx \lambda_i$ , such that it is still positive definite and lies in the first factor of  $U$ . Then

$$(\tilde{Q}_1, \dots, \tilde{Q}_k) := \sum_i \tilde{\lambda}_i (f_1^{(i)}, \dots, f_k^{(i)})$$

is a tuple of isospectral rational positive-definite forms in  $U$  and, up to a constant  $\tilde{Q}_i$ , are integral.  $\square$

In 1984, Suwa-Bier, a student of Kneser, obtained the following impressive result.

**Theorem 6.6** (Suwa-Bier [55]). *The choir numbers are finite.*

With this in mind, it is natural to investigate the asymptotic behavior of the choir numbers. We have already seen in Lemma 2.33 that the choir numbers are nondecreasing. In light of the four-dimensional examples of isospectral, nonisometric lattices, we obtain the following lower bound that we have not been able to find in the literature.

**Theorem 6.7.** *If  $b_m \geq k$ , then*

$$b_n \geq \binom{\lfloor n/m \rfloor + k - 1}{k - 1}.$$

*In particular,  $b_n \geq \lfloor n/4 \rfloor + 1$ , and  $b_n$  tends towards infinity.*

*Proof.* In dimension  $mn$ , for some positive integer  $n$ , we construct  $\binom{n+k-1}{k-1}$  pairwise isospectral nonisometric flat tori. This would prove the statement by Lemma 2.33. In dimension  $m$  we have  $k$  flat tori  $\mathbb{T}_{\Gamma_1}, \dots, \mathbb{T}_{\Gamma_k}$  that are isospectral and nonisometric. Consider the sequence of  $mn$ -dimensional lattices

$$\Omega_{i_1, \dots, i_k} := \Gamma_1^{i_1} \times \dots \times \Gamma_k^{i_k},$$

with nonnegative  $i_j$  such that  $i_1 + \dots + i_k = n$ . There are  $\binom{n+k-1}{k-1}$  different choices of the sequence  $i_1, \dots, i_k$ . As a direct consequence of Theorem 2.31 and Proposition 2.30, the flat tori  $\mathbb{T}_{\Omega_{i_1, \dots, i_k}}$  all share a common Laplace spectrum but are pairwise nonisometric.  $\square$

The exact values of the choir numbers remains an open problem. Do they have polynomial or exponential growth? It appears possible to use Suwa-Bier's techniques to obtain an upper bound for  $b_n$ . This could be an interesting bachelor's or master's thesis. Although it may be difficult to determine the exact expression for  $b_n$  as a function of  $n$ , perhaps the following question is more tractable.

**Question 6.8.** What is the asymptotic behavior of  $b_n$  as  $n \rightarrow \infty$ ?

**6.2. The fourth choir number.** Since the third choir number has already been determined, the next step is to consider the fourth one. We could start by trying to show that  $b_4 = 2$ , by looking at triplets of quadratic forms instead of pairs. Let  $\mathcal{C}_n$  be a set containing a unique representative of each  $n$ -dimensional positive-definite form. We believe it is possible, albeit tedious, in the spirit of the Eisenstein and Schiemann reductions to obtain  $\mathcal{C}_4$  with this property. Let  $k, n$  be positive integers. We define

$$\mathcal{D}_{n,k} := \{(f_1, \dots, f_k) \in (\mathcal{C}_n)^k : f_i \text{ are isospectral for all } i = 1, \dots, k\},$$

so that for instance  $\mathcal{D}_{4,3}$  consists of triplets  $(f, g, h) \in \mathcal{C}_4 \times \mathcal{C}_4 \times \mathcal{C}_4$ . It is possible to modify Schiemann's algorithm to deal with triplets instead of pairs. Our termination criterion in this case would be that for a polyhedral cone  $T$ , each triplet  $(f, g, h) \in T$  should have either  $f = g$ ,  $f = h$ , or  $g = h$ , since if the elements of a covering would have this property, then there are no triplets of different forms that all share representation numbers. The first main difficulty in working with triplets is that each iteration would go through three MIN sets instead of two. For this reason, the algorithm might be too slow, but considering the massive increase in modern computation power since Schiemann's thesis approximately 30 years ago, this might not pose too much difficulty. However, it could be that the algorithm never terminates. In that case, we would propose a computer search to obtain a triplet of isospectral nonisometric flat tori and perform the analogous algorithm with  $\mathcal{D}_{4,4}$  to try to prove  $b_4 = 3$ . This process could be repeated iteratively. In this way, using  $\mathcal{D}_{n,k}$ , we could write a theoretically functioning program in the spirit of Schiemann's symphony for determining  $b_n$  for each  $n$ . The first step would be to answer

**Question 6.9.** What is the precise value of  $b_4$ ?

**6.3. The classification of quadratic forms.** The classification of even quadratic forms could be applied to shed light on the spectral geometry of flat tori, for example via Proposition 6.5. One such classification of even quadratic forms is through *genera*. Two even quadratic forms are of the same genus if they are  $p$ -adically equivalent for each  $p$ . One strong connection between genera and spectra is that the genus is determined by the spectrum precisely when the dimension is less than or equal to 4 [9, p. 114]. The classification of *regular* quadratic forms is related to Schiemann's work. In three dimensions, building on the work of Hsia [26], Schiemann together with coauthors made a significant contribution [27]. A positive-definite integral quadratic form  $q$  of  $k$  variables is *regular* if, for every positive integer  $a$  for which the congruence  $f(x) \equiv a \pmod{n}$  has a solution for each positive integer  $n$ , the equation  $f(x) = a$  has a solution  $x \in \mathbb{Z}^k$ . This definition is nearly a century old, originally due to Dickson [13]. A quarter century later, Watson proved [59] that there are finitely many inequivalent primitive regular ternary quadratic forms. In 1997, Jagy, Kaplansky, and Schiemann [27] produced a list containing representatives of all the *possible* equivalence classes of such forms; there are 913. They proved that 891 of the list are indeed regular, leaving the remaining 22 as an open problem. In 2011, Oh [42] proved eight of the remaining 22 are regular. In 2020, Oh and Kim classified all 49 regular ternary triangular forms; we refer to [31] for the details. These types of problems are generally approached with number theoretic techniques and tools, but it may be interesting to approach them from an analytic or geometric perspective. What are the spectral and geometric implications for the associated flat tori? What are the geometric features for the associated lattices?

**6.4. The geometry of lattices and  $k$ -spectra.** Our third perspective on the isospectral problem for flat tori concerns the geometry of lattices. The  $k$ -spectrum is a geometric invariant of a lattice for which many questions are open.

**Definition 6.10** ( $k$ -spectrum). Let  $\Gamma$  be a lattice, and let  $\Lambda \subseteq \Gamma$  be a sublattice. Write  $[\Lambda]$  for the equivalence class of sublattices in  $\Gamma$  with respect to congruence.

We define the  $k$ -spectrum of  $\Gamma$  to be the set

$$\mathcal{L}^k(\Gamma) := \{([\Lambda], m_\Lambda) : \Lambda \text{ is a } k\text{-rank sublattice of } \Gamma\}.$$

Here,  $m_\Lambda = \#\Lambda$  is the number of sublattices of  $\Gamma$  that are congruent to  $\Lambda$ .

Two full-rank lattices in  $\mathbb{R}^n$  are congruent if and only if their  $n$ -spectra agree. Two lattices are isospectral if and only if their 1-spectra agree. The  $k$ -spectrum is in this sense a generalization of the length and Laplace spectra. It gives rise to new problems, perhaps the most natural being

**Question 6.11.** For which triplets of positive integers  $(n, k, m)$  with  $n \geq k$ ,  $m$ , does  $\mathcal{L}^k(\Gamma_1) = \mathcal{L}^k(\Gamma_2)$  imply  $\mathcal{L}^m(\Gamma_1) = \mathcal{L}^m(\Gamma_2)$  for any  $n$ -rank lattices  $\Gamma_1, \Gamma_2$ ?

It follows from our work here that the answer is positive for some triplets, for example  $(3, 1, 3)$  as in Section 5, and negative for others, such as  $(4, 1, 4)$  as in Section 4. In the unpublished report [7], Claes showed that the answer is yes if  $n = m = 3$ , and  $k = 2$ . We show below that in any dimension  $n$ , the  $k$  spectra determines the  $m$  spectra for  $m \leq k$ .

**Proposition 6.12.** *Let  $n \geq k \geq m \geq 1$ , and let  $\Gamma_i$  be full-rank lattices in  $\mathbb{R}^n$ . If the  $k$ -spectra of  $\Gamma_1$  and  $\Gamma_2$  agree, then their  $m$ -spectra also agree.*

*Proof.* The  $k$ -spectra agree if and only if there is a bijection

$$\phi : \Gamma_1^k \rightarrow \Gamma_2^k,$$

such that  $\phi$  maps any set of linearly independent vectors  $u_1, \dots, u_k \in \Gamma_1$  to a set of linearly independent  $v_1, \dots, v_k \in \Gamma_2$ , where the parallelotope spanned by  $u_i$  is congruent to that which is spanned by  $v_i$ , meaning that we can order  $v_i$  such that  $(\langle u_i, u_j \rangle)_{ij} = (\langle v_i, v_j \rangle)_{ij}$ . Now consider the function  $\phi' : \Gamma_1^{k-1} \rightarrow \Gamma_2^{k-1}$ , sending the sets  $u_1, \dots, u_{k-1} \in \Gamma_1$  to  $v_1, \dots, v_{k-1} \in \Gamma_2$  corresponding to  $\phi$ . Since the upper left  $(k-1) \times (k-1)$  submatrices of the above  $k \times k$  Gram matrices are equal,  $\phi'$  satisfies precisely the condition that the  $(k-1)$ -spectra of  $\Gamma_1$  and  $\Gamma_2$  agree.  $\square$

As a consequence of the preceding proposition, the number of  $n$ -dimensional noncongruent lattices whose  $k$ -spectra all agree is bounded above by  $b_n$ , for any  $k$ . One could also phrase the  $k$ -spectrum as a property of flat tori or quadratic forms. In terms of quadratic forms, there is a connection to *Siegel modular forms*; see [45, Chapter 1]. For a certificate similar to Corollary 3.7 in terms of  $k$ -spectra, see [46]. We hope that readers investigating these and related problems will keep all three perspectives in mind and thereby reap the benefits of techniques from analysis, number theory, and geometry.

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