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# Topological Manin pairs and $(n, s)$ -type series

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## Abstract

Lie subalgebras of  $L = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x]$ , complementary to the diagonal embedding  $\Delta$  of  $\mathfrak{g}[[x]]$  and Lagrangian with respect to some particular form, are in bijection with formal classical  $r$ -matrices and topological Lie bialgebra structures on the Lie algebra of formal power series  $\mathfrak{g}[[x]]$ . In this work we consider arbitrary subspaces of  $L$  complementary to  $\Delta$  and associate them with so-called series of type  $(n, s)$ . We prove that Lagrangian subspaces are in bijection with skew-symmetric  $(n, s)$ -type series and topological quasi-Lie bialgebra structures on  $\mathfrak{g}[[x]]$ . Using the classification of Manin pairs we classify up to twisting and coordinate transformations all quasi-Lie bialgebra structures. Series of type  $(n, s)$ , solving the generalized classical Yang-Baxter equation, correspond to subalgebras of  $L$ . We discuss their possible utility in the theory of integrable systems.

**Keywords** Lie bialgebras · quasi-Lie bialgebras · Manin pairs · Yang-Baxter equations ·  $r$ -matrices · Lie algebra splittings

**Mathematics Subject Classification** 17B62 · 17B38 (Primary) · 17B80 (Secondary)

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Dedicated to the memory of Yuri Manin.

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### 1 Introduction

Let  $F$  be an algebraically closed field of characteristic 0 equipped with the discrete topology and  $\mathfrak{g}$  be a simple Lie algebra over  $F$ . We define the Lie algebra  $\mathfrak{g}[[x]]$  to be the space  $\mathfrak{g} \otimes F[[x]]$  with the bracket

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg$$

and we equip it with the  $(x)$ -adic topology. The continuous dual of  $\mathfrak{g}[[x]]$  is denoted by  $\mathfrak{g}[[x]]'$  and it is endowed with the discrete topology.

A topological Manin pair is a pair  $(L, \mathfrak{g}[[x]])$  where

1.  $L$  is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form  $B$ ;
2.  $\mathfrak{g}[[x]] \subset L$  is a Lagrangian subalgebra with respect to  $B$ ;
3. for any continuous functional  $T: \mathfrak{g}[[x]] \rightarrow F$  there is  $f \in L$  such that  $T = B(f, -)$ .

Topological Manin pairs were classified in [1] using the tools from [8]. More precisely, if  $(L, \mathfrak{g}[[x]])$  is a topological Manin pair, then  $L$  is isomorphic, as a Lie algebra with form, to either  $L(\infty)$  or  $L(n, \alpha)$ . In this paper we consider only the "non-degenerate" case, namely  $L \cong L(n, \alpha)$ .

As a Lie algebra

$$L(n, \alpha) = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x].$$

The bilinear form  $B$  on  $L(n, \alpha)$  is completely determined by the sequence  $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ . For example, when  $n = 0$  the form is given by

$$B(a \otimes f, b \otimes g) = \kappa(a, b) \text{res}_0 \{ \alpha(x) fg \},$$

where  $\kappa$  is the Killing form on  $\mathfrak{g}$  and  $\alpha(x) := 1 + \alpha_{-2}x + \alpha_{-3}x^2 + \dots \in F((x))$ . In case  $n > 0$  the form is given by a similar formula; see Sect. 2.

It was established in [1], that the following objects are in one-to-one correspondence

- Lagrangian subalgebras  $W \subseteq L(n, 0), 0 \leq n \leq 2$ , complementary to the diagonal

$$\Delta := \{ (f, [f]) \mid f \in \mathfrak{g}[[x]] \},$$

i.e.  $\Delta \dot{+} W = L(n, 0)$ ;

- non-degenerate topological Lie bialgebra structures on  $\mathfrak{g}[[x]]$  and
- formal solutions to the classical Yang-Baxter equation (CYBE) in the form

$$\frac{y^n \Omega}{x - y} + g(x, y) = \Omega \sum_{k \geq 0} x^{-k-1} y^{k+n} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]], \tag{1}$$

where  $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$  is the quadratic Casimir element and  $g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ .

Furthermore, the proof of the above-mentioned correspondence reveals that series Eq. (1) can be viewed as a generating series for the corresponding subalgebra  $W$ . The present paper can be thus considered as a continuation of [1], where we extend the preceding correspondence using series of type  $(n, s)$ .

To define a series of type  $(n, s)$  fix a basis  $\{b_i\}_{i=1}^d$  of  $\mathfrak{g}$ , orthonormal with respect to its Killing form  $\kappa$ , and interpret  $y^n\Omega/(x - y)$  as a series

$$\frac{y^n\Omega}{x - y} = \sum_{k=0}^{\infty} \sum_{i=1}^d w_{k,i} \otimes b_i y^k \in ((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]) \otimes \mathfrak{g})[[y]]. \tag{2}$$

This expression might be understood as a Taylor series expansion. Elements  $w_{k,i} \in \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$  are presented explicitly in Eq. (20). A series of type  $(n, s)$  is a series of the form

$$\frac{s(x)y^n\Omega}{x - y} + g(x, y) \in ((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]) \otimes \mathfrak{g})[[y]], \tag{3}$$

where  $s \in F[[x]]^\times$  and  $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ ; See Definition 3.2. For each series  $r$  of type  $(n, s)$  we define another series  $\bar{r}$  of the same type as follows

$$\bar{r} := \frac{s(y)x^n\Omega}{x - y} - \tau(g(y, x)), \tag{4}$$

where  $\tau$  is the  $F[[x, y]]$ -linear extension of the map  $a \otimes b \mapsto b \otimes a$ .

Each series of type  $(n, s)$  produces a subspace of  $\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$  complementary to the diagonal embedding  $\Delta$  of  $\mathfrak{g}[[x]]$ . The following results generalize the above-mentioned correspondence from [1].

**Theorem A** *Let  $n \in \mathbb{Z}_{\geq 0}$  and  $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$  be an arbitrary sequence with the corresponding series  $\alpha(x) := x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0x^{-1} + \dots \in F((x))$ . For any  $(n, s)$ -type series*

$$r = \sum_{k=0}^{\infty} \sum_{i=1}^d f_{k,i} \otimes b_i y^k \in ((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]) \otimes \mathfrak{g})[[y]] \tag{5}$$

define the space

$$W(r) := \text{span}_F\{f_{k,i} \mid k \geq 0, 1 \leq i \leq d\} \subseteq \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]. \tag{6}$$

The following results are true:

1.  $W$  defines a bijection between series of type  $(n, \frac{1}{x^n\alpha(x)})$  and subspaces  $V \subset L(n, \alpha)$  complementary to the diagonal  $\Delta$ , i.e.  $L(n, \alpha) = \Delta \dot{+} V$ ;
2. For any series  $r$  of type  $(n, \frac{1}{x^n\alpha(x)})$  we have  $W(r)^\perp = W(\bar{r})$  inside  $L(n, \alpha)$ ;

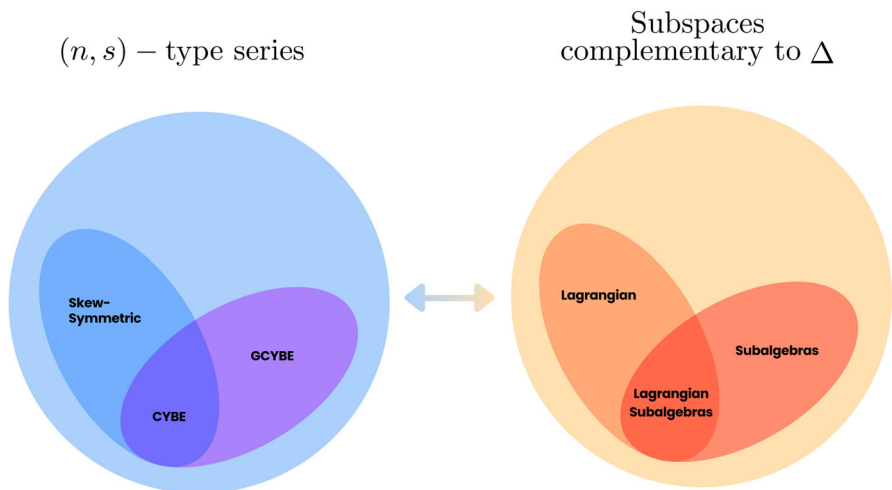


Fig. 1 Series-subspaces correspondence

3. Any series  $r$  of type  $(n, \frac{1}{x^n \alpha(x)})$  satisfies  $GCYB(r) = \psi$  (see Definition 3.5 for the meaning of  $GCYB(r)$ ), where  $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x_1, x_2, x_3]]$  is defined by

$$B(v_1 \otimes v_2 \otimes v_3, \psi) = B(v_1, [v_2, v_3])$$

for all  $v_1 \in W(\bar{r})$ ,  $v_2, v_3 \in W(r)$ .

In particular, considering the case when  $r$  is skew-symmetric, meaning  $r = \bar{r}$ , or when  $\psi = 0$  we get the following correspondences.

**Corollary B** Let  $n \in \mathbb{Z}_{\geq 0}$ ,  $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$  and  $W$  be the map from Theorem A. Then

1.  $W$  defines a bijection between skew-symmetric  $(n, \frac{1}{x^n \alpha(x)})$ -type series and Lagrangian subspaces  $V \subseteq L(n, \alpha)$ , complementary to the diagonal  $\Delta$ ;
2.  $W$  defines a bijection between  $(n, \frac{1}{x^n \alpha(x)})$ -type series solving the GCYBE and subalgebras  $V \subseteq L(n, \alpha)$  complementary to the diagonal  $\Delta$ .

Observe that an  $(n, s)$ -type series produces a subspace of  $L(n, \alpha)$  for any sequence  $\alpha$ . However, to obtain the compatibility with the form, given by  $\alpha$ , we need the equality  $s(x) = 1/(x^n \alpha(x))$ . In this case, the components  $f_{k,i}$  and  $b_i y^k$  of the series become dual bases for  $W(r)$  and  $\Delta$  respectively.

The requirement on a series  $r$  of type  $(n, s)$  to solve the CYBE is equivalent to being skew-symmetric and to solve the GCYBE. Together with Corollary B this implies that Lagrangian subalgebras  $W \subset L(n, \alpha)$ , satisfying  $W \dot{+} \Delta = L(n, \alpha)$ , are in bijection with  $(n, 1/(x^n \alpha(x)))$ -type series solving the classical Yang-Baxter equation. These correspondences are schematically depicted in Fig. 1.

**Remark 1.1** Let  $r$  be a series of type  $(n, s)$ . Applying the projection  $(a, b) \otimes c \mapsto a \otimes c$  onto the left component to  $r$  we obtain the series

$$r_{\text{proj}} = \frac{s(x)y^n \Omega}{x - y} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})(x) \llbracket y \rrbracket. \tag{7}$$

Conversely, starting with a series  $r_{\text{proj}}$  of the form Eq. (7), we can obtain an  $(n, s)$ -type series  $r$  by taking two Taylor series expansions of  $r_{\text{proj}}$  at  $x = 0$  and  $y = 0$  respectively and then constructing  $r$  by combining the coefficients of  $b_i y^k, k \geq 0$ , in these expansions. These two constructions are inverse to each other and hence both  $r$  and its projection  $r_{\text{proj}}$  contain exactly the same information. Consequently, all the statements made for  $(n, s)$ -type series can be stated in terms of their projections onto the left component and vice versa. In contrast with [1], in this paper we give preference to series of type  $(n, s)$  rather than to their projections, because the statement that series of type  $(n, s)$  generate subspaces of  $L(n, \alpha)$  becomes transparent.

Reinterpreting the results of [1] in terms of  $(n, s)$ -type series we see that skew-symmetric series of type  $(n, 1/(x^n \alpha(x)))$ , that also solve the GCYBE, exist only for  $n = 0, 1$  and  $n = 2$  with  $\alpha_0 = 0$ .

Lagrangian subalgebras of  $L(n, \alpha)$ , complementary to  $\Delta$ , correspond to topological Lie bialgebra structures on  $\mathfrak{g} \llbracket x \rrbracket$ . If we instead consider Lagrangian subspaces (not necessarily subalgebras) of  $L(n, \alpha)$ , we get so called (non-degenerate) topological quasi-Lie bialgebra structures on  $\mathfrak{g} \llbracket x \rrbracket$ . A topological quasi-Lie bialgebra structure on  $\mathfrak{g} \llbracket x \rrbracket$  consists of

- a skew-symmetric continuous linear map  $\delta: \mathfrak{g} \llbracket x \rrbracket \rightarrow (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$  and
- a skew-symmetric element  $\varphi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y, z \rrbracket$ ,

which are subject to the following three conditions

- $\delta([a, b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] - [b \otimes 1 + 1 \otimes b, \delta(a)]$ , i.e.  $\delta$  is a 1-cocycle;
- $\frac{1}{2} \text{Alt}((\delta \otimes 1)\delta(a)) = [a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a, \varphi]$ ;
- $\text{Alt}((\delta \otimes 1 \otimes 1)\varphi) = 0$ ,

where  $\text{Alt}(x_1 \otimes \dots \otimes x_n) := \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$ .

Following [5] we prove the following direct relation between  $\delta, \varphi$  and skew-symmetric  $(n, s)$ -type series  $r$ .

**Proposition C** *There is a bijection between topological quasi-Lie bialgebras and skew-symmetric  $(n, s)$ -type series. Let  $r$  be the  $(n, s)$ -type series corresponding to  $(\mathfrak{g} \llbracket x \rrbracket, \delta, \varphi)$ , then, under the identification  $\mathfrak{g} \llbracket x \rrbracket \cong \Delta$ , we have the following identities:*

- $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$  for any  $a \in \mathfrak{g} \llbracket x \rrbracket$  and
- $\text{CYB}(r) = -\varphi$ .

The same is true if  $r$  is interpreted as an element in  $(\mathfrak{g} \otimes \mathfrak{g})(x) \llbracket y \rrbracket$ .

In view of this result we call skew-symmetric  $(n, s)$ -type series quasi- $r$ -matrices.

Repeating the ideas from [7] and [5] we show that topological quasi-Lie bialgebras can be twisted similar to topological Lie bialgebras. More precisely, if  $\delta$  is a quasi-Lie bialgebra structure on  $\mathfrak{g} \llbracket x \rrbracket$ , given by the Lagrangian subspace  $W$ , and  $s :=$

$\sum_i a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$  is an arbitrary skew-symmetric tensor, then

$$W_s := \left\{ \sum_i B(b^i, w)a_i - w \mid w \in W \right\} \tag{8}$$

is another (twisted) Lagrangian subspace complementary to the diagonal. This observation implies, that in order to classify all topological quasi-Lie bialgebra structures on  $\mathfrak{g}[[x]]$  up to twisting it is enough to find one single Lagrangian subspace within each  $L(n, \alpha)$ . Moreover, it was shown in [1] that substitutions of the form  $x \mapsto x + a_2x^2 + a_3x^3 + \dots, a_i \in F$ , allow us to assume that  $\alpha$  has the form

$$\alpha = (\dots, 0, \alpha_0, 0, \dots, 0).$$

Lagrangian subspaces for such  $L(n, \alpha)$  are constructed in Sect. 4.1.

Using Theorem A and Proposition C we explain how twisting of a Lagrangian subspace  $W \subset L(n, \alpha)$  is seen at the level of  $\delta$  and the corresponding quasi- $r$ -matrix  $r$ .

**Corollary D** *Let  $(\mathfrak{g}[[x]], \delta, \varphi)$  be a topological quasi-Lie bialgebra structure corresponding to the quasi- $r$ -matrix  $r$ . If we twist  $W(r)$  with a skew-symmetric tensor  $s$  we obtain another topological quasi-Lie bialgebra  $(\mathfrak{g}[[x]], \delta_s, \varphi_s)$ , such that*

1.  $W(r)_s = W(r - s)$ ;
2.  $\delta_s = \delta + ds$ ;
3.  $\varphi_s = \varphi + \text{CYB}(s) - \frac{1}{2}\text{Alt}((\delta \otimes 1)s)$ .

Therefore, to describe all quasi- $r$ -matrices up to twisting it is enough to find one single quasi- $r$ -matrix for each  $L(n, \alpha)$ . We achieve that goal in Sect. 4.2 by writing out explicitly series of type  $(n, s)$  for subspaces from Sect. 4.1.

The results above, in particular, show that if  $r$  is a quasi- $r$ -matrix and  $\delta(a) := [a \otimes 1 + 1 \otimes a, r]$ , then the condition

$$\text{Alt}((\delta \otimes 1 \otimes 1)\text{CYB}(r)) = 0 \tag{9}$$

is trivially satisfied.

We conclude the paper by using Theorem A for construction of Lie algebra splittings  $\Delta \dot{+} W = L(n, \alpha)$  and the corresponding  $(n, s)$ -type series, which we call generalized  $r$ -matrices. These constructions are important in the theory of integrable systems because of their use in the Adler-Konstant-Symes (AKS) scheme and the so-called  $r$ -matrix approach; see [4, 6]. The subalgebra splittings of  $L(0, 0)$  as well as their physical applications were considered in e.g. [9, 10].

Our first result shows that in order to obtain new generalized  $r$ -matrices from subalgebra splittings  $L(n, \alpha) = \Delta \dot{+} W$  with  $n > 2$ , the subalgebra  $W$  must be unbounded. Otherwise the situation can be reduced to the splitting of  $L(2, \alpha)$ .

**Proposition E** *Let  $L(n, \alpha) = \Delta \dot{+} W$  for some subalgebra  $W \subset L(n, \alpha)$  and  $n > 2$ . Assume  $W$  is bounded, i.e. there is an integer  $N > 0$  such that*

$$x^{-N} \mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N \mathfrak{g}[x^{-1}],$$

where  $W_+$  is the projection of  $W \subset L(n, \alpha) = \mathfrak{g}((x)) \oplus \mathfrak{g}[x]/x^n \mathfrak{g}[x]$  on the first component  $\mathfrak{g}((x))$ . Then we have the inclusion

$$\{0\} \times [x^2] \mathfrak{g}[x]/x^n \mathfrak{g}[x] \subseteq W$$

and the image  $\tilde{W}$  under the canonical projection  $L(n, \alpha) \rightarrow L(2, \alpha)$  is a subalgebra satisfying  $L(2, \alpha) = \Delta + \tilde{W}$ .

Despite this result we think that bounded subalgebras  $W \subset L(n, \alpha)$  complementary to  $\Delta$  are still interesting, because in the case  $\alpha \neq 0$  they lead to unbounded orthogonal complements  $W^\perp$  which are also important in view of the AKS scheme. We give examples of subalgebras of  $L(n, \alpha)$  with unbounded orthogonal complements.

## 2 Topological Manin pairs

Let  $F$  be an algebraically closed field of characteristic 0,  $\mathfrak{g}$  be a finite-dimensional simple  $F$ -Lie algebra and  $\mathfrak{g}[[x]] := \mathfrak{g} \otimes F[[x]]$  be the Lie algebra with the bracket defined by

$$[a \otimes f, b \otimes g] := [a, b] \otimes fg,$$

for all  $a, b \in \mathfrak{g}$  and  $f, g \in F[[x]]$ . From now on, we always endow  $F$  with the discrete topology and view  $\mathfrak{g}[[x]]$  as a topological Lie algebra with the  $(x)$ -adic topology.

A *topological Manin pair* is a pair  $(L, \mathfrak{g}[[x]])$ , where  $L$  is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form  $B$ , such that

1.  $\mathfrak{g}[[x]] \subseteq L$  is a Lagrangian Lie subalgebra with respect to  $B$ ;
2. for any continuous functional  $T : \mathfrak{g}[[x]] \rightarrow F$  there exists an element  $f \in L$  such that  $T = B(f, -)$ .

The statements of [8, Proposition 2.9] and [1, Proposition 3.12] give a description of all topological Manin pairs. For precise formulation we need to repeat the definitions of some specific Lie algebras with forms from [1, Section 3.2] and [8, Section 2].

**Definition 2.1** We define the Lie algebra  $L(\infty) := \mathfrak{g} \otimes A(\infty)$ , where  $A(\infty)$  is the unital commutative algebra with underlying space  $\sum_{i \geq 0} Fa_i + F[[x]]$  and multiplication given by

$$a_i a_j := 0, \quad a_i x^j := a_{i-j} \text{ for } i \geq j \text{ and } a_i x^j := 0 \text{ otherwise.}$$

Let  $t : A \rightarrow F$  be the functional, given by  $t(a_0) := 1, t(a_i) := 0, i \geq 1$  and  $t(F[[x]]) := 0$ . We equip  $L(\infty)$  with the symmetric non-degenerate invariant bilinear form

$$B \left( a \otimes \left( \sum_{i \geq 0} c_i a_i, f(x) \right), b \otimes \left( \sum_{i \geq 0} t_i a_i, g(x) \right) \right)$$

$$:= \kappa(a, b) \mathfrak{t} \left( g(x) \sum_{i \geq 0} c_i a_i + f(x) \sum_{i \geq 0} t_i a_i \right). \tag{10}$$

**Definition 2.2** Let  $n \geq 1$  and  $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$  be an arbitrary sequence. Consider the algebra

$$A(n, \alpha) := F((x)) \oplus F[x]/(x^n).$$

Abusing the notation we denote the element  $x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0x^{-1} + \dots \in F((x))$  with the same letter  $\alpha$ . Define the functional  $\mathfrak{t}: A(n, \alpha) \rightarrow F$  by

$$\mathfrak{t}(f, [p]) := \text{res}_0 \{ \alpha(f - p) \}.$$

Taking the tensor product of  $A(n, \alpha)$  with  $\mathfrak{g}$  we get the Lie algebra

$$L(n, \alpha) := \mathfrak{g} \otimes A(n, \alpha) \tag{11}$$

which we equip with the form

$$B(a \otimes (f, [p]), b \otimes (g, [q])) := \kappa(a, b) \mathfrak{t}(fg, [pq]). \tag{12}$$

It is known that the bilinear form  $B$  is symmetric non-degenerate and invariant.

**Definition 2.3** Take an arbitrary sequence  $\alpha = (\alpha_i \in F \mid -\infty < i \leq -2)$  and let  $A(0, \alpha) := F((x))$ . We define the functional  $\mathfrak{t}: A(0, \alpha) \rightarrow F$  by

$$\mathfrak{t}(f) := \text{res}_0 \{ \alpha f \},$$

where  $\alpha = 1 + \alpha_{-2}x + \dots \in F((x))$ . We equip the Lie algebra  $L(0, \alpha) := \mathfrak{g} \otimes A(0, \alpha)$  with the bilinear form

$$B(a \otimes f, b \otimes g) := \kappa(a, b) \mathfrak{t}(fg), \tag{13}$$

which is again symmetric non-degenerate and invariant. From now on we identify  $F((x))$  with  $F((x)) \times \{0\}$  and write  $(f, 0)$  for elements in  $A(0, \alpha)$ .

**Definition 2.4** A series of the form  $\varphi = x + a_2x^2 + a_3x^3 + \dots \in F[[x]]$  is called a *coordinate transformation*. Coordinate transformations form a group  $\text{Aut}_0 F[[x]]$  under substitution which we view as a subgroup of automorphisms of  $F[[x]]$ .

An element  $\varphi \in \text{Aut}_0 F[[x]]$  induces an automorphism of  $A(n, \alpha)$  by  $f/g \mapsto \varphi(f)/\varphi(g)$  and  $[p] \mapsto [\varphi(p)]$  that changes the functional  $\mathfrak{t}$  to  $\mathfrak{t} \circ \varphi$ . We write  $A(n, \alpha)^{(\varphi)}$  for the algebra  $A(n, \alpha)$  with the functional  $\mathfrak{t} \circ \varphi$ . It is not hard to see that for any  $\varphi \in \text{Aut}_0 F[[x]]$  there is a sequence  $\beta$  such that  $A(n, \alpha)^{(\varphi)} = A(n, \beta)$ .

Let  $(L, \mathfrak{g}[[x]])$  be a topological Manin pair. According to [8, Proposition 2.9] as a Lie algebra with form  $L \cong L(\infty)$  or  $L \cong L(n, \alpha)$ , for some  $n \geq 0$  and some sequence  $\alpha$ . Here we identify  $\mathfrak{g}[[x]]$  with the diagonal

$$\Delta := \{(f, [f]) \mid f \in \mathfrak{g}[[x]]\} \subset L(n, \alpha).$$

Moreover, we can assume that all the elements  $\alpha_i$  in the sequence  $\alpha$ , except maybe  $\alpha_0$ , are 0 by virtue of the following result.

**Proposition 2.5** [1, Proposition 3.12] *Let  $n \geq 0$  and  $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$  be a sequence. There exists a  $\varphi \in \text{Aut}_0 F[[x]]$  such that  $A(n, \alpha) \cong A(n, \beta)^{(\varphi)}$ , where  $\beta$  is the sequence satisfying  $\beta_i = 0$  for all  $i \neq 0$  and  $\beta_0 = \alpha_0$ .*

**Remark 2.6** Observe that the result of Proposition 2.5 can be interpreted in terms of a formal differential equation. Consider an arbitrary  $\alpha(x) = x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0x^{-1} + \dots \in F((x))$  and  $\beta(x) = x^{-n} + \alpha_0x^{-1}$ . Then the functionals  $t_\alpha$  and  $t_\beta$  defined on  $A(n, \alpha)$  and  $A(n, \beta)$  respectively are given by

$$t_\alpha(f, [p]) = \text{res}_0\{\alpha(f - p)\} \quad \text{and} \quad t_\beta(f, [p]) = \text{res}_0\{\beta(f - p)\}$$

The equality  $A(n, \alpha)^{(\varphi)} = A(n, \beta)$  can be expressed as

$$\text{res}_0\{\beta(x)f(x)\} = \text{res}_0\{\alpha(x)f(\varphi(x))\} = \text{res}_0\{\alpha(\psi(x))f(x)\psi'(x)\}, \tag{14}$$

where  $\psi \in \text{Aut}_0(F[[x]])$  is the compositional inverse of  $\varphi$ , i.e.  $\varphi(\psi(x)) = x$ . Since the residue pairing is non-degenerate on  $F((x))$ , we obtain

$$\alpha(\psi(x))\psi'(x) = \beta(x). \tag{15}$$

In particular, the transformation  $\varphi$  is the compositional inverse of the solution to Eq. (15).

### 3 Series of type $(n, s)$ and subspaces of $L(n, \alpha)$

Let  $\{b_i\}_{i=1}^d$  be an orthonormal basis of  $\mathfrak{g}$  with respect to the Killing form  $\kappa$ . We write  $\Omega$  for the quadratic Casimir element  $\sum_{i=1}^d b_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ . It satisfies the identity  $[a \otimes 1 + 1 \otimes a, \Omega] = 0$  for all  $a \in \mathfrak{g}$ .

In this section we describe a bijection between subspaces  $W \subset L(n, \alpha)$  complementary to  $\Delta$  and certain series. The following definition introduces convenient spaces containing these series.

**Definition 3.1** We put  $A_1(n, \alpha) := A(n, \alpha) = F((x_1)) \oplus F[x_1]/(x_1^n)$  and then define inductively the algebras

$$A_m(n, \alpha) := A_{m-1}(n, \alpha)((x_m)) \oplus A_{m-1}(n, \alpha)[x_m]/x_m^n A_{m-1}(n, \alpha), \quad m > 1. \tag{16}$$

The functional  $t$  defined on  $A(n, \alpha)$  extends inductively to a functional on  $A_m(n, \alpha)$ . More precisely,

$$t \left( \sum_{k \geq -N} f_k x_m^k, \sum_{\ell=0}^{n-1} [g_\ell x_m^\ell] \right) := \sum_{k \geq -N} t(f_k) t(x_m^k, 0) + \sum_{\ell=0}^{n-1} t(g_\ell) t(0, [x_m]^\ell), \tag{17}$$

where  $f_k, g_\ell \in A_{m-1}(n, \alpha)$ . Since  $t(x^n F[[x]]) = 0$ , the sum on the right-hand side of Eq. (17) is finite and well-defined. This allows us to extend the form  $B$  on  $L(n, \alpha)$  to a symmetric non-degenerate bilinear form on the  $\mathfrak{g}$ -module

$$L_m(n, \alpha) := \mathfrak{g}^{\otimes m} \otimes A_m(n, \alpha) \tag{18}$$

by letting

$$B((a_1 \otimes \dots \otimes a_m) \otimes f, (b_1 \otimes \dots \otimes b_m) \otimes g) := t(fg) \prod_{k=1}^m \kappa(a_k, b_k), \tag{19}$$

for all  $a_1, \dots, a_m, b_1, \dots, b_m \in \mathfrak{g}$  and  $f, g \in A_m(n, \alpha)$ .

Fix some integer  $n \geq 0$ . We interpret the quotient  $y^n \Omega / (x - y)$  in the following way

$$\begin{aligned} \frac{y^n \Omega}{x - y} &= \sum_{k=0}^{n-1} \sum_{i=1}^d b_i(0, -[x]^{(n-1)-k}) \otimes b_i(y^k, [y]^k) \\ &+ \sum_{k=n}^{\infty} \sum_{i=1}^d b_i(x^{(n-1)-k}, 0) \otimes b_i(y^k, 0) \\ &= \sum_{k=0}^{\infty} \sum_{i=1}^d w_{k,i} \otimes b_i(y^k, [y]^k) \in (L(n, \alpha) \otimes \mathfrak{g}) [[y, [y]]] \subset L_2(n, \alpha), \end{aligned} \tag{20}$$

where  $\alpha$  is an arbitrary sequence and we write  $b_i(x^\ell, [x]^m)$  meaning  $b_i \otimes (x^\ell, [x]^m)$ .

**Definition 3.2** Since  $(L(n, \alpha) \otimes \mathfrak{g}) [[y, [y]]]$  is an  $F[[x]] \cong F[[x, [x]]]$ -module and

$$(\mathfrak{g} \otimes \mathfrak{g})[[x, y]] \cong (\Delta \otimes \mathfrak{g})[[y, [y]]] \subset (L(n, \alpha) \otimes \mathfrak{g}) [[y, [y]]]$$

the series

$$r(x, y) = \frac{s(x)y^n \Omega}{x - y} + g(x, y), \tag{21}$$

where  $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$  and  $s \in F[[x]]^\times$ , is also inside  $(L(n, \alpha) \otimes \mathfrak{g}) [[y, [y]]]$ . Series of the form Eq. (21) are called *series of type  $(n, s)$* .

**Remark 3.3** Every series

$$r(x, y) = \frac{h(x, y)\Omega}{x - y} + g(x, y) \in L_2(n, \alpha),$$

where  $h \in F[[x, y]]$ ,  $h(x, x) \neq 0$  and  $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ , has a unique representation as a series of type  $(n, s)$ . Indeed, write  $h(x, x) = x^n s(x)$  for some  $s \in F[[x]]^\times$ . Then  $h(x, y) - y^n s(x) = (x - y)f(x, y)$  for some  $f \in F[[x, y]]$ . This implies that we can rewrite  $r$  in the  $(n, s)$  form

$$r(x, y) = \frac{s(x)y^n\Omega}{x - y} + f(x, y)\Omega + g(x, y). \tag{22}$$

In the construction of  $f$  we are using the fact that for any  $F$ -vector space  $V$  and any element  $h \in V[[x, y]]$

$$h(z, z) = 0 \implies h(x, y) = (x - y)f(x, y) \tag{23}$$

for some  $f \in V[[x, y]]$ .

**Definition 3.4** For each series  $r$  of type  $(n, s)$  we define another series  $\bar{r}$  of the same type  $(n, s)$  by

$$\bar{r}(x, y) := \frac{s(y)x^n\Omega}{x - y} - \tau(g(y, x)) \in (L(n, \alpha) \otimes \mathfrak{g})[[y, [y]]], \tag{24}$$

where  $\tau$  is the  $F[[x, y]]$ -linear extension of the map  $a \otimes b \mapsto b \otimes a$ . To see that this is an  $(n, s)$ -type series its enough to apply the argument from Remark 3.3. Series of type  $(n, s)$ , satisfying  $r = \bar{r}$ , are called *skew-symmetric*.

**Definition 3.5** The *generalized classical Yang-Baxter equation (GCYBE)* is the equation for an  $(n, s)$ -type series of the form

$$\begin{aligned} \text{GCYB}(r) := & [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] \\ & + [r^{13}(x_1, x_3), \bar{r}^{23}(x_2, x_3)] = 0. \end{aligned} \tag{25}$$

Here  $(-)^{13}: L_2(n, \alpha) \rightarrow (U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes A_3(n, \alpha)$  is the inclusion map given by

$$\begin{aligned} a \otimes b \otimes & \left( \sum_{k \geq -N} F(x_1, [x_1])x_2^k, \sum_{m=0}^{n-1} G(x_1, [x_1])[x_2]^m \right) \\ \mapsto a \otimes 1 \otimes b \otimes & \left( \sum_{k \geq -N} F(x_1, [x_1])x_3^k, \sum_{m=0}^{n-1} G(x_1, [x_1])[x_3]^m \right). \end{aligned}$$

Other inclusions are defined in a similar manner. The commutators are then taken in the associative  $A_3(n, \alpha)$ - algebra  $(U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes A_3(n, \alpha)$ .

Before formulating the main theorem of the section we note that if  $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$  is an arbitrary sequence and  $\alpha(x) = x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0x^{-1} + \dots \in F((x))$  is the corresponding series, then  $x^n\alpha(x) \in F[[x]]^\times$ .

**Theorem 3.6** *Let  $n \in \mathbb{Z}_{\geq 0}$  and  $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$  be an arbitrary sequence with the corresponding series  $\alpha(x) \in F((x))$ . Consider the map*

$$W : L_2(n, \alpha) \longrightarrow \{V \subset L(n, \alpha) \mid V \text{ is a subspace}\}$$

given by

$$\begin{aligned} & \sum_{i,j} b_i \otimes b_j \otimes \left( \sum_{k \geq -N_i} (f_k^{ij}, [p_k^{ij}])x^k, \sum_{m=0}^{n-1} (g_m^{ij}, [q_m^{ij}])[x]^m \right) \\ & \mapsto \text{span}_F \left\{ b_i (f_k^{ij}, [p_k^{ij}]) \mid k \geq -N, 1 \leq i, j \leq d \right\}. \end{aligned}$$

The following results are true:

1.  $W$  defines a bijection between series of type  $\left(n, \frac{1}{x^n\alpha(x)}\right)$  and subspaces  $V \subseteq L(n, \alpha)$  complementary to the diagonal  $\Delta$ , i.e.  $L(n, \alpha) = \Delta \dot{+} V$ ;
2. For any series  $r$  of type  $\left(n, \frac{1}{x^n\alpha(x)}\right)$  we have  $W(r)^\perp = W(\bar{r})$  inside  $L(n, \alpha)$ ;
3. Any series  $r$  of type  $\left(n, \frac{1}{x^n\alpha(x)}\right)$  satisfies  $\text{GCYB}(r) = \psi$ , where

$$\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x_1, [x_1]], (x_2, [x_2]), (x_3, [x_3])]]$$

is defined by

$$B(v_1 \otimes v_2 \otimes v_3, \psi) = B(v_1, [v_2, v_3])$$

for all  $v_1 \in W(\bar{r})$ ,  $v_2, v_3 \in W(r)$ .

**Proof** Fix an  $\left(n, \frac{1}{x^n\alpha(x)}\right)$ -type series inside  $(L(n, \alpha) \otimes \mathfrak{g})[[y, [y]]]$

$$\begin{aligned} r(x, y) &= \frac{1}{x^n\alpha(x)} \frac{y^n \Omega}{x - y} + g(x, y) \\ &= \sum_{k=0}^{\infty} \sum_{i=1}^d s_{k,i} \otimes b_i(y^k, [y]^k) + \sum_{k=0}^{\infty} \sum_{i=1}^d g_{k,i} \otimes b_i(y^k, [y]^k). \end{aligned}$$

It is easy to see that

$$U := \text{span}_F \{w_{k,i} \mid k \geq 0, 1 \leq k \leq d\} \subset L(n, \alpha),$$

where  $w_{k,i}$  are defined in Eq. (20), satisfies the condition  $\Delta \dot{+} U = L(n, \alpha)$ . Since  $s := \frac{1}{x^n \alpha(x)}$  is invertible, we have  $sU \dot{+} s\Delta = sU \dot{+} \Delta = L(n, \alpha)$ . In other words, the space

$$sU = \text{span}_F \{s_{k,i} = sw_{k,i} \mid k \geq 0, 1 \leq k \leq d\} \subset L(n, \alpha) \tag{26}$$

is also complementary to the diagonal. Finally, since  $g_{k,i} \in \Delta$  the space

$$W(r) = \text{span}_F \{sw_{k,i} + g_{k,i} \mid k \geq 0, 1 \leq k \leq d\} \subset L(n, \alpha)$$

is complementary to the diagonal. Conversely, if  $V \subset L(n, \alpha)$  satisfies  $V \dot{+} \Delta = L(n, \alpha)$ , then for each  $k \geq 0$  and  $1 \leq i \leq d$  we can find a unique  $g_{k,i} \in \Delta$  such that  $sw_{k,i} + g_{k,i} \in V$ . Define the  $(n, s)$  series  $r_V$  by

$$r_V(x, y) = \sum_{k \geq 0} \sum_{i=1}^d (sw_{k,i} + g_{k,i}) \otimes b_i(y^k, [y]^k).$$

It is now clear, that  $W(r_V) = V$ . These constructions establish the bijection in part 1.

To prove the second statement, observe that

$$B(sw_{k,i}, b_j(y^\ell, [y]^\ell)) = \delta_{i,j} \delta_{k,\ell}. \tag{27}$$

Furthermore, the straightforward calculation shows that

$$B(sw_{k,i}, sw_{\ell,j}) = \begin{cases} -\text{res}_0 \{sx^{(n-1)-k-\ell-1}\} & \text{if } i = j \text{ and } 0 \leq k, \ell \leq n - 1, \\ \text{res}_0 \{sx^{(n-1)-k-\ell-1}\} & \text{if } i = j \text{ and } k, \ell \geq n, \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} -s_{k+\ell-n+1} & \text{if } i = j, 0 \leq k, \ell \leq n - 1 \text{ and } k + \ell \geq n - 1, \\ s_{k+\ell-n+1} & \text{if } i = j \text{ and } k, \ell \geq n, \\ 0 & \text{otherwise,} \end{cases}$$

where  $s(x) = \sum_{k=0}^\infty s_k x^k$ . We write

$$\begin{aligned} \bar{r}(x, y) &= \frac{s(y)x^n \Omega}{x - y} - \tau(g(y, x)) = \frac{s(x)y^n \Omega}{x - y} - \frac{(s(x)y^n - s(y)x^n)\Omega}{x - y} - \tau(g(y, x)) \\ &= \sum_{k \geq 0} \sum_{i=1}^d (sw_{k,i} + \bar{g}_{k,i}) \otimes b_i(y^k, [y]^k). \end{aligned}$$

Consider the quotient

$$\begin{aligned} \frac{(s(x)y^n - s(y)x^n)\Omega}{x - y} &= \frac{y^n(s(x) - s(y))\Omega}{x - y} - \frac{s(y)(x^n - y^n)\Omega}{x - y} \\ &= \sum_{k \geq 0} \sum_{i=1}^d s_k \left( \sum_{\ell=1}^k b_i(x^{k-\ell}, [x]^{k-\ell}) \otimes b_i(y^{(n-1)+\ell}, [y]^{(n-1)+\ell}) \right. \\ &\quad \left. - \sum_{\ell=1}^n b_i(x^{n-\ell}, [x]^{n-\ell}) \otimes b_i(y^{k+\ell-1}, [y]^{k+\ell-1}) \right). \end{aligned}$$

The coefficient of  $b_i(x^k, [x]^k) \otimes b_i(y^\ell, [y]^\ell)$  in the expression above is

$$\begin{aligned} &-s_{k+\ell-(n-1)} \text{ if } 0 \leq k, \ell \leq n - 1 \text{ and } k + \ell \geq n - 1, \\ &s_{k+\ell-(n-1)} \text{ if } k, \ell \geq n, \end{aligned}$$

which coincides with  $B(sw_{k,i}, sw_{\ell,i})$ . If we now expand the coefficients  $g_{k,i}$  in the following way

$$g_{k,i} = \sum_{\ell \geq 0} \sum_{j=1}^d g_{k,i}^{\ell,j} b_j(x^\ell, [x]^\ell),$$

the coefficients  $\bar{g}_{k,i}$  can be rewritten as

$$\bar{g}_{k,i} = - \sum_{\ell \geq 0} \sum_{j=1}^d (g_{\ell,j}^{k,i} + B(sw_{k,i}, sw_{\ell,j})) b_j(x^k, [x]^k) \otimes b_j(y^\ell, [y]^\ell).$$

Combining all the results above we obtain the desired equality

$$\begin{aligned} B(sw_{k,i} + g_{k,i}, sw_{\ell,j} + \bar{g}_{\ell,j}) &= B(sw_{k,i}, sw_{\ell,j}) + B(sw_{k,i}, \bar{g}_{\ell,j}) \\ &\quad + B(g_{k,i}, sw_{\ell,j}) + B(g_{k,i}, \bar{g}_{\ell,j}) \\ &= B(sw_{k,i}, sw_{\ell,j}) + (-g_{k,i}^{\ell,j} - B(sw_{k,i}, sw_{\ell,j})) + g_{k,i}^{\ell,j} + 0 \\ &= 0 \end{aligned}$$

which completes the proof of the second statement.

Using the same technique as in [2, Section 1], one can prove that

$$\psi := \text{GCYB}(r) \in (\Delta \otimes \mathfrak{g} \otimes \mathfrak{g})[[x_2, [x_2]], (x_3, [x_3])]]$$

for any series  $r$  of type  $(n, s)$ . Define  $r_{k,i} := sw_{k,i} + g_{k,i}$  and  $\bar{r}_{k,i} := sw_{k,i} + \bar{g}_{k,i}$  and rewrite GCYB( $r$ ) as

$$\begin{aligned} \psi &= \sum_{k, \ell \geq 0} \sum_{i, j=1}^d [r_{k,i}, r_{\ell,j}] \otimes b_i(x_2^k, [x_2]^k) \otimes b_j(x_3^\ell, [x_3]^\ell) \\ &+ \sum_{k \geq 0} \sum_{i=1}^d r_{k,i} \otimes \left( [b_i(x_2^k, [x_2]^k) \otimes (1, 1), r(x_2, x_3)] \right. \\ &\left. + [(1, 1) \otimes b_i(x_3^k, [x_3]^k), \bar{r}(x_2, x_3)] \right). \end{aligned} \tag{28}$$

Applying  $B(\bar{r}_{k_1, i_1} \otimes r_{k_2, i_2} \otimes r_{k_3, i_3}, -)$  to the equation above, we get

$$B(\bar{r}_{k_1, i_1} \otimes r_{k_2, i_2} \otimes r_{k_3, i_3}, \psi) = B(\bar{r}_{k_1, i_1}, [r_{k_2, i_2}, r_{k_3, i_3}]). \tag{29}$$

This gives the last statement because  $W(r)$  and  $W(\bar{r})$  are generated by  $r_{k,i}$  and  $\bar{r}_{k,i}$  respectively. □

**Corollary 3.7** *Let  $n \in \mathbb{Z}_{\geq 0}$ ,  $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$  and  $W$  be as in Theorem 3.6. Then*

1.  $W$  defines a bijection between skew-symmetric  $\left(n, \frac{1}{x^n \alpha(x)}\right)$ -type series and Lagrangian subspaces  $V \subseteq L(n, \alpha)$  complementary to the diagonal  $\Delta$ ;
2.  $W$  defines a bijection between  $\left(n, \frac{1}{x^n \alpha(x)}\right)$ -type series solving GCYBE and subalgebras  $V \subseteq L(n, \alpha)$  complementary to the diagonal  $\Delta$ .

As we can see from the proof of Theorem 3.6 the element  $\psi$  in  $\text{GCYB}(r) = \psi$  represents the obstruction for  $W(r)$  from being a Lie subalgebra. This observation raises an interesting question that we do not consider in this paper: what elements  $\psi$  can appear on the right-hand side of the above-mentioned equation.

Observe that if  $r$  is a series of type  $(n, s)$  and it satisfies

$$\begin{aligned} \text{CYB}(r) &:= [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] \\ &+ [r^{13}(x_1, x_3), r^{23}(x_2, x_3)] = \psi \end{aligned} \tag{30}$$

for some  $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]]$ , then  $r$  is automatically skew-symmetric and hence solves the first equation as well. To prove that one can e.g. repeat the argument from [1, Lemma 5.2]. In other words, for a fixed  $\psi$  solutions to  $\text{CYB}(r) = \psi$  form a subclass of solutions to  $\text{GCYB}(r) = \psi$ . In particular, solutions to  $\text{CYB}(r) = 0$  are exactly the skew-symmetric solutions to  $\text{GCYB}(r) = 0$ . We call the equation  $\text{CYB}(r) = \psi$  *Manin-Yang-Baxter equation*.

**Remark 3.8** As our notation suggest, we could have interpreted  $y^n \Omega / (x - y)$  as

$$\frac{y^n \Omega}{x - y} = \sum_{k \geq 0} \sum_{i=1}^d b_i x^{-k-1} \otimes b_i y^{n+k} \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]$$

and performed all the arithmetic calculations in this form. To restore an  $(n, s)$ -type series from

$$\frac{s(x)y^n\Omega}{x-y} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]] \quad (31)$$

we can simply view  $s(x) \in F[[x]]^\times$  and  $g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$  as elements in  $F[[x, [x]]]^\times$  and  $(\mathfrak{g} \otimes \mathfrak{g})[[x, [x]], (y, [y])]$  respectively and reinterpret the singular part  $y^n\Omega/(x-y)$  as it was done in Eq. (20).

Conversely, to get a series of the form Eq. (31) from a series of type  $(n, s)$  we can just project the latter onto the first component.

In other words, we have a bijection between  $(n, s)$ -type series in  $L_2(n, \alpha)$  and their projections Eq. (31) onto the first component given by different interpretations of the singular part  $y^n\Omega/(x-y)$ .

Although, all arithmetic operations can be performed in the form Eq. (31), the construction of  $W(r)$  and statements like  $\Delta \cap W(r) = 0$  require us to pass to the interpretation Eq. (20). This is our main motivation to work directly with  $(n, s)$ -type series in  $L_2(n, \alpha)$  instead of their projections.

In view of Remark 3.8, we have a new proof of [1, Corollary 5.5].

**Corollary 3.9** *Classical (formal)  $r$ -matrices, i.e. skew-symmetric elements*

$$\frac{s(x)y^n\Omega}{x-y} + g(x, y) = \frac{1}{x^n\alpha(x)} \frac{y^n\Omega}{x-y} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]], \quad (32)$$

*solving GCYBE, are in bijection with skew-symmetric series of type  $(n, s)$  solving GCYBE and hence in bijection with Lagrangian Lie subalgebras of  $L(n, \alpha)$  complementary to the diagonal  $\Delta$ .*

The result of [1, Theorem 5.6] can be now formulated in the following way.

**Corollary 3.10** *Skew-symmetric series of type  $\left(n, \frac{1}{x^n\alpha(x)}\right)$  that also solve GCYBE exist only for  $n = 0, 1$  and  $n = 2$  with  $\alpha_0 = 0$ .*

## 4 Quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$

We remind that  $F$  is a discrete algebraically closed field of characteristic 0 and  $\mathfrak{g}[[x]]$  is an  $F$ -Lie algebra equipped with the  $(x)$ -adic topology.

As we now know, series of type  $(n, 1/(x^n\alpha(x)))$  solving CYBE Eq. (30) are in bijection with Lagrangian subalgebras  $W \subset L(n, \alpha)$  complementary to the diagonal. On the other hand, such Lagrangian subalgebras are in bijection with non-degenerate topological Lie bialgebra structures. See [1] for their definition and classification.

It turns out, that if we drop the condition on  $W$  being a subalgebra, we get so called (non-degenerate) topological quasi-Lie bialgebras. This section is devoted to their classification up to topological twists and coordinate transformations.

**Definition 4.1** A topological quasi-Lie bialgebra structure on  $\mathfrak{g}[[x]]$  consists of

- a skew-symmetric continuous linear map  $\delta: \mathfrak{g}[[x]] \rightarrow (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$  and
- a skew-symmetric element  $\varphi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]]$ ,

which are subject to the following conditions

1.  $\delta([a, b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] - [b \otimes 1 + 1 \otimes b, \delta(a)]$ , i.e.  $\delta$  is a 1-cocycle;
2.  $\frac{1}{2}\text{Alt}((\delta \otimes 1)\delta(a)) = [a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a, \varphi]$ ;
3.  $\text{Alt}((\delta \otimes 1 \otimes 1)\varphi) = 0$ ,

where  $\text{Alt}(x_1 \otimes \dots \otimes x_n) := \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$ .

**Lemma 4.2** There is a one-to-one correspondence between triples  $(L, \mathfrak{g}[[x]], W)$ , where  $(L, \mathfrak{g}[[x]])$  is a topological Manin pair and  $W \subset L$  is a Lagrangian subspace satisfying  $W \dot{+} \mathfrak{g}[[x]] = L$ , and quasi-Lie bialgebra structures on  $\mathfrak{g}[[x]]$ .

**Proof** We start with a topological Manin pair  $(L, \mathfrak{g}[[x]])$ . If  $W \subset L$  is a Lagrangian subspace complementary to  $\mathfrak{g}[[x]]$ , then it is easy to see that  $W \cong \mathfrak{g}[[x]]'$ . Therefore, we have an isomorphism of vector spaces

$$L \cong \mathfrak{g}[[x]] \dot{+} \mathfrak{g}[[x]]'.$$

The form on  $L$  under this isomorphism becomes standard evaluation form  $\langle -, - \rangle$  on  $\mathfrak{g}[[x]] \dot{+} \mathfrak{g}[[x]]'$ . We fix such an isomorphism.

Let us define two linear functions

$$p_1: \mathfrak{g}[[x]]' \otimes \mathfrak{g}[[y]]' \rightarrow \mathfrak{g}[[x]] \text{ and } p_2: \mathfrak{g}[[x]]' \otimes \mathfrak{g}[[y]]' \rightarrow \mathfrak{g}[[x]]'$$

by  $[f, g] = p_1(f \otimes g) + p_2(f \otimes g)$ . We put

$$\delta := p_2^\vee: (\mathfrak{g}[[x]]')^\vee \cong \mathfrak{g}[[x]] \rightarrow (\mathfrak{g}[[x]]' \otimes \mathfrak{g}[[y]]')^\vee \cong (\mathfrak{g} \otimes \mathfrak{g})[[x, y]],$$

and let  $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]]$  be the unique element satisfying the condition

$$\langle h, [f, g] \rangle = \langle h, p_1(f \otimes g) \rangle = \langle f \otimes g \otimes h, \psi \rangle \text{ for all } f, g, h \in \mathfrak{g}[[x]]'. \tag{33}$$

The skew-symmetry of  $p_2$  implies the skew-symmetry of  $\delta$ , whereas the skew-symmetry of  $p_1$  and the invariance of the evaluation form yield the skew-symmetry of  $\psi$ .

Next, we observe that for all  $a, b \in \mathfrak{g}[[x]]$  and  $f, g \in \mathfrak{g}[[x]]'$  we have

$$\begin{aligned} \langle [a, f], g \rangle &= \langle a, [f, g] \rangle = \langle a, p_2(f \otimes g) \rangle = \langle \delta(a), f \otimes g \rangle = \langle (f \otimes 1)\delta(a), g \rangle, \\ \langle [a, f], b \rangle &= -\langle f, [a, b] \rangle = -\langle f \circ \text{ad}_a, b \rangle. \end{aligned}$$

In other words, the invariance of the form forces the following equality to hold

$$[a, f] = -f \circ \text{ad}_a + (f \otimes 1)\delta(a). \tag{34}$$

Using Eq. (34) and non-degeneracy of the form we show that  $\delta$  is a 1-cocycle:

$$\begin{aligned}
 \langle \delta([a, b]), f \otimes g \rangle &= \langle [a, b], p_2(f \otimes g) \rangle = \langle [a, b], [f, g] \rangle = \langle [[a, b], f], g \rangle \\
 &= \langle -[[b, f], a] - [[f, a], b], g \rangle \\
 &= \langle [f \circ \text{ad}_b - (f \otimes 1)\delta(b), a] - [f \circ \text{ad}_a - (f \otimes 1)\delta(a), b], g \rangle \\
 &= -\langle a, [f \circ \text{ad}_b, g] \rangle + \langle b, [f \circ \text{ad}_a, g] \rangle + \langle (f \otimes \text{ad}_a)\delta(b), g \rangle \\
 &\quad - \langle (f \otimes \text{ad}_b)\delta(a), g \rangle \\
 &= \langle [a \otimes 1 + 1 \otimes a, \delta(b)] - [b \otimes 1 + 1 \otimes b, \delta(a)], f \otimes g \rangle.
 \end{aligned} \tag{35}$$

The 1-cocycle condition implies that  $\delta$  is continuous as it was noted in [1, Remark 3.16].

For conditions 2 and 3 from the definition of a topological quasi-Lie bialgebra consider the Jacobi identity for  $f, g, h \in \mathfrak{g}[[x]]'$ :

$$\begin{aligned}
 0 &= [p_1(f \otimes g), h] + [p_1(g \otimes h), f] + [p_1(h \otimes f), g] \\
 &\quad + p_1(p_2(f \otimes g) \otimes h) + p_1(p_2(g \otimes h) \otimes f) + p_1(p_2(h \otimes f) \otimes g) \\
 &\quad + p_2(p_2(f \otimes g) \otimes h) + p_2(p_2(g \otimes h) \otimes f) + p_2(p_2(h \otimes f) \otimes g).
 \end{aligned} \tag{36}$$

We denote by  $\circ$  the summation over circular permutations of symbols  $f, g$  and  $h$ , e.g.  $\circ \langle p_1(f \otimes g), h \rangle = \langle p_1(f \otimes g), h \rangle + \langle p_1(g \otimes h), f \rangle + \langle p_1(h \otimes f), g \rangle$ . Applying  $\langle -, a \rangle$  to Eq. (36) for an arbitrary  $a \in \mathfrak{g}[[x]]'$  gives

$$\begin{aligned}
 \langle p_2(p_2 \otimes 1)(\circ f \otimes g \otimes h), a \rangle &= -\langle \circ [p_1(f \otimes g), h], a \rangle \\
 \langle p_2 \otimes 1(\circ f \otimes g \otimes h), \delta(a) \rangle &= \circ \langle -h \circ \text{ad}_a, p_1(f \otimes g) \rangle \\
 \langle \circ f \otimes g \otimes h, (\delta \otimes 1)\delta(a) \rangle &= \circ \langle f \otimes g \otimes (-h \circ \text{ad}_a), \psi \rangle \\
 \langle f \otimes g \otimes h, \text{Alt}((\delta \otimes 1)\delta(a))/2 \rangle &= -\langle f \otimes g \otimes h, [1 \otimes 1 \otimes a + 1 \otimes a \otimes 1 \\
 &\quad + a \otimes 1 \otimes 1, \psi] \rangle,
 \end{aligned}$$

where the very last identity holds because of the skew-symmetry of  $\psi$ . Multiplying this equality by 2 we get the relation

$$\langle f \otimes g \otimes h, \text{Alt}((\delta \otimes 1)\delta(a)) + 2[1 \otimes 1 \otimes a + 1 \otimes a \otimes 1 + a \otimes 1 \otimes 1, \psi] \rangle = 0.$$

Letting  $\varphi := -\psi$  we obtain the second identity from the definition of a topological quasi-Lie bialgebra structure. Applying instead  $\langle s, - \rangle, s \in \mathfrak{g}[[x]]'$  to the Jacobi identity Eq. (36) we get the desired

$$\text{Alt}((\delta \otimes 1 \otimes 1)\psi) = 0.$$

Therefore,  $(\mathfrak{g}[[x]], \delta, \varphi)$  is a topological quasi-Lie bialgebra.

For the converse direction, we put  $L := \mathfrak{g}[[x]] \dot{+} \mathfrak{g}[[x]]'$  with the standard evaluation form; we let  $p_1$  be the unique element in  $\text{Hom}_{F\text{-Vect}}(\mathfrak{g}[[x]]' \otimes \mathfrak{g}[[x]]', \mathfrak{g}[[x]])$  satisfying

Eq. (33) with  $\psi := -\varphi$ ; we define  $p_2 := \delta'$ , i.e. the dual map of  $\delta$ . The Lie bracket between two elements in  $\mathfrak{g}[[x]]'$  is given by the sum  $p_1 + p_2$ . Defining  $[a, f]$  as in Eq. (34) the evaluation form becomes invariant and we get a topological Manin pair  $(L, \mathfrak{g}[[x]])$  with the Lagrangian subspace  $\mathfrak{g}[[x]]'$ . These constructions are clearly inverse to each other. □

Combining the classification of Manin pairs mentioned in Sect. 2 with Corollary 3.7 and Lemma 4.2 we get the following description of all topological quasi-Lie bialgebra structures on  $\mathfrak{g}[[x]]$ .

**Lemma 4.3** *There is a bijection between topological quasi-Lie bialgebra structures on  $\mathfrak{g}[[x]]$  and Lagrangian subspaces  $W \subset L(n, \alpha)$  or  $L(\infty)$  complementary to the diagonal  $\Delta$ , where  $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$  is an arbitrary sequence and  $n \geq 0$ . Moreover, such Lagrangian subspaces  $W \subset L(n, \alpha)$  are in bijection with skew-symmetric sequences of type  $(n, 1/(x^n\alpha(x)))$ .*

In view of this result we call skew-symmetric series of type  $(n, s)$  as well as their projections onto the first component *quasi-r-matrices*. Quasi-Lie bialgebra structures can also be described using their associated quasi-r-matrices in the following way.

**Proposition 4.4** *Assume  $(\mathfrak{g}[[x]], \delta, \varphi)$  is a topological quasi-Lie bialgebra and let  $r \in L_2(n, \alpha)$  be the corresponding quasi-r-matrix given by the bijection from Lemma 4.3. Under the identification  $\mathfrak{g}[[x], [x]] \cong \mathfrak{g}[[x]]$  we have the following identities:*

- $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$  for any  $a \in \mathfrak{g}[[x]]$  and
- $\text{CYB}(r) = -\varphi$ .

The same is true for the projection  $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]$ .

**Proof** We start, as in the proof of Lemma 4.2, by fixing an identification  $L(n, \alpha) = \Delta \dot{+} W(r) \cong \mathfrak{g}[[x]] \dot{+} \mathfrak{g}[[x]]'$ . Let  $\{v_{k,i}\}$  be a basis for  $\mathfrak{g}[[x]]'$  dual to  $\{\varepsilon_{k,i} := b_i y^k\}$ . Then  $r = \sum_{k \geq 0} \sum_{i=1}^d v_{k,i} \otimes \varepsilon_{k,i}$  and we have

$$\begin{aligned} [a \otimes 1 + 1 \otimes a, r] &= \sum_{k \geq 0} \sum_{i=1}^d [a, v_{k,i}] \otimes \varepsilon_{k,i} + v_{k,i} \otimes [a, \varepsilon_{k,i}] \\ &= \sum_{k \geq 0} \sum_{i=1}^d (-v_{k,i} \circ \text{ad}_a + (v_{k,i} \otimes 1)\delta(a)) \otimes \varepsilon_{k,i} + v_{k,i} \otimes [a, \varepsilon_{k,i}]. \end{aligned}$$

Applying  $\langle v_{\ell,j} \otimes v_{m,t}, - \rangle$  to the equality above we get

$$\begin{aligned} \langle v_{\ell,j} \otimes v_{m,t}, [a \otimes 1 + 1 \otimes a, r] \rangle &= \sum_{k \geq 0} \sum_{i=1}^d \langle v_{\ell,j} \otimes v_{m,t}, (v_{k,i} \otimes 1)\delta(a) \otimes \varepsilon_{k,i} \rangle \\ &= \langle v_{\ell,j}, (v_{m,t} \otimes 1)\delta(a) \rangle \\ &= \langle v_{\ell,j} \otimes v_{m,t}, -\delta(a) \rangle. \end{aligned}$$

Applying instead  $\langle \varepsilon_{\ell,j} \otimes v_{m,t}, - \rangle$  to the same equality we obtain

$$\begin{aligned} \langle \varepsilon_{\ell,j} \otimes v_{m,t}, [a \otimes 1 + 1 \otimes a, r] \rangle &= \sum_{k \geq 0} \sum_{i=1}^d \langle \varepsilon_{\ell,j} \otimes v_{m,t}, (-v_{k,i} \circ \text{ad}_a) \otimes \varepsilon_{k,i} \\ &\quad + v_{k,i} \otimes [a, \varepsilon_{k,i}] \rangle \\ &= -\langle \varepsilon_{\ell,j}, v_{m,t} \circ \text{ad}_a \rangle + \langle v_{m,t}, [a, \varepsilon_{\ell,j}] \rangle \\ &= 0. \end{aligned}$$

This implies the desired equality  $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$ . The identity  $\text{CYB}(r) = -\varphi$  follows from the skew-symmetry of  $r$ , Theorem 3.6 and the fact that  $\varphi = -\psi$  according to the proof of Lemma 4.2.  $\square$

**Remark 4.5** Assume  $r \in (\mathfrak{g} \otimes \mathfrak{g})(\llbracket x \rrbracket) \llbracket y \rrbracket$  is a series such that

$$[f(x) \otimes 1 + 1 \otimes f(y), r(x, y)] \in (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket \tag{37}$$

for all  $f \in \mathfrak{g} \llbracket x \rrbracket$ . Write  $r = s(x^{-1}, y) + g(x, y)$ , where  $s \in x^{-1}(\mathfrak{g} \otimes \mathfrak{g})[x^{-1}] \llbracket y \rrbracket$  and  $g \in (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$ . Then, because of Eq. (37), we must have

$$[a \otimes 1 + 1 \otimes a, s(x^{-1}, y)] = 0$$

for all  $a \in \mathfrak{g}$ . Since the  $\mathfrak{g}$ -invariant elements of  $\mathfrak{g} \otimes \mathfrak{g}$  are precisely the multiples of the quadratic Casimir element  $\Omega$ , we have the identity  $s(x^{-1}, y) = p(x^{-1}, y)\Omega$  for some  $p \in x^{-1}F[x^{-1}] \llbracket y \rrbracket$ . Furthermore, the condition

$$[ax \otimes 1 + 1 \otimes ay, p(x^{-1}, y)\Omega] = [a(x - y) \otimes 1, p(x^{-1}, y)\Omega] \in (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$$

implies  $(x - y)p(x^{-1}, y) \in F \llbracket x, y \rrbracket$ , meaning that there exists an  $s \in F \llbracket y \rrbracket$  such that  $p(x^{-1}, y) = s(y)/(x - y)$ . In other words,  $r$  has the form Eq. (21). This result can be considered as another motivation to study series of type  $(n, s)$ .

Observe that if we know one Lagrangian subspace  $W_0$  inside  $L \cong \mathfrak{g} \llbracket x \rrbracket + \mathfrak{g} \llbracket x \rrbracket'$  then any other Lagrangian subspace can be constructed from  $W_0$  through twisting. More precisely, if  $s = \sum_i a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$  is a skew-symmetric tensor, then we can associate with it a (twisted) Lagrangian subspace

$$W_s := \left\{ \sum_i B(b^i, w)a_i - w \mid w \in W \right\} \subseteq L \tag{38}$$

complementary to  $\mathfrak{g} \llbracket x \rrbracket$ . The converse is also true; for proof see [3]. In other words, the following statement holds.

**Lemma 4.6** *There is a bijection between Lagrangian subspaces  $W \subseteq L(n, \alpha)$  or  $L(\infty)$  and skew-symmetric tensors in  $(\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$ .*

Combining Lemma 4.4, Eq. (38) and the algorithm for constructing a quasi- $r$ -matrix from a Lagrangian subspace  $W \subset L(n, \alpha)$ ,  $W \dot{+} \Delta = L(n, \alpha)$ , we obtain the following twisting rules for Lagrangian subspaces, quasi-Lie bialgebra structures and quasi- $r$ -matrices.

**Lemma 4.7** *Let  $(\mathfrak{g}[[x]], \delta, \varphi)$  be a topological quasi-Lie bialgebra structure corresponding to the quasi- $r$ -matrix  $r$ . If we twist  $W(r)$  with a skew-symmetric tensor  $s$  as described in Eq. (38) we obtain another topological quasi-Lie bialgebra  $(\mathfrak{g}[[x]], \delta_s, \varphi_s)$ , such that*

1.  $W(r)_s = W(r - s)$ ;
2.  $\delta_s = \delta + ds$ ;
3.  $\varphi_s = \varphi + \text{CYB}(s) - \frac{1}{2}\text{Alt}((\delta \otimes 1)s)$ ,

where  $ds(a) := [a \otimes 1 + 1 \otimes a, s]$ .

**Remark 4.8** Since any quasi- $r$ -matrix  $r$  defines a topological quasi-Lie bialgebra structure  $\delta(a) = [a \otimes 1 + 1 \otimes a, r]$  on  $\mathfrak{g}[[x]]$ , the third condition in Definition 4.1 is trivially satisfied. In other words,

$$\text{Alt}((\delta \otimes 1 \otimes 1)\text{CYB}(r)) = 0$$

for any quasi- $r$ -matrix  $r$ .

Lemma 4.6 and Lemma 4.7 state that, in order to obtain a description of topological quasi-Lie bialgebra structures on  $\mathfrak{g}[[x]]$  up to twisting it is enough to find a single Lagrangian subspace  $W_0$ , complementary to  $\mathfrak{g}[[x]]$ , inside  $L(\infty)$  and each  $L(n, \alpha)$ . The same is true for the associated quasi- $r$ -matrices

The case  $L(\infty)$  is trivial, because by definition  $\mathfrak{g}[[x]]' = \bigoplus_{j \geq 0} \mathfrak{g} \otimes a_j \subseteq L(\infty)$  is a Lagrangian subalgebra (see Lemma 2.1). Similar to the Lie bialgebra case, topological quasi-Lie bialgebras corresponding to the Manin pair  $(L(\infty), \mathfrak{g}[[x]])$  are called *degenerate*.

Let us now focus on *non-degenerate* topological quasi-Lie bialgebra structures, i.e. the ones corresponding to the Manin pairs  $(L(n, \alpha), \Delta)$ . By Proposition 2.5 for each Manin pair  $(L(n, \alpha), \Delta)$  there exists an appropriate coordinate transformation that makes it into  $(L(n, \beta), \Delta)$ , where  $\beta_0 = \alpha_0$  and all other  $\beta_i = 0$ . This means, that to classify all non-degenerate topological quasi-Lie bialgebras on  $\mathfrak{g}[[x]]$ , up to coordinate transformations and twisting, it is enough to construct a Lagrangian subspace  $W_0$  within each  $L(n, \alpha_0) := L(n, (\dots, 0, \alpha_0, 0, \dots, 0))$  complementary to  $\Delta$ . Equivalently, it is enough to find a quasi- $r$ -matrix of type  $(n, \alpha_0)$  for any  $n \geq 0$  and  $\alpha_0 \in F$ .

### 4.1 Lagrangian subspaces of $L(n, \alpha_0)$

As before we let  $\{b_i\}_{i=1}^d$  be an orthonormal basis for  $\mathfrak{g}$  with respect to the Killing form  $\kappa$ . The form  $B$  on  $L(n, \alpha_0)$  has the following explicit form

$$\begin{aligned}
 & B(a \otimes (f, [p]), b \otimes (g, [q])) \\
 &= \begin{cases} \kappa(a, b) \{ \text{coeff}_{n-1}(fg - pq) - \alpha_0 \text{coeff}_0(fg - pq) \} & \text{if } n \geq 2, \\ \kappa(a, b) \text{coeff}_{n-1}(fg - pq) & \text{if } n = 0, 1. \end{cases}
 \end{aligned} \tag{39}$$

We now present an explicit construction for a Lagrangian subspace of  $L(n, \alpha_0)$  complementary to  $\Delta$  for arbitrary  $n \geq 0$  and  $\alpha_0 \in F$ . Using the twisting procedure from Lemma 4.7, this subspace can be twisted in order to obtain all other Lagrangian subspaces of  $L(n, \alpha_0)$  complementary to  $\Delta$ .

$n = 0$ : When  $n = 0$ , the subalgebra  $W_0 := x^{-1}\mathfrak{g}[[x^{-1}]] \subseteq \mathfrak{g}((x))$  is known to be Lagrangian.

$n = 1$ : For  $n = 1$  it is easy to see that the subspace

$$W_0 := \text{span}_F \{ b_i(1, -1), b_i(x^{-k}, 0) \mid k \geq 1, 1 \leq i \leq d \} \subset L(1, \alpha_0) \tag{40}$$

is Lagrangian and complementary to the diagonal  $\Delta$ .

$n = 2k$ : For even  $n \geq 2$  and arbitrary  $\alpha_0 \in F$  the subspace  $W_0 \subset L(n, \alpha_0)$  spanned by the elements

$$\begin{aligned}
 & b_i \left\{ (x^{(n-1)-m}, 0) - \alpha_0 (x^{2(n-1)-m}, 0) + \alpha_0^2 (x^{3(n-1)-m}, 0) - \alpha_0^3 (x^{4(n-1)-m}, 0) + \dots \right\}, \\
 & 0 \leq m \leq \frac{n}{2} - 1, \\
 & b_i \left( 0, -[x]^{(n-1)-\ell} \right), \quad \frac{n}{2} \leq \ell < n - 1, \\
 & b_i \left( 0, -1 + \frac{\alpha_0}{2} [x]^{n-1} \right), \\
 & b_i(x^{-k}, 0), k \geq 1,
 \end{aligned}$$

is Lagrangian and complementary to the diagonal.

$n = 2k + 1$ : Modifying slightly the basis for even case we obtain the following basis for  $W_0 \subset L(n, \alpha_0)$  with odd  $n \geq 3$ :

$$\begin{aligned}
 & b_i \left\{ (x^{(n-1)-m}, 0) - \alpha_0 (x^{2(n-1)-m}, 0) + \alpha_0^2 (x^{3(n-1)-m}, 0) - \alpha_0^3 (x^{4(n-1)-m}, 0) + \dots \right\}, \\
 & 0 \leq m \leq \frac{n-1}{2} - 1, \\
 & b_i \left\{ \left( x^{\frac{n-1}{2}}, -[x]^{\frac{n-1}{2}} \right) - \alpha_0 \left( x^{\frac{3(n-1)}{2}}, 0 \right) + \alpha_0^2 \left( x^{\frac{5(n-1)}{2}}, 0 \right) - \alpha_0^3 \left( x^{\frac{7(n-1)}{2}}, 0 \right) + \dots \right\}, \\
 & b_i(0, -[x]^{(n-1)-\ell}), \quad \frac{n-1}{2} + 1 \leq \ell < n - 1, \\
 & b_i \left( 0, -1 + \frac{\alpha_0}{2} [x]^{n-1} \right), \\
 & b_i(x^{-k}, 0), k \geq 1.
 \end{aligned}$$

The subspaces above were constructed by "guessing". However, there is an abstract procedure that produces Lagrangian subspaces for arbitrary  $n$  and  $\alpha$ . We present it here for completeness.

The easiest skew-symmetric  $(n, s)$ -type series is given by

$$\begin{aligned}
 r(x, y) &:= \frac{1}{2} \left( \frac{s(x)y^n \Omega}{x-y} + \frac{s(y)x^n \Omega}{x-y} \right) = \frac{s(x)y^n \Omega}{x-y} + \frac{\Omega}{2} \left( \frac{s(y)x^n - s(x)y^n}{x-y} \right) \\
 &= \frac{s(x)y^n \Omega}{x-y} - \frac{1}{2} \sum_{k,\ell=0}^{\infty} \sum_{i,j=1}^d B(sw_{k,i}, sw_{\ell,j}) b_i(x, [x]^k) \otimes b_j(y, [y]^\ell),
 \end{aligned}$$

where we recall that

$$B(sw_{k,i}, sw_{\ell,j}) = \begin{cases} -s_{k+\ell-n+1} & \text{if } i = j, 0 \leq k, \ell \leq n-1 \text{ and } k + \ell \geq n-1, \\ s_{k+\ell-n+1} & \text{if } i = j \text{ and } k, \ell \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 3.7 the subspace

$$\begin{aligned}
 W(r) &= \text{span}_F \left\{ sw_{k,i} - \frac{1}{2} \sum_{\ell=0}^{\infty} B(sw_{\ell,i}, sw_{k,i}) b_i(x, [x]^\ell) \mid k \geq 0, 1 \leq d \leq n \right\} \\
 &= \text{span}_F \left\{ sw_{k,i} + \frac{1}{2} \left( \sum_{\ell=0}^{n-1} s_{k+\ell-n+1} b_i(x, [x]^\ell) - \sum_{\ell=n}^{\infty} s_{k+\ell-n+1} b_i(x, [x]^\ell) \right) \mid \right. \\
 &\quad \left. k \geq 0, 1 \leq d \leq n \right\}
 \end{aligned}$$

is Lagrangian and complementary to the diagonal. Here we used the convention that  $s_k = 0$  for  $k < 0$ . Calculating the basis explicitly for some particular  $s$  requires some effort and it may not look as friendly as the ones given above.

### 4.2 Quasi- $r$ -matrices

The goal of this section is to describe the quasi- $r$ -matrices corresponding to the Lagrangian subspaces described in the previous section. The twisting procedure from Lemma 4.7 then yields all other quasi- $r$ -matrices.

The proof of Theorem 3.6 gives us an algorithm for constructing a series of type  $(n, s(x) := 1/(x^n \alpha(x)))$  from a subspace  $W \subset L(n, \alpha)$  complementary to the diagonal. More precisely, the desired series is given by

$$\sum_{k \geq 0} \sum_{i=1}^d v_{k,i} \otimes b_i(y^k, [y]^k), \tag{41}$$

where

$$W = \text{span}_F\{v_{k,i} \mid k \geq 0, 1 \leq i \leq d\} \text{ and } B(v_{k,i}, b_j(y^\ell, [y]^\ell)) = \delta_{i,j} \delta_{k,\ell},$$

i.e.  $\{v_{k,i}\}$  is a basis of  $V$  dual to  $\{b_i(y^k, [y]^k)\}$ . Indeed, non-degeneracy of the form  $B$  then implies that  $v_{k,i}$  has the desired form  $v_{k,i} = sw_{k,i} + g_{k,i}$  for some  $g_{k,i} \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ .

Applying this idea to  $W_0$ 's constructed in the preceding section we get the following series.

$n = 0$ : The classical  $r$ -matrix (equivalently  $(0, 1)$ -type series) corresponding to  $W_0 := x^{-1} \mathfrak{g}[[x^{-1}]] \subseteq \mathfrak{g}((x))$  is the Yang's matrix  $\Omega/(x - y)$ .

$n = 1$ : The quasi- $r$ -matrix corresponding to  $\text{span}_F\{b_i(1, -1), b_i(x^{-k}, 0) \mid k \geq 1, 1 \leq i \leq d\} \subset L(1, \alpha_0)$  is

$$\begin{aligned} & \frac{y\Omega}{x - y} + \frac{1}{2} \sum_{i=1}^d b_i(1, -1) \otimes b_i(1, 1) \in L_2(1, 1) \text{ with the projection} \\ & \frac{y\Omega}{x - y} + \frac{1}{2} \Omega \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]. \end{aligned}$$

$n = 2k$ : For even  $n \geq 2$  and arbitrary  $\alpha_0 \in F$  we have the following quasi- $r$ -matrix

$$\begin{aligned} & \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{\Omega}{1 + \alpha_0 x^{n-1}} \sum_{0 \leq m < \frac{n}{2}} x^{(n-1)-m} y^m \\ & + \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})} \left( y^{2(n-1)} \right. \\ & \left. + \sum_{\frac{n}{2} \leq \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} - \frac{1}{2} x^{n-1} y^{n-1} \right). \end{aligned}$$

$n = 2k + 1$ : In the odd case  $n \geq 3$  the series corresponding to  $W_0 \subset L(n, \alpha_0)$  is

$$\begin{aligned} & \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{\Omega}{1 + \alpha_0 x^{n-1}} \left( x^{\frac{n-1}{2}} y^{\frac{n-1}{2}} + \sum_{0 \leq m < \frac{n-1}{2}} x^{(n-1)-m} y^m \right) \\ & + \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})} \left( y^{2(n-1)} \right. \\ & \left. + \sum_{\frac{n-1}{2} < \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} - \frac{1}{2} x^{n-1} y^{n-1} \right). \end{aligned}$$

### 5 Lie algebra splittings of $L(n, \alpha)$ and generalized $r$ -matrices

By Corollary 3.7 we have a bijection between subalgebras of  $L(n, \alpha)$  and series of type  $(n, 1/(x^n\alpha(x)))$  solving GCYBE. Therefore, we can construct new solutions to GCYBE by finding subalgebras of  $L(n, \alpha)$  complementary to the diagonal. However, as the following result shows, the most interesting new solutions should arise from unbounded subalgebras of  $L(n, \alpha)$ ,  $n > 2$ .

**Proposition 5.1** *Let  $L(n, \alpha) = \Delta \dot{+} W$  for some subalgebra  $W \subset L(n, \alpha)$  and  $n > 2$ . Assume  $W$  is bounded, i.e. there is an integer  $N > 0$  such that*

$$x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}],$$

where  $W_+$  is the projection of  $W \subset L(n, \alpha) = \mathfrak{g}((x)) \oplus \mathfrak{g}[x]/x^n\mathfrak{g}[x]$  on the first component  $\mathfrak{g}((x))$ . Then there is an element  $\sigma \in \text{Aut}_{F[x]-\text{LieAlg}}(\mathfrak{g}[x])$  such that

$$\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq (\sigma \times \sigma)W \subseteq x\mathfrak{g}[x^{-1}] \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$$

and the image  $\tilde{W}$  under the canonical projection  $L(n, \alpha) \rightarrow L(2, \alpha)$  is a subalgebra satisfying  $L(2, \alpha) = \Delta \dot{+} \tilde{W}$ .

In the language of  $(n, s)$ -type series: Let

$$r = \frac{s(x)y^n\Omega}{x - y} + g(x, y)$$

be the generalized  $r$ -matrix corresponding to a bounded  $W \subset L(n, \alpha)$ ,  $n \geq 2$ . Then there is  $p(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]$  of degree at most one in  $x$  and an element  $\sigma \in \text{Aut}_{F[x]-\text{LieAlg}}(\mathfrak{g}[x])$  such that

$$(\sigma(x) \otimes \sigma(y))r(x, y) = y^{n-2} \underbrace{\left( \frac{s(x)y^2\Omega}{x - y} + p(x, y) \right)}_{r'(x, y)},$$

where  $r'$  is a generalized  $r$ -matrix in  $L_2(2, \alpha)$ .

**Proof** The condition  $x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}]$  means exactly that  $W_+$  is an order. Moreover, since  $W$  is complementary to the diagonal, we have  $W_+ + \mathfrak{g}[x] = \mathfrak{g}[x, x^{-1}]$ . It was shown in [11] that such orders, up to the action of some  $\sigma \in \text{Aut}_{F[x]-\text{LieAlg}}(\mathfrak{g}[x])$ , are contained in a maximal order  $\mathfrak{M}$  associated to the so called fundamental simplex  $\Delta_{\text{st}}$ . These maximal orders are explicitly described in [11] and satisfy  $\mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$ . Therefore, we have  $\sigma W_+ \subseteq \mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$ . Moreover, we have the identity

$$(\sigma \times \sigma)W \dot{+} \Delta = L(n, \alpha),$$

implying the inclusion  $\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq (\sigma \times \sigma)W$ . The remaining parts follow straightforward from the construction Theorem 3.6. □

Unfortunately, we have not found a new example of an unbounded subalgebra of  $L(n, \alpha)$ . However, we present an infinite family of bounded subalgebras. We believe these examples are still interesting because their orthogonal complements, which are important in the view of Adler-Kostant-Symes scheme, are unbounded if  $\alpha \neq 0$ .

Consider the subspaces of  $L(n, \alpha_0)$ ,  $n > 0$ :

$$W_0 = \text{span}_F \{b_i(x^{-k}, 0), b_i(1, 0), b_i(0, -[x]^\ell) \mid k \geq 1, 1 \leq \ell \leq n - 1\},$$

$$W_1 = \text{span}_F \{b_i(x^{-k}, 0), b_i(0, -1), b_i(0, -[x]^\ell) \mid k \geq 1, 1 \leq \ell \leq n - 1\}.$$

These are clearly subalgebras. The corresponding generalized  $r$ -matrices are

$$r_0 = \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{y^{n-1} \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})}$$

$$+ \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})} \left( y^{2(n-1)} + \sum_{0 \leq \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right)$$

$$= \frac{y^{n-1}}{1 + \alpha_0 y^{n-1}} \left( \frac{y \Omega}{x - y} + \Omega \right),$$

$$r_1 = \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})}$$

$$\times \left( y^{2(n-1)} + \sum_{0 < \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right)$$

$$= \frac{1}{1 + \alpha_0 y^{n-1}} \frac{y^n \Omega}{x - y}.$$

By considering decompositions  $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \mathfrak{s}_2$  of  $\mathfrak{g}$  into direct sums of subalgebras we can get an infinite family of generalized  $r$ -matrices "in between"  $r_0$  and  $r_1$ . More precisely, let  $\{s_{1,i}\}_{i=1}^{d_1}$  and  $\{s_{2,j}\}_{j=1}^{d_2}$  be bases for  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  respectively. Such a decomposition leads to another subalgebra of  $L(n, \alpha_0)$ :

$$W_{01} := \text{span}_F \left\{ b_i(x^{-k}, 0), s_{1,m}(1, 0), s_{2,j}(0, 1), b_i(0, -[x]^\ell) \mid k \geq 1, 1 \leq \ell \leq n - 1, 1 \leq i \leq d, \right.$$

$$\left. 1 \leq m \leq d_1, 1 \leq j \leq d_2 \right\}.$$

Rewrite the elements  $b_i$  in terms of  $s_{1,m}$  and  $s_{2,j}$ :

$$b_i = \sum_{m=1}^{d_1} \lambda_{1,m}^i s_{1,m} + \sum_{j=1}^{d_2} \lambda_{2,j}^i s_{2,j},$$

where  $\lambda_{1,m}^i, \lambda_{2,j}^i \in F$ . Finding a basis in  $W_{12}$  dual to  $\{b_i(y^m, [y]^m)\} \subset \Delta$  and then projecting the generating series for  $W_{01}$  onto the first component we obtain the following generalized  $r$ -matrix

$$\begin{aligned}
 r_{01} &= \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})} \\
 &\times \left( y^{2(n-1)} + \sum_{0 < \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right) \\
 &+ \frac{y^{n-1}}{1 + \alpha_0 y^{n-1}} \sum_{i=1}^d \sum_{m=1}^{d_1} \lambda_{1,m}^i s_{1,m} \otimes b_i \\
 &= \frac{y^{n-1}}{1 + \alpha_0 y^{n-1}} \left( \frac{y \Omega}{x - y} + \sum_{i=1}^d \sum_{m=1}^{d_1} \lambda_{1,m}^i s_{1,m} \otimes b_i \right). \tag{42}
 \end{aligned}$$

Clearly  $r_{01}$  coincides with  $r_0$  when  $\mathfrak{s}_1 = \mathfrak{g}$  and  $r_1$  if  $\mathfrak{s}_2 = \mathfrak{g}$ . The corresponding orthogonal complements are

$$W_0^\perp = W(\overline{r_0}) = \text{span}_F \left\{ b_i \left( 0, [x]^{n-1} \right), b_i \left( \frac{x^{-k(n-1)-m}}{1 + \alpha_0 x^{n-1}}, 0 \right) \mid k \geq -1, 0 < m < n - 1 \right\},$$

$$W_1^\perp = W(\overline{r_1}) = \text{span}_F \left\{ b_i \left( \frac{x^{-k(n-1)-m}}{1 + \alpha_0 x^{n-1}}, 0 \right) \mid k \geq -1, 0 \leq m < n - 1 \right\},$$

$$\begin{aligned}
 W_{01}^\perp = W(\overline{r_{01}}) &= \mathfrak{s}_1^\perp \left( \frac{x^{n-1}}{1 + \alpha_0 x^{n-1}}, 0 \right) \dot{+} \mathfrak{s}_2^\perp \left( 0, [x]^{n-1} \right) \\
 &\dot{+} \text{span}_F \left\{ b_i \left( \frac{x^{-k(n-1)-m}}{1 + \alpha_0 x^{n-1}}, 0 \right) \mid k \geq -1, 0 < m < n - 1 \right\}, \tag{43}
 \end{aligned}$$

which are unbounded because of the factor  $1/(1 + \alpha_0 x^{n-1})$ .

Note that a series of type  $(n, s)$  defines a subspace inside  $L(n, \alpha)$  for any  $\alpha$ , because the subalgebra property is not affected by the form. With the previous examples in mind we can prove the following statement.

**Lemma 5.2** *Let  $B_0$  and  $B_\alpha$  be the bilinear forms on  $L(n, 0)$  and  $L(n, \alpha)$  respectively. For a series  $r$  of type  $(n, s)$  we have*

$$W(r)^\perp_{B_\alpha} = \frac{1}{x^n \alpha(x)} W(r)^\perp_{B_0} \subset L(n, \alpha). \tag{44}$$

**Proof** Set  $u(x) := 1/(x^n \alpha(x))$ . Write

$$r = \sum_{k \geq 0} \sum_{i=1}^d (s w_{k,i} + g_{k,i}) \otimes b_i(y^k, [y]^k) \quad \text{and} \quad \bar{r} = \sum_{k \geq 0} \sum_{i=1}^d (s w_{k,i} + \overline{g_{k,i}}) \otimes b_i(y^k, [y]^k).$$

Then by Theorem 3.6 and definition Eq. (12)  $B_\alpha(sw_{k,i} + g_{k,i}, u(sw_{\ell,j} + \overline{g_{\ell,j}})) = B_0(sw_{k,i} + g_{k,i}, sw_{\ell,j} + \overline{g_{\ell,j}}) = 0$ .  $\square$

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## Declarations

**Conflict of interest** The authors have no competing interests to declare that are relevant to the content of this article.

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## References

1. Abedin, R., Maximov, S., Stolin, A., Zelmanov, E.: Topological Lie bialgebra structures and their classification over  $\mathfrak{g}[[x]]$  (2022). [arXiv: 2203.01105](https://arxiv.org/abs/2203.01105) [math.RA]
2. Abedin, R.: Geometrization of solutions of the generalized classical Yang-Baxter equation and a new proof of the Belavin-Drinfeld trichotomy (2021) [arXiv: 2107.10722](https://arxiv.org/abs/2107.10722) [math.AG]
3. Abedin, R., Maximov, S.: Classification of classical twists of the standard Lie bialgebra structure on a loop algebra. In: *J. Geom. Phys.* 164 (2021), pp. 104149, 26. issn: 0393-0440. <https://doi.org/10.1016/j.geomphys.2021.104149>
4. Adler, M., van Moerbeke, P., Vanhaecke, P.: Algebraic integrability, Painlevé geometry and lie algebras. A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg (2004)
5. Alekseev, A., Kosmann-Schwarzbach, Y.: Manin Pairs and Moment Maps. In: *Journal of Differential Geometry* 56.1, pp. 133-165 (2000)
6. Babelon, O., Bernard, D., Talon, M.: Introduction to Classical Integrable Systems. Cambridge University Press (2003)
7. Drinfeld, V.G.: Quasi-Hopf algebras. *Algebra i Analiz* 6, 114–148 (1989)
8. Montaner, F., Stolin, A., Zelmanov, E.: Classification of Lie bialgebras over current algebras. In: *Selecta Math. (N.S.)* 16.4, pp. 935-962. issn: 1022-1824. (2010) <https://doi.org/10.1007/s00029-010-0038-7>
9. Skrypnyk, T.: Infinite-dimensional Lie algebras, classical r-matrices, and Lax operators: Two approaches. In: *Journal of Mathematical Physics* 54.10, p. 103507 (2013)
10. Skrypnyk, T.: Integrable quantum spin chains, non-skew symmetric r-matrices and quasigraded Lie algebras. In: *J. Geom. Phys.* 57 (2006)
11. Stolin, A.: On rational solutions of Yang-Baxter equations. Maximal orders in loop algebra. In: *Comm. Math. Phys.* 141.3, pp. 533-548. issn: 0010-3616. (1991) <http://projecteuclid.org/euclid.cmp/1104248392>

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