THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

COMPUTING VECTOR-VALUED MODULAR FORMS OF CONGRUENCE TYPES AND OF SOME EXTENSION TYPES

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ABSTRACT

This thesis explores applications of vector-valued modular forms of congruence and extension types to scalar-valued modular forms for congruence subgroups with a character, higher order modular forms, and iterated Eichler-Shimura integrals of depth one and two, including considerable generalizations thereof.

In Paper I (co-authored with Martin Raum), we present an algorithm for computing bases for spaces of vector-valued modular forms of congruence type and of weight at least 2 in terms of products of components of vector-valued Eisenstein series. Since the Fourier series expansions of these Eisenstein series are available, our algorithm can be used to compute Fourier series expansions of any vector-valued modular form belonging to these spaces. It complements two available algorithms that (as opposed to ours) are limited to inductions of Dirichlet characters, and vector-valued modular forms of Weil type. Our algorithm is based on a representation theoretical interpretation of a theorem due to Raum and Xià. After a heuristic evaluation of the time-complexity, we compare our algorithm to the two available ones, highlighting the trade-offs between generality and performance.

In Paper II (co-authored with Martin Raum and Albin Ahlbäck), we show that all Eichler integrals, and all "generalized second order modular forms" can be expressed as linear combinations of corresponding generalized second order Eisenstein series with coefficients in classical modular forms. We compute the Fourier series expansions of generalized second order Eisenstein series in level one, and provide their tail estimates via convexity bounds for additively twisted L-functions. As an application, we illustrate a bootstrapping procedure that yields numerical evaluations of, for instance, Eichler integrals from merely the associated cocycle.

Finally, in Paper III (co-authored with Martin Raum), we provide an explicit vector-valued modular form whose top components are given by the depth two iterated Eichler-Shimura integral $I_{f,g}$, where f and g are cusp forms of weight $k \in \mathbb{Z}_{\geq 2}$. We show that this vector-valued modular form gives rise to a scalar-valued iterated Eichler integral of depth two, denoted by $\mathcal{E}_{f,g}$, that can be seen as a higher-depth generalization of the scalar-valued Eichler integral $\mathcal{E}_f$ of depth one. As an aside, our argument provides an alternative explanation of an orthogonality relation satisfied by period polynomials originally due to Paşol and Popa. We show that $\mathcal{E}_{f,g}$ can be expressed in terms of sums of products of components of vector-valued Eisenstein series with classical modular forms after multiplication with a suitable power of the discriminant modular form Δ. This allows for effective computation of $\mathcal{E}_{f,g}$.

SAMMANFATTNING

Denna avhandling utforskar tillämpningar av vektorvärda modulära former av kongruens- och utvidgningstyper på skalarvärda modulära former för kongruensdelgrupper med karaktärer, högre ordningens modulära former, itererade Eichler-Shimura integraler av djup ett och två och därtill betydande generaliseringar.

I ARTIKEL I (skriven tillsammans med Martin Raum), presenterar vi en algoritm för att beräkna baser av rum av vektorvärda modulära former av vikt minst 2 och av kongruenstyp, i termer av produkter av komponenter av vektorvärda Eisensteinserier. Eftersom Fourierserieutvecklingarna till dessa Eisensteinserier finns tillgängliga, kan vår algoritm också användas för att beräkna Fourierserieutvecklingar av godtyckliga vektorvärda modulära former i dessa rum. Vår algoritm komplementerar två tillgängliga algoritmer som (till skillnad från vår) är begränsade till induktioner av Dirichletkaraktärer och vektorvärda modulära former av Weiltyp. Algoritmen bygger på en representationsteoretisk tolkning av en sats ursprungligen bevisad av Raum och Xià. Efter en heuristisk uppskattning av tidskomplexiteten jämför vi vår algoritm med de två tillgängliga, och betonar avvägningarna mellan generalitet och prestanda.

I ARTIKEL II (skriven tillsammans med Martin Raum och Albin Ahlbäck), visar vi att alla Eichlerintegraler och alla "generaliserade andra ordningens modulära former" kan uttryckas som linjärkombinationer av motsvarande generaliserade andra ordningens Eisensteinserier med koefficienter i klassiska modulära former. Vi beräknar Fourierserieutvecklingarna till generaliserade andra ordningens Eisensteinserier i nivå ett och ger svansuppskattningar genom konvexitetsuppskattningar för additivt vridna L-funktioner. Som en tillämpning betraktar vi en "bootstrapping"-teknik som kan användas för numerisk beräkning av exempelvis Eichlerintegraler, givet endast den associerade kocykeln.

Slutligen presenterar vi i ARTIKEL III (skriven tillsammans med Martin Raum), en explicit vektorvärd modulär form vars övre komponenter ges av den itererade Eichler-Shimura integralen av djup två $I_{f,g}$, där f och g är spetsformer av vikt $k \in \mathbb{Z}_{\geq 2}$. Vi visar att denna vektorvärda modulära form ger upphov till en skalarvärd itererad Eichlerintegral av djup två som vi betecknar med $\mathcal{E}_{f,g}$ och som kan ses som en generalisering till högre djup av den skalarvärda Eichlerintegralen $\mathcal{E}_f$ av djup ett. Som en parentes ger vårt argument en alternativ förklaring till en ortogonalitetsrelation som uppfylls av periodpolynom som ursprungligen bevisades av Paşol och Popa. Vi visar att $\mathcal{E}_{f,g}$ kan uttryckas som en summa av produkter av komponenter av vektorvärda Eisensteinserier med klassiska modulära former efter multiplikation med en passande potens av diskriminantformen Δ. Detta kan användas för att effektivt beräkna $\mathcal{E}_{f,g}$.

LIST OF APPENDED PAPERS

[I] Tobias Magnusson and Martin Raum. *On the Computation of General Vector-valued Modular Forms*. arXiv 2202.06676. To appear in *Mathematics of Computation*. DOI: 10.48550/ARXIV.2202.06676. URL: https://arxiv.org/abs/2202.06676.

[II] Albin Ahlbäck, Tobias Magnusson, and Martin Raum. *Eichler integrals and generalized second order Eisenstein series*. arXiv 2203.15462. DOI: 10.48550/ARXIV.2203.15462. URL: https://arxiv.org/abs/2203.15462.

[III] Tobias Magnusson and Martin Raum. *Scalar-valued depth two Eichler-Shimura Integrals of Cusp Forms*. arXiv 2209.00488. Revised December 7, 2022. DOI: 10.48550/ARXIV.2209.00488. URL: https://arxiv.org/abs/2209.00488.

CONTRIBUTIONS

Paper I: This paper is an extension of my licentiate thesis [35]. I wrote the initial draft, and contributed to the code-base for `ModularForms.jl`. We wrote the final version together.

Paper II: This paper was co-authored with Martin's and my bachelor student Albin Ahlbäck and grew out of his bachelor thesis project. I contributed primarily to Sections 1, 2, 4, and 5, and helped with writing the proofs in Section 3.

Paper III: Martin came up with the original idea of investigating modular forms associated to depth two extension types with $\mathbf{1}$, $\mathrm{sym}^d(X)$, and $\mathbf{1}$ on the diagonal. I then carried out most of the research largely autonomously and wrote the first drafts. We created the final version together, and were then able to simplify many proofs.

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There are of course countless other people who have contributed to me being able to write this thesis. You know who you are, and I thank you too.

DISPOSITION

This thesis is divided into four parts:

(I) a general background to the theory of classical modular forms (Chapters 1 and 2),

(II) an introduction to the theory of vector-valued modular forms (Chapter 3),

(III) summaries of the appended papers (Chapters 4 and 5), and finally

(IV) the appended papers themselves.

Chapter 1 is intended for a general mathematical audience and can be skimmed over for readers with an understanding of the basic theory of classical modular forms. Chapter 2 is intended for readers who are familiar with classical modular forms of level one and provides a summary of the theory of modular forms for subgroups of the full modular group and of modular forms of higher order. Having read Chapter 2 (or being an expert), the reader is prepared to read Chapter 3, which provides an introduction to the parts of the theory of vector-valued modular forms that are necessary to understand the appended papers. The remaining parts are self-explanatory.

CONTENTS

Part I

WHY MODULAR FORMS?

1

SCALAR-VALUED MODULAR FORMS OF LEVEL ONE

1.1 THE LATTICE VIEW OF MODULAR FORMS

Our story starts with two-dimensional complex lattices. They are defined as follows. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Then the lattice associated to (ω_1, ω_2) is given by

$$L(\omega_1, \omega_2) = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2. \tag{1.1}$$

We can visualize it as in Figure 1.1.

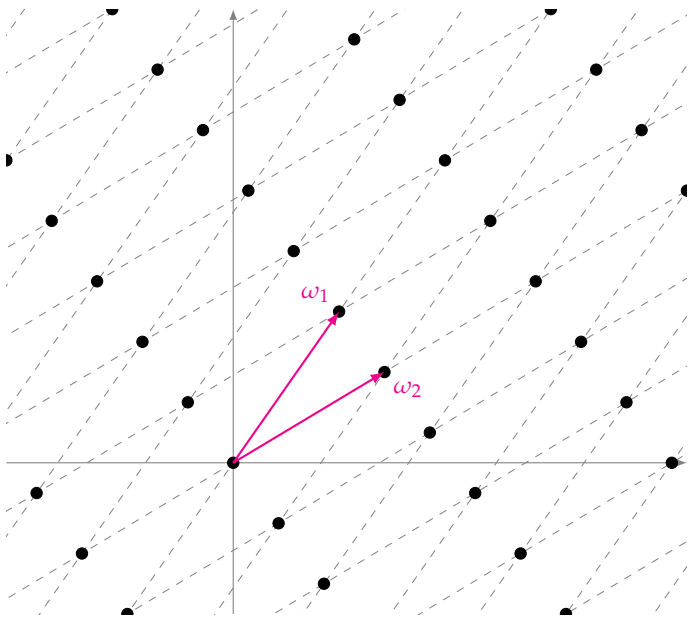

Figure 1.1: A visualization of a two-dimensional complex lattice spanned by ω_1 and ω_2. The elements of the lattice are the black dots.

We say that (ω_1, ω_2) is a basis for $L(\omega_1, \omega_2)$. A natural question to ask is – when are two lattices equal? The following proposition answers this.

Proposition 1.1. Let $(\omega_1, \omega_2), (\omega_1', \omega_2') \in \mathbb{C}^2$ be lattice bases satisfying that $\mathrm{Im}(\omega_1/\omega_2) > 0$ and $\mathrm{Im}(\omega_1'/\omega_2') > 0$.[1] Then

$$L(\omega_1, \omega_2) = L(\omega_1', \omega_2'), \tag{1.2}$$

if and only if

$$\gamma \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}, \tag{1.3}$$

for some 2×2 matrix γ with integer entries, satisfying that $\det(\gamma) = 1$.

Proof. Let us first suppose that $L(\omega_1, \omega_2) = L(\omega_1', \omega_2')$. Since $\omega_1, \omega_2 \in L(\omega_1', \omega_2')$ and $\omega_1', \omega_2' \in L(\omega_1, \omega_2)$, there exists matrices $A, B \in \mathbb{Z}^{2\times 2}$ satisfying

$$\omega = A\omega' \quad \text{and} \quad \omega' = B\omega, \tag{1.4}$$

where $\omega = (\omega_1, \omega_2)^T$ and $\omega' = (\omega_1', \omega_2')^T$. This implies that $\omega = AB\omega$ and $\omega' = BA\omega'$, and since the entries of ω and ω' are linearly independent, also that

$$AB = BA = I, \tag{1.5}$$

so that A is invertible with $B = A^{-1}$. We also have that $\det(A), \det(B) \in \mathbb{Z}$ and that $\det(B) = 1/\det(A)$. This implies that $\det(A) = \det(B)$ and that $\det(A) \in \{1, -1\}$. To find out which sign is correct, let us write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1.6}$$

for some $a, b, c, d \in \mathbb{Z}$. We then have that

$$\mathrm{Im}(\omega_1/\omega_2) = \frac{|\omega_2'|^2}{|\omega_2|^2}\mathrm{Im}(\omega_1'/\omega_2')(ad - bc), \tag{1.7}$$

which implies that $ad - bc > 0$ and so $\det(A) = 1$. Conversely, let us now suppose that $\omega = A\omega'$ for a matrix A on the given form. Then we have immediately that $\omega_1, \omega_2 \in L(\omega_1', \omega_2')$ and so $L(\omega_1, \omega_2) \subseteq L(\omega_1', \omega_2')$. Since $A^{-1} \in \mathbb{Z}^{2\times 2}$, we obtain in the same way that $L(\omega_1', \omega_2') \subseteq L(\omega_1, \omega_2)$. □

The set of matrices $A \in \mathbb{Z}^{2\times 2}$ with $\det(A) = 1$ will feature prominently in the rest of this thesis, and is given a special name. It is called the *special linear group of degree two over* $\mathbb{Z}$, and is denoted by $\mathrm{SL}_2(\mathbb{Z})$. Since the determinant is multiplicative, it is

1 The set of complex numbers $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ is called the upper half-plane and is denoted by $\mathbb{H}$.

clear that $\mathrm{SL}_2(\mathbb{Z})$ is a group with respect to matrix multiplication. We record here that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices[2]

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{1.8}$$

More generally, if R is any commutative ring with unity, the special linear group over R is given by

$$\mathrm{SL}_2(R) = \{A \in R^{2\times 2} : \det(A) = 1\}. \tag{1.9}$$

We also write $\Gamma_\infty = \langle T, S^2 \rangle$. This is called the parabolic subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Given an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we write $a(\gamma) = a$, $b(\gamma) = b$, $c(\gamma) = c$, and $d(\gamma) = d$.

Besides being aesthetically pleasing, lattices are central in many wide-ranging parts of mathematics. Just to mention a few:

1. Optimal sphere packings are often described in terms of lattices (where the lattice points represent the centers of the spheres). In particular, Maryna Viazovska and her co-authors showed in [31] and [29] that the lattice $E_8 \subseteq \mathbb{C}^8$ and $\Lambda_{24} \subseteq \mathbb{C}^{24}$ (see the papers for their definitions) provide optimal sphere packings in 8-dimensional and 24-dimensional Euclidean spaces, respectively. Their work made extensive use of modular forms and related objects.

2. In the study of Lie groups and Lie algebras lattices feature as *root lattices* of root systems [3, 8].

3. Given a two-dimensional complex lattice L, the quotient space $\mathbb{C}/L$ is in a well-defined sense isomorphic to an elliptic curve [24].

4. Computationally hard problems on lattices form the basis for lattice-based cryptography, such as NTRU [9] or "Ring-learning with errors" [23, 25]. The latter of these is projected to be important in post-quantum cryptography [39].

Given a class A of interesting objects, it is often insightful to study "well-behaved" functions from A to some other class of better understood objects – such as $\mathbb{R}^n$ or $\mathbb{C}^n$. As a silly (but important) example, we might consider $\mathbb{R}^n$ itself to be interesting, and study *linear* functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. These can of course be identified with the set of matrices $\mathbb{R}^{m\times n}$, and venturing further along these lines is, as the reader well knows, the goal of linear algebra.

On a more number theoretic note, we might consider $\mathbb{N}$ to be interesting and study *multiplicative* functions[3] from $\mathbb{N}$ to $\mathbb{C}$. By considering their associated L-functions,

2 Proving this is a fun exercise in applying Euclid's algorithm.

3 That is, functions $f : \mathbb{N} \to \mathbb{C}$ satisfying that $f(ab) = f(a)f(b)$ for coprime numbers a and b. Some important examples of these are the Möbius function μ, Euler's totient function ϕ, and the divisor function σ_k.

and studying them using tools from complex analysis, one can draw extraordinary conclusions concerning the distribution of prime numbers – such as the prime number theorem and Dirichlet's theorem on primes in arithmetic progression.

In our case, the class of interesting objects is the set $\mathcal{L}$ of all two-dimensional complex lattices, that is, the set[4]

$$\mathcal{L} = \{\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 : \omega_1, \omega_2 \in \mathbb{C}^\times \text{ and } \omega_1, \omega_2 \text{ not on a line}\}, \tag{1.10}$$

and we shall study "well-behaved" functions $f : \mathcal{L} \to \mathbb{C}$.

It is not a coincidence that we mentioned *linear* functions above. Indeed, we will restrict ourselves to functions on $\mathcal{L}$ that satisfy a criterion that looks a lot like linearity. However, linearity requires a notion of both scaling and addition, and in contrast to $\mathbb{R}^n$, $\mathcal{L}$ has no obvious additive structure. That being said, its elements can be scaled. That is, if $\lambda \in \mathbb{C}^\times$ and $L \in \mathcal{L}$, then

$$\lambda L = \{\lambda z : z \in L\} \in \mathcal{L}. \tag{1.11}$$

We can thus study functions $f : \mathcal{L} \to \mathbb{C}$ that are intertwined with scaling. That is, functions $f : \mathcal{L} \to \mathbb{C}$ satisfying that

$$f(\lambda L) = \lambda f(L) \text{ for all } \lambda \in \mathbb{C}^\times \text{ and } L \in \mathcal{L}. \tag{1.12}$$

Upon closer inspection, we find that these functions are not very interesting. Indeed, suppose that $f : \mathcal{L} \to \mathbb{C}$ is intertwined with scaling, then since $-L = L$, we have that

$$f(L) = f(-L) = -f(L) \text{ for all } L \in \mathcal{L}. \tag{1.13}$$

This evidently implies that $f = 0$. In order to get something interesting, we thus have to relax the invariance condition (1.12) somewhat. Let us therefore introduce a parameter $k \in \mathbb{Z}$ and consider functions $f : \mathcal{L} \to \mathbb{C}$ satisfying that

$$f(\lambda L) = \lambda^{-k} f(L) \text{ for all } \lambda \in \mathbb{C}^\times \text{ and } L \in \mathcal{L}. \tag{1.14}$$

Of course, if k is odd, then for the same reason as (1.13), we obtain only the function $f = 0$. If k is even though, we obtain an entire world of functions, bringing with them a sweeping wealth of arithmetic information – we obtain *modular forms*.

Modular forms are also required to satisfy some analytically desireable properties, so as to not make them all too unwieldy. To be precise, we have the following definition.

Definition 1.1 (Lattice modular forms)**.** Let $\Lambda : \mathbb{H} \to \mathcal{L}$ be given by $\Lambda(\tau) = \mathbb{Z}\tau + \mathbb{Z}$. Let also $k \in \mathbb{Z}$. Then a lattice modular form of weight k is a function $f : \mathcal{L} \to \mathbb{C}$ such that

4 Recall that $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

1. for all $\lambda \in \mathbb{C}^\times$ and $L \in \mathcal{L}$, it holds that
$$f(\lambda L) = \lambda^{-k} f(L), \tag{1.15}$$
2. $f \circ \Lambda$ is holomorphic, and
3. there exists a number $a \in \mathbb{R}$ such that
$$|f(\Lambda(\tau))| = O(\mathrm{Im}(\tau)^a) \quad \text{as} \quad \mathrm{Im}(\tau) \to \infty, \tag{1.16}$$
uniformly in $\mathrm{Re}(\tau)$.

The set of lattice modular forms of weight k forms a $\mathbb{C}$-vector space, denoted by LM_k.

As a first example of a non-trivial element in LM_{2k}, where $k \in \mathbb{Z}$, we consider the classical Eisenstein series of weight $2k$, defined by the (a priori formal) series
$$G^*_{2k}(L) = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^{2k}}. \tag{1.17}$$
Given that $G^*_{2k}(L)$ converges absolutely for any $L \in \mathcal{L}$, we have for $\lambda \in \mathbb{C}^\times$ that
$$G^*_{2k}(\lambda L) = \sum_{\omega \in L \setminus \{0\}} \frac{1}{(\lambda\omega)^{2k}} = \lambda^{-2k} \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^{2k}} = \lambda^{-2k} G^*_{2k}(L), \tag{1.18}$$
so it stands to reason that $G^*_{2k} \in \mathrm{LM}_{2k}$ if the convergence is sufficiently "nice". When $k \geq 2$ it is, but to prove it, it is useful to first change to a more analytic point of view.

1.2 THE ANALYTIC VIEW OF MODULAR FORMS

Any lattice $L \in \mathcal{L}$ can be rescaled so as to be put into the form $\mathbb{Z}\tau + \mathbb{Z}$, for some complex number $\tau \in \mathbb{H}$. Combined with the scaling condition (1.15), this suggests that we should be able to map lattice modular forms to similarly behaved functions from $\mathbb{H}$ to $\mathbb{C}$.

This is indeed possible, and the map will be a vector space isomorphism, but the incurred cost is that the scaling condition (1.15) will take on a somewhat less appealing form. Let us bring in a new definition.

Definition 1.2 (Modular forms)**.** For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$, we let $\gamma\tau$ be given by
$$\gamma\tau = \frac{a\tau + b}{c\tau + d}. \tag{1.19}$$
This defines a group action – often referred to as the Möbius action. Let now $k \in \mathbb{Z}$ and let $f : \mathbb{H} \to \mathbb{C}$ be a function. Then we say that f is a modular form of weight k if

1. for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ it holds that
$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \tau \in \mathbb{H}, \tag{1.20}$$
2. f is holomorphic, and
3. there exists a number $a \in \mathbb{R}$ such that
$$|f(\tau)| = O(\mathrm{Im}(\tau)^a) \quad \text{as} \quad \mathrm{Im}(\tau) \to \infty, \tag{1.21}$$
uniformly in $\mathrm{Re}(\tau)$.

The set of modular forms of weight k forms a $\mathbb{C}$-vector space, which we denote by M_k.

Let now $\psi : \mathrm{LM}_k \to \mathrm{M}_k$ be given by
$$\psi(f) = f \circ \Lambda, \tag{1.22}$$
where $\Lambda : \mathbb{H} \to \mathcal{L}$ is given as in Definition 1.1. We then have the following proposition.

Proposition 1.2. The function ψ is a vector space isomorphism.

Proof. It is clear that ψ is linear. Furthermore, for $f \in \mathrm{LM}_k$ the function $\psi(f)$ immediately satisfies conditions 2. and 3. of Definition 1.2. Hence, we only have to verify that ψ is bijective and that for $f \in \mathrm{LM}_k$, the function $\psi(f)$ satisfies condition 1. of Definition 1.2. Let us start with the latter.

Let $\tau \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then for $f \in \mathrm{LM}_k$ we have that
$$\begin{aligned} f \circ \Lambda(\gamma\tau) &= f\Big(\mathbb{Z}\frac{a\tau + b}{c\tau + d} + \mathbb{Z}\Big) \\ &= f\Big(\frac{1}{c\tau + d}(\mathbb{Z}(a\tau + b) + \mathbb{Z}(c\tau + d))\Big) \\ &= (c\tau + d)^k f(\mathbb{Z}(a\tau + b) + \mathbb{Z}(c\tau + d)). \end{aligned} \tag{1.23}$$

However, we have that
$$\begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = \gamma \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \tag{1.24}$$
so that by Proposition 1.1, we have that
$$\mathbb{Z}(a\tau + b) + \mathbb{Z}(c\tau + d) = \mathbb{Z}\tau + \mathbb{Z}. \tag{1.25}$$

To finish the proof, we construct a linear two-sided inverse $\xi : \mathrm{M}_k \to \mathrm{LM}_k$ to ψ. For $f \in \mathrm{M}_k$, we let $\xi(f)$ be given by
$$\xi(f)(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2), \tag{1.26}$$

where (without loss of generality) we assume that $\mathrm{Im}(\omega_1/\omega_2) > 0$.[5] Given that $\xi(f)$ is well-defined, then ξ is evidently linear. Hence, let $f \in \mathrm{M}_k$ be given and say that $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ for some $\omega_1, \omega_2, \omega_1', \omega_2' \in \mathbb{C}$, where $\mathrm{Im}(\omega_1/\omega_2) > 0$ and $\mathrm{Im}(\omega_1'/\omega_2') > 0$. Then Proposition 1.1 tells us that there exists a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(\omega_1, \omega_2)^T = (\omega_1', \omega_2')^T$. We thus obtain that

$$\begin{aligned} \omega_2'^{-k} f(\tfrac{\omega_1'}{\omega_2'}) &= (c\omega_1 + d\omega_2)^{-k} f\Big(\frac{a\frac{\omega_1}{\omega_2} + b}{c\frac{\omega_1}{\omega_2} + d}\Big) \\ &= (c\omega_1 + d\omega_2)^{-k} (c\tfrac{\omega_1}{\omega_2} + d)^k f(\tfrac{\omega_1}{\omega_2}) \\ &= \omega_2^{-k} f(\tfrac{\omega_1}{\omega_2}). \end{aligned} \tag{1.27}$$

Hence, for $f \in \mathrm{M}_k$, $\xi(f)$ is well-defined as a function on $\mathcal{L}$. It is also clear that $\xi(f)$ satisfies conditions 1. through to 3. of Definition 1.1.

Finally, we make sure that ξ is a two-sided inverse of ψ. Let $f \in \mathrm{M}_k$ and $g \in \mathrm{LM}_k$. Then we have for $\tau \in \mathbb{H}$, that

$$\psi(\xi(f))(\tau) = \xi(f)(\mathbb{Z}\tau + \mathbb{Z}) = f(\tau), \tag{1.28}$$

and for $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \in \mathcal{L}$ with $\mathrm{Im}(\omega_1/\omega_2) > 0$, that

$$\begin{aligned} \xi(\psi(g))(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) &= \omega_2^{-k} \psi(g)(\omega_1/\omega_2) \\ &= \omega_2^{-k} g(\mathbb{Z}\tfrac{\omega_1}{\omega_2} + \mathbb{Z}) \\ &= g(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2). \end{aligned} \tag{1.29}$$

This finishes the proof. □

Remark 1.1. The new "scaling" condition (1.20) can be rephrased in terms of a right action of $\mathrm{SL}_2(\mathbb{Z})$ on functions from $\mathbb{H}$ to $\mathbb{C}$, parametrized by the weight, called the slash action. It is given as follows: let $k \in \mathbb{Z}$, then for a function $f : \mathbb{H} \to \mathbb{C}$ and an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define a new function $f\big|_k\gamma : \mathbb{H} \to \mathbb{C}$ by

$$(f\big|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma\tau), \quad \tau \in \mathbb{H}. \tag{1.30}$$

Hence a function $f : \mathbb{H} \to \mathbb{C}$ satisfies (1.20) if and only if

$$f\big|_k\gamma = f \quad \text{for all} \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}). \tag{1.31}$$

The slash action will feature prominently throughout the rest of this thesis.

5 If $\mathrm{Im}(\omega_1/\omega_2) < 0$, then just swap the order of the basis elements.

To see how our new point of view may bear fruit, let us again consider the Eisenstein series G_{2k}^*. Its equivalent in M_{2k} takes the form

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^{2k}}. \tag{1.32}$$

Our next proposition shows that G_{2k} indeed is an example of a modular form of weight $2k$.

Proposition 1.3. Let $k \in \mathbb{Z}_{\geq 2}$. Then it holds that G_{2k} converges absolutely and locally uniformly on $\mathbb{H}$, and that $G_{2k}(\tau)$ is bounded as $\tau \to i\infty$. Furthermore, it holds that $G_{2k} \in \mathrm{M}_{2k}$.

Proof. For $A, B \in \mathbb{R}_{>0}$ we let

$$\Omega_{A,B} = \{\tau \in \mathbb{H} : |\mathrm{Re}(\tau)| \leq A \text{ and } \mathrm{Im}(\tau) > B\}. \tag{1.33}$$

Let now $A, B \in \mathbb{R}_{>0}$ be arbitrary. It is a fact, which we will not prove here, that there exists a constant $C \in \mathbb{R}_{>0}$ depending on A and B, such that for all $\tau \in \Omega_{A,B}$ and $\delta \in \mathbb{R}$, we have that

$$|\tau+\delta| \geq C\max\{1, |\delta|\}. \tag{1.34}$$

We now have for $\tau \in \Omega_{A,B}$, that

$$\begin{aligned} |m\tau+n|^{-2k} &= |m|^{-2k}|\tau+\tfrac{n}{m}|^{-2k} \\ &\leq C|m|^{-2k}\max\{1, |\tfrac{n}{m}|\}^{-2k} \\ &= C\max\{|m|, |n|\}^{-2k}. \end{aligned} \tag{1.35}$$

We also have that

$$\begin{aligned} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \max\{|m|,|n|\}^{-2k} &= \lim_{N\to\infty} \sum_{\substack{|m|,|n|\leq N \\ (m,n)\neq(0,0)}} \max\{|m|,|n|\}^{-2k} \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} n^{-2k}(2(2n+1)+2(2n-1)) \\ &= 8\sum_{n\geq 1} \frac{1}{n^{2k-1}} = 8\zeta(2k-1) < \infty. \end{aligned} \tag{1.36}$$

The second equality is best understood geometrically, as in Figure 1.2.

Weierstraß' M-test now implies that G_{2k} converges absolutely and uniformly on $\Omega_{A,B}$. Since any compact subset of $\mathbb{H}$ is a subset of $\Omega_{A,B}$ for some $A, B \in \mathbb{R}_{>0}$, it follows that G_{2k} is holomorphic on $\mathbb{H}$.

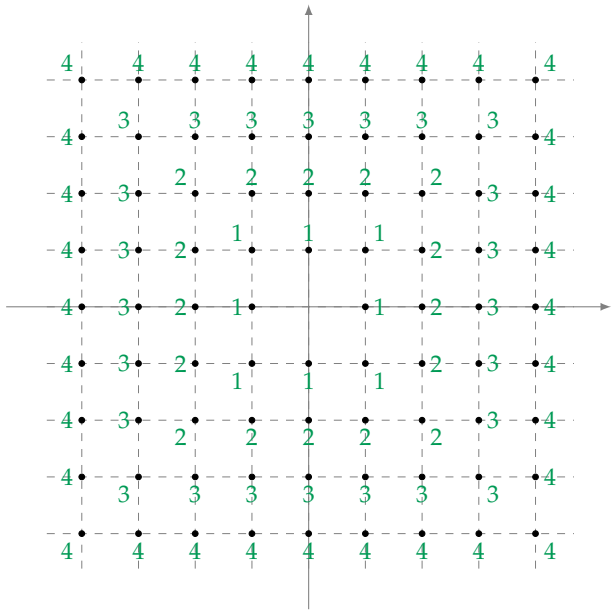

Figure 1.2: The value of $\max\{|m|,|n|\}$ (in green) when $|m|,|n| \leq 4$. We obtain that

$$\sum_{|m|,|n|\leq 4, |m|,|n|\neq 0} \max\{|m|,|n|\}^{-2k} = 1^{-2k}(2\cdot 3 + 2\cdot(3-2)) + \cdots + 4^{-2k}(2\cdot 9 + 2\cdot(9-2)).$$

Let now $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ be arbitrary. Then

$$\begin{aligned} G_{2k}(\gamma\tau) &= \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{(c\tau+d)^k}{(m(a\tau+b)+n(c\tau+d))^k} \\ &= (c\tau+d)^k \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m(a\tau+b)+n(c\tau+d))^k} \\ &= (c\tau+d)^{2k} G_{2k}(\tau). \end{aligned} \tag{1.37}$$

In the last equality we use the fact that

$$\mathbb{Z}(a\tau+b)+\mathbb{Z}(c\tau+d) = \mathbb{Z}\tau+\mathbb{Z},$$

familiar from Proposition 1.1. Finally, to show that $|G_{2k}(\tau)|$ is bounded as $\tau \to i\infty$, we note that for all $\tau \in \Omega_{1,1}$ we have that

$$|\tau+\delta| \geq \frac{1}{3}\max\{1,|\delta|\}. \tag{1.38}$$

Combining this with the fact that

$$G_{2k}(\tau+1) = G_{2k}(T\tau) = G_{2k}(\tau) \quad \text{for } \tau \in \mathbb{H}, \tag{1.39}$$

we find that $|G_{2k}(\tau)|$ is bounded on $\{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > 1\}$. □

The techniques applied in the proof of Proposition 1.3 reoccur under different guises throughout the theory of modular forms, so it is good to be well-acquainted with it.

Let now $k \in \mathbb{Z}$ and let $f \in \mathrm{M}_k$ be a modular form. Then since

$$f(\tau+1) = f(T\tau) = f(\tau), \tag{1.40}$$

the function f admits a Fourier series expansion on the form

$$f(\tau) = \sum_{n \geq 0} c(f;n) e^{2\pi i n \tau}, \tag{1.41}$$

where $c(f;n) \in \mathbb{C}$ are the (uniquely determined) Fourier series coefficients of f. This gives us a convenient way to uniquely represent any modular form. For a function $f : \mathbb{H} \to \mathbb{C}$ with a Fourier series expansion on the form $f(\tau) = \sum_{n \in \mathbb{Z}} c(f;n) e^{2\pi i n \tau}$, $\tau \in \mathbb{H}$, we write

$$\mathrm{ord}(f) = \min\{n \in \mathbb{Z} : c(f;n) \neq 0\}. \tag{1.42}$$

Note that for functions $f, g : \mathbb{H} \to \mathbb{C}$ with Fourier series expansions as above, we have that $\mathrm{ord}(f \cdot g) = \mathrm{ord}(f) + \mathrm{ord}(g)$.

We now define the subspace $\mathrm{S}_k \subseteq \mathrm{M}_k$ of weight k cusp forms by

$$\mathrm{S}_k = \{f \in \mathrm{M}_k : \mathrm{ord}(f) \geq 1\}. \tag{1.43}$$

As an example of a Fourier series expansion, we find using Lipschitz' summation formula [19] that

$$G_{2k}(\tau) = 2\zeta(2k)\Big(1 - \frac{4k}{B_{2k}} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi i n \tau}\Big), \tag{1.44}$$

where

$$B_{2k} = \frac{(-1)^{k+1} 2 \cdot (2k)!}{(2\pi)^{2k}} \zeta(2k), \tag{1.45}$$

are the Bernoulli numbers, and where

$$\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}, \tag{1.46}$$

is the $(2k-1)$th divisor function. In other words, the Fourier series expansion of G_{2k} can be seen as a generating series for σ_{2k-1}. Since $\zeta(2k) \neq 0$, we also see that G_{2k} is not a cusp form. In fact, the space $\mathbb{C}G_k$ of weight k Eisenstein series (where $G_k = 0$ for k odd) is complementary to S_k, and their sum is equal to M_k. To be precise, we have the following proposition.

Proposition 1.4. Let $k \in \mathbb{Z}_{\geq 4}$ be even. Then we have the following short exact sequence of vector spaces

$$0 \to S_k \xrightarrow{\iota(f)=f} M_k \xrightarrow{\nu(f)=c(f;0)} \mathbb{C} \to 0. \tag{1.47}$$

Furthermore, it holds that $M_k = S_k \oplus \mathbb{C}G_k$.

Proof. It is clear that ι is injective and that $\ker \nu = S_k = \operatorname{im} \iota$. To see that ν is surjective, we note that $\nu(G_k) = 2\zeta(k) \neq 0$, and so for an arbitrary $z \in \mathbb{C}$ it holds that $\nu(\frac{z}{2\zeta(k)} G_k) = z$, showing that ν is surjective. Since every short exact sequence of vector spaces splits, we have that

$$M_k = \iota(S_k) \oplus u(\mathbb{C}), \tag{1.48}$$

where $u(z) = z \cdot G_k$ is a right inverse to ν. However, $\iota(S_k) = S_k$ and $u(\mathbb{C}) = \mathbb{C}G_k$, and thus we are done. □

An important cusp form is the discriminant modular form Δ. It is defined by

$$\Delta(\tau) = (2\pi)^{-12}(g_2(\tau)^3 - 27g_3(\tau)^2), \quad \tau \in \mathbb{H}, \tag{1.49}$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$. It is clear that $g_2^3, g_3^2 \in M_{12}$ and thus $\Delta \in M_{12}$. Furthermore, comparing to (1.44), one finds that $c(\Delta;0) = 0$ and $c(\Delta;1) = 1$, so that $\operatorname{ord}(\Delta) = 1$ and $\Delta \in S_{12}$. One can show, though this requires machinery outside of our purview, that $\Delta(\tau)$ is the discriminant of a certain cubic polynomial whose coefficients depend on τ and whose roots are distinct for any τ. This implies that $\Delta(\tau) \neq 0$ for all $\tau \in \mathbb{H}$.

The discriminant modular form is a very useful computational tool, owing in large part to the following proposition.

Proposition 1.5. Let $k \in \mathbb{Z}$. Then the map $\phi : M_{k-12} \to S_k$ given by

$$\phi(f) = \Delta \cdot f, \tag{1.50}$$

is an isomorphism of vector spaces.

Proof. It is clear that ϕ is linear, and thus we only have to show that it is bijective. Suppose that $\Delta \cdot f = 0$ for some $f \in M_{k-12}$. Then since $\Delta(\tau) \neq 0$ for all $\tau \in \mathbb{H}$, we have that $f(\tau) = 0$ for all $\tau \in \mathbb{H}$. This shows that ϕ is injective.

As for surjectivity, we construct a right-inverse to ϕ. Let $g \in S_k$. Then since $\mathrm{ord}(\Delta) = 1$, we have that $\mathrm{ord}(g/\Delta) \geq 0$, where $(g/\Delta)(\tau) = g(\tau)/\Delta(\tau)$. Hence g/Δ satisfies the growth condition (1.21) and is holomorphic. It is also clear that $(g/\Delta)|_{k-12}\gamma = g/\Delta$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and hence $g/\Delta \in M_{k-12}$. The linear map $\psi : S_k \to M_{k-12}$ given by $\psi(g) = g/\Delta$ thus satisfies that $\phi \circ \psi = \mathrm{id}_{S_k}$. □

The following result, which we will not prove here, will be needed in the forthcoming discussion.

Proposition 1.6. Let $k \in \mathbb{Z}_{\leq -1} \cup \{2\}$. Then $M_k = \{0\}$.

Proof. See [18] or [2]. □

Remark 1.2. The standard proof of Proposition 1.6 relies on a non-trivial result called the *valence formula*, which provides a relation between the orders of points in the *fundamental domain* $\mathcal{F} = \{z \in \mathbb{C} : |z| \geq 1, |\mathrm{Re}(z)| \leq 1/2\}$. It should be noted that the valence formula is strongly connected to the dimension formula of scalar-valued modular forms for subgroups of $\mathrm{SL}_2(\mathbb{Z})$, see Proposition 2.3 of Chapter 2.

So far, we have only seen two examples of elements in M_k. In the next section, we will see how we can use these to compute any modular form in M_k.

1.3 STURM'S BOUND, THE DIMENSION, AND A BASIS

The following proposition shows that we do not need *all* Fourier series coefficients to uniquely identify a modular form.[6]

Proposition 1.7 (Sturm's bound)**.** Let $k \in \mathbb{Z}_{\geq 4}$ be even, and let $f \in M_k$. If $c(f;n) = 0$ for all $0 \leq n \leq \lfloor k/12 \rfloor$, then it holds that $f = 0$.

Proof. We have that $\mathrm{ord}(f) \geq \lfloor k/12 \rfloor + 1$, and so

$$\mathrm{ord}(f^{12}) = 12 \cdot \mathrm{ord}(f) \geq 12(\lfloor k/12 \rfloor + 1) > k. \tag{1.51}$$

We have that the function $1/\Delta$ is holomorphic. It is also 1-periodic, and thus $\mathrm{ord}(1/\Delta)$ is well-defined. Consequently, we have that

$$0 = \mathrm{ord}(1) = \mathrm{ord}(\Delta \cdot \frac{1}{\Delta}) = \mathrm{ord}(\Delta) + \mathrm{ord}(\frac{1}{\Delta}), \tag{1.52}$$

so that $\mathrm{ord}(1/\Delta) = -\mathrm{ord}(\Delta)$. We thus conclude that

$$\mathrm{ord}(f^{12} \cdot \Delta^{-k}) = \mathrm{ord}(f^{12}) - k \cdot \mathrm{ord}(\Delta) > 0. \tag{1.53}$$

6 This is a simplified version of a proposition first proved by Jacob Sturm. This version serves only to "set the scene" for our later discussions.

Hence $f^{12} \cdot \Delta^{-k} \in S_0$. However, for $g \in S_0$, we have that $g/\Delta \in M_{-12}$ so that by Proposition 1.5, we have that $g/\Delta = 0$. This implies that $g = 0$. Consequently $S_0 = \{0\}$, and so $f^{12} = 0$, but then $f = 0$. □

Proposition 1.7 implies that the map

$$M_k \ni f \mapsto (c(f;0), c(f;1), \ldots, c(f;\lfloor k/12 \rfloor)) \in \mathbb{C}^{\lfloor k/12 \rfloor + 1}, \tag{1.54}$$

is an injective linear map, and therefore it holds that

$$\dim(M_k) \leq \left\lfloor \frac{k}{12} \right\rfloor + 1. \tag{1.55}$$

However, combining Propositions 1.5 and 1.6, we can say more.

Proposition 1.8 (Dimension of M_k)**.** Let $k \in \mathbb{Z}$. Then

$$\dim(M_k) = \begin{cases} 0 & \text{if } k < 0 \text{ or } 2 \nmid k, \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv_{12} 2, \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv_{12} 2. \end{cases} \tag{1.56}$$

Proof. Suppose first that $k \in \{4, 6, 8, 10\}$ and let $f \in M_k$. Then for some $\alpha \in \mathbb{C}^\times$ we have that $f - \alpha G_k \in S_k$. By Proposition 1.5, we thus have that

$$\Delta^{-1} \cdot (f - \alpha G_k) \in M_{k-12} = \{0\}, \tag{1.57}$$

so that $f = \alpha G_k$. It is thus clear that $M_k = \mathbb{C} G_k$. As for $k = 0$, we have that $1 \in M_0$, and so for $f \in M_0$ we have that $f - 1 \cdot c(f;0) \in S_0$ and thus

$$\Delta^{-1} \cdot (f - c(f;0)) \in M_{-12} = \{0\}, \tag{1.58}$$

implying that $f = c(f;0)$. This implies that $M_0 = \mathbb{C}$. In summary, we have the following values of $\dim(M_k)$ thus far:

k	0	2	4	6	8	10
$\dim(M_k)$	1	0	1	1	1	1

For $k \geq 12$, we apply induction and Proposition 1.5. Let $q \in \mathbb{Z}_{\geq 0}$ and $r \in \{0, 4, 6, 8, 10\}$ and suppose that the theorem holds for $k = 12q + r$. Let $f \in M_{12(q+1)+r}$. Then for some $\alpha \in \mathbb{C}^\times$ we have that $f - \alpha G_{12(q+1)+r} \in S_{12(q+1)+r}$, and thus by Proposition 1.5, it holds that

$$\Delta^{-1} \cdot (f - \alpha G_{12(q+1)+r}) \in M_{12q+r}, \tag{1.59}$$

and so $f \in \Delta \, \mathrm{M}_{12q+r} \oplus \mathbb{C} G_{12(q+1)+r}$. This implies that

$$\mathrm{M}_{12(q+1)+r} = \Delta \, \mathrm{M}_{12q+r} \oplus \mathbb{C} G_{12(q+1)+r}, \tag{1.60}$$

whence by the induction assumption we have that

$$\dim(\mathrm{M}_{12(q+1)+r}) = \begin{cases} q+1 & \text{if } r = 2 \\ q+2 & \text{if } r \in \{0,4,6,8,10\}. \end{cases} \tag{1.61}$$

Since $\lfloor \frac{12(q+1)+r}{12} \rfloor = q+1$, this proves the theorem. □

The time we spent on Eisenstein series turns out to have been well-spent – we can in fact express every element in M_k as a linear combination of products of Eisenstein series.

Proposition 1.9. Let $k \in \mathbb{Z}_{\geq 4}$. Then

$$\mathcal{A}_k = \{G_4^a \cdot G_6^b : a, b \in \mathbb{Z}_{\geq 0} \text{ and } 4a + 6b = k\}, \tag{1.62}$$

is a basis for M_k.

Proof. Let us first show that $\operatorname{span}_{\mathbb{C}} \mathcal{A}_k = \mathrm{M}_k$. Propositions 1.8 and 1.4 directly implies that

$$\begin{aligned} &\mathrm{M}_4 = \mathbb{C} G_4, && \mathrm{M}_6 = \mathbb{C} G_6, && \mathrm{M}_8 = \mathbb{C} G_4^2 \\ &\mathrm{M}_{10} = \mathbb{C} G_4 \cdot G_6, && \mathrm{M}_{12} = \mathbb{C} G_6^2 \oplus \mathbb{C}\Delta, && \mathrm{M}_{14} = \mathbb{C} G_4^2 \cdot G_6. \end{aligned} \tag{1.63}$$

Since $\Delta \in \operatorname{span}_{\mathbb{C}} \mathcal{A}_{12}$, we have that (1.63) settles $\operatorname{span}_{\mathbb{C}} \mathcal{A}_k = \mathrm{M}_k$ for $k \in \{4,6,8,10,12,14\}$. For higher weights, we again apply induction and Proposition 1.5. Let $q \in \mathbb{Z}_{\geq 0}$ and $r \in \{4,6,8,10,12,14\}$ be given, and suppose that $\operatorname{span}_{\mathbb{C}} \mathcal{A}_k = \mathrm{M}_k$ holds for $k = 12q + r$. Let $f \in \mathrm{M}_{12(q+1)+r}$ and let $a, b \in \mathbb{Z}_{\geq 0}$ satisfy $4a + 6b = k + 12$. Since $\operatorname{ord}(G_4^a \cdot G_6^b) = 0$, we have that there exists some $\alpha \in \mathbb{C}^\times$ satisfying that $f - \alpha G_4^a \cdot G_6^b \in \mathrm{S}_{k+12}$. Proposition 1.5 thus implies that

$$f - \alpha G_4^a \cdot G_6^b \in \Delta \, \mathrm{M}_k. \tag{1.64}$$

Applying the induction assumption, we obtain that

$$\Delta \, \mathrm{M}_k \subseteq \operatorname{span}_{\mathbb{C}} \mathcal{A}_{k+12}, \tag{1.65}$$

and thus $f \in \operatorname{span}_{\mathbb{C}} \mathcal{A}_{k+12}$. By induction, we now obtain that $\operatorname{span}_{\mathbb{C}} \mathcal{A}_k = \mathrm{M}_k$ for all even weights $k \in \mathbb{Z}_{\geq 4}$.

As for linear independence, let $k \in \mathbb{Z}_{\geq 4}$ be even, and suppose that we have a relation

$$\sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c_{a,b} G_4^a G_6^b = 0, \tag{1.66}$$

for some coefficients $c_{a,b} \in \mathbb{C}$. By multiplying (1.66) with $G_4 \cdot G_6$, G_4^2, G_6, G_4, or $G_4^2 \cdot G_6$, when the remainder of k modulo 12 is 2, 4, 6, 8, or 10; respectively, we may assume that $k \equiv_{12} 0$. This implies[7] that $3 \mid a$. We also have that $G_6^{k/6} = G_6^{2a/3+b}$, implying that

$$G_4^a G_6^{k/6} = G_4^a G_6^b G_6^{2a/3}, \tag{1.67}$$

and so $(G_4^3/G_6^2)^{a/3} = G_4^a G_6^b / G_6^{k/6}$. Dividing (1.66) by $G_6^{k/6}$ we thus obtain that

$$\sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c_{a,b} (G_4^3/G_6^2)^{a/3} = 0. \tag{1.68}$$

This implies that the polynomial

$$p = \sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c_{a,b} X^{a/3} \in \mathbb{C}[X], \tag{1.69}$$

has infinitely many zeros, and so $p = 0$. Therefore all the coefficients $c_{a,b}$ are equal to zero, and we conclude that the elements of $\mathcal{A}_k$ are linearly independent. □

Let us now see an example of the power of Proposition 1.9, and of M_k in general. For $k \geq 4$, we let $E_k = \frac{1}{2\zeta(k)} G_k$. Since $1 = c(E_4 \cdot E_6; 0) = c(E_{10}; 0)$, Proposition 1.9 implies that $E_{10} = E_4 \cdot E_6$. This implies that

$$1 - 264 \sum_{n\geq 1} \sigma_9(n) e^{2\pi i n\tau} = \left(1 + 240 \sum_{n\geq 1} \sigma_3(n) e^{2\pi i n\tau}\right) \cdot \left(1 - 504 \sum_{n\geq 1} \sigma_5(n) e^{2\pi i n\tau}\right). \tag{1.70}$$

Expanding the product, we obtain the following identity [8]

$$11\,\sigma_9(n) = 21\,\sigma_5(n) - 10\,\sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m). \tag{1.71}$$

7 Suppose that $12l = 4a + 6b$. Then $6l = 2a + 3b$. If b is even, this implies that $6 \mid 2a$ so that $3 \mid a$. If b is odd, this implies that $3 \mid 2a$, so that by Euclid's lemma we have that $3 \mid$.

8 This can be taken as a justification for studying modular forms. They seem to "magically" encode arithmetic information.

We can of course apply the same technique to E_8, E_{12}, and E_{14}, to obtain similar identities involving σ_7, σ_{11} and σ_{13}; respectively. We would like to remark that we are not aware of any proof of (1.71) or identities like it, relying solely on elementary means. However, with just a little understanding of M_k, we obtain non-trivial arithmetic identities with negligible amounts of extra work.

Using the Fast Fourier Transform (FFT), we can quickly compute the Fourier expansions of the elements of $\mathcal{A}_k$. Combining this with Propositions 1.7 and 1.9, and row-reduction, we obtain a means to express any element $f \in \mathrm{M}_k$ as a sum of products of Eisenstein series.

We can describe this algorithmically as follows:

Algorithm 1.1: Determine if a given truncated Fourier series expansion corresponds to an element of M_k, and if so express it in the basis $\mathcal{A}_k$.

input : An even integer $k \in \mathbb{Z}_{\geq 4}$, and a vector $v \in \mathbb{C}^{\lfloor k/12 \rfloor + 1}$.
output: Either $\perp$, or coefficients $\alpha_p \in \mathbb{C}$, $p \in \mathcal{A}_k$, satisfying $v_i = \sum_{p \in \mathcal{A}_k} \alpha_p \cdot c(p; i-1)$ for all i.

$N \leftarrow \lfloor k/12 \rfloor$;
$D \leftarrow \dim(\mathrm{M}_k)$;
$A \leftarrow \mathbf{0}_{(N+1)\times(D+1)} \in \mathbb{C}^{(N+1)\times(D+1)}$;
$A_{\cdot,D+1} \leftarrow v$;
$B \leftarrow \mathbf{0}_D \in \mathbb{C}[X,Y]^D$;
for $\lceil \frac{k}{6} \rceil \leq n \leq \lfloor \frac{k}{4} \rfloor$ **do**
  $a \leftarrow -\frac{k}{2} + 3n$;
  $b \leftarrow \frac{k}{2} - 2n$;
  $B_n \leftarrow X^a Y^b$;
  compute $p_n \leftarrow (c(G_4^a G_6^b; 0), \ldots, c(G_4^a G_6^b; N))^T$ with FFT;
  $A_{\cdot,n} \leftarrow p_n$;
end
$A_{\mathrm{rref}} \leftarrow \mathtt{rref}(A)$;
$r \leftarrow \mathrm{rank}(A)$;
if $r \neq D$ **then**
  return $\perp$;
else
  $\alpha \leftarrow (A_{\mathrm{rref}})_{\leq D, D+1}$;
  $P \leftarrow \sum_{i=1}^{D} \alpha_i B_i$;
  return P;
end

Before ending this chapter, let us briefly explain some of the design choices of Algorithm 1.1. In a general purpose computer language, there is no default way of representing an element of M_k. However, we know that they can be viewed as

(a) polynomials in $\mathbb{C}[X, Y]$, where X corresponds to G_4 and Y corresponds to G_6,

(b) and as Fourier series expansions truncated at the Sturm bound, that is, elements of

$$\frac{\mathbb{C}[\![q]\!]}{q^{\lfloor k/12 \rfloor + 1}\mathbb{C}[\![q]\!]}, \tag{1.72}$$

where the formal variable q corresponds to $e^{2\pi i \tau}$.

The vector B relies on interpretation (a), and line 10 relies on interpretation (b).

Furthermore, the enumeration on line 6 is motivated by the fact that the solutions to the linear Diophantine equation $4a + 6b = k$ is given by $(a, b) = (-\frac{k}{2} + 3n, \frac{k}{2} - 2n)$ where $n \in \mathbb{Z}$. The condition $a, b \geq 0$ is equivalent to $\lceil \frac{k}{6} \rceil \leq n \leq \lfloor \frac{k}{4} \rfloor$.

2 GENERAL MODULAR FORMS

In the previous chapter, we sketched out a more or less complete picture of the theory of the space of weight k modular forms, M_k. We saw that it was intimately tied to the Eisenstein series G_k, and that M_k captured certain arithmetic information; see for example (1.71).

Since studying M_k led to non-trivial arithmetic insights, it stands to reason that we should obtain yet more insights of such nature by generalizing the definition of M_k in various ways. There are couple of obvious routes one can take for this. For example, one can:

(i) modify the group on which the invariance condition is satisfied,

(ii) modify the invariance condition itself,

(iii) modify the domain, or

(iv) modify the range,

or modify any combination of the above. To mention a few well-known examples we have that (i) and (ii) lead to modular forms of higher level and modular forms of higher order; that (ii) and (iii) lead to Jacobi forms; that (i), (ii), (iii) and (iv) lead to Siegel and Bianchi modular forms; that (i), (ii), and (iii) lead to Hilbert modular forms; and that (i), (ii), and (iv) lead to vector-valued modular forms.

As the title of this thesis indicates, we will pay particular attention to vector-valued modular forms. We will show how they capture modular forms of higher level, modular forms of higher order, and considerable generalizations thereof. Work by Eichler and Zagier [6, 17] also shows that vector-valued modular forms capture Jacobi forms. Beside the advantage of having a unified theory for classes of modular forms that traditionally have been treated as disjoint, our approach also allows us to use a unified framework to compute bases for these types of modular forms.

Furthermore, as we shall see later, our unified computational framework bears a very distinct resemblance to the computational framework for M_k that we outlined in the previous chapter.

In this chapter, we provide an outline of the theory of modular forms of higher level and of modular forms of higher order. This serves as preparation for the material that we present in the latter parts of the thesis.

2.1 MODULAR FORMS ON SUBGROUPS

Let us first define precisely what we mean by a modular form on a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. In order to obtain a finite-dimensional space, we restrict our scope to subgroups with finite index in $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1. Let $k \in \mathbb{Z}$ and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup, and let $\nu : \Gamma \to \mathbb{C}^\times$ be a multiplicative character. Then we say that a function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k for Γ with character ν if

1. for all $\gamma \in \Gamma$ it holds that
$$(f\big|_k\gamma)(\tau) = \nu(\gamma)f(\tau), \quad \tau \in \mathbb{H}, \tag{2.1}$$
2. f is holomorphic, and
3. there exists a number $a \in \mathbb{R}$ such that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have that
$$|(f\big|_k\gamma)(\tau)| = O(\mathrm{Im}(\tau)^a) \quad \text{as} \quad \mathrm{Im}(\tau) \to \infty, \tag{2.2}$$
uniformly in $\mathrm{Re}(\tau)$.

We remind the reader that the weight k slash action $\big|_k$ was defined in Remark 1.1. If in addition, we have for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ that
$$(f\big|_k\gamma)(\tau) \to 0 \quad \text{as} \quad \mathrm{Im}(\tau) \to \infty, \tag{2.3}$$
uniformly in τ, we call f a cusp form. The space of weight k modular forms and cusp forms for Γ with character ν, is denoted by $\mathrm{M}_k(\Gamma, \nu)$ and $\mathrm{S}_k(\Gamma, \nu)$. Furthermore, we write $\mathrm{M}_k(\Gamma) = \mathrm{M}_k(\Gamma, \mathbf{1}_\Gamma)$ and $\mathrm{S}_k(\Gamma) = \mathrm{S}_k(\Gamma, \mathbf{1}_\Gamma)$, where $\mathbf{1}_\Gamma$ is the trivial character, defined by $\mathbf{1}_\Gamma(\gamma) = 1$ for all $\gamma \in \Gamma$

Arguably, the only unexpected difference is in the growth condition (2.2). To understand what it comes from, we need talk a little bit about *cusps*.

We extend the Möbius action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ to $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ by letting
$$\gamma q = \frac{aq+b}{cq+d}, \quad \gamma(-\frac{d}{c}) = i\infty, \quad \gamma(i\infty) = \frac{a}{c}, \qquad q \in \mathbb{Q}, \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.4}$$
Note that $\lim_{t\to+\infty} \gamma(it) = a/c$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, so the above extension is well-defined.

We call the elements in $\mathbb{Q} \cup \{i\infty\}$ the cusps of $\mathbb{H}^*$ and given a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, we call the orbits in $\Gamma\backslash\mathbb{Q} \cup \{i\infty\}$ the cusp classes associated to Γ. Let $q = a/c \in \mathbb{Q}$ with $\gcd(a, c) = 1$. Then there exists integers $b, d \in \mathbb{Z}$ such that $ad - bc = 1$ and thus $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(i\infty) = a/c$. This means that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the

cusps, or equivalently, that there is only one cusp class, namely $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{Q}\cup\{i\infty\} = \{\mathrm{SL}_2(\mathbb{Z})i\infty\}$.

In fact, for a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, there are only finitely many cusp classes associated to Γ. To be precise, we have the following proposition.

Proposition 2.1. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of index $m \in \mathbb{Z}_{\geq 1}$. Then it holds that

$$|\Gamma\backslash\mathbb{Q}\cup\{i\infty\}| \leq m. \tag{2.5}$$

Proof. We prove this by establishing a bijective correspondence with a certain double quotient. To be precise, we let

$$\begin{aligned} F : \Gamma\backslash\mathbb{Q}\cup\{i\infty\} &\to \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty \quad \text{be defined by} \\ F(\Gamma q) &= \Gamma\gamma_q\Gamma_\infty, \end{aligned} \tag{2.6}$$

where $\gamma_q \in \mathrm{SL}_2(\mathbb{Z})$ is any element satisfying $\gamma_q(i\infty) = q$.

We first prove that F is well-defined. Let therefore $q_1, q_2 \in \mathbb{Q}\cup\{i\infty\}$ satisfy that $\Gamma q_1 = \Gamma q_2$. Then $q_1 = \gamma q_2$ for some $\gamma \in \Gamma$. There exists elements $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ such that $q_i = \gamma_i(\infty)$ for $i \in \{1,2\}$. Consequently, we have that $\gamma_1^{-1}\gamma\gamma_2(i\infty) = i\infty$ implying that $\gamma_1^{-1}\gamma\gamma_2 \in \Gamma_\infty$. Then $\gamma\gamma_2\Gamma_\infty = \gamma_1\Gamma_\infty$, and so $\Gamma\gamma_2\Gamma_\infty = \Gamma\gamma_1\Gamma_\infty$, and thus F is well-defined.

Repeating the same argument in reverse, we see that F is injective. Finally, to see that F is surjective, we just note that for an arbitrary element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have that $F(\Gamma\frac{a}{c}) = \Gamma\gamma\Gamma_\infty$, where we identify $a/0$ with $i\infty$.

It is clear that $|\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty| \leq |\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})|$, and thus the result follows. □

We now have enough of an understanding of the cusps of $\mathbb{H}^*$ to explain the difference in the growth condition. We summarize it in the following proposition.

Proposition 2.2. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup, and let $\{\gamma_1, \dots, \gamma_r\} \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a complete set of representatives for $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$. Let also $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function. Then f satisfies the growth condition (2.2) if and only if there exists a number $a \in \mathbb{R}$, such that for all $1 \leq i \leq r$, it holds that

$$|f(\gamma_i\tau)| = O(\mathrm{Im}(\tau)^a) \quad \text{as} \quad \mathrm{Im}(\tau) \to \infty, \tag{2.7}$$

uniformly in $\mathrm{Re}(\tau)$.

Proof. If the growth condition (2.2) is satisfied, then the condition (2.7) is immediately satisfied. As for the converse, let $\gamma = \mathrm{SL}_2(\mathbb{Z})$. Then for some $1 \leq i \leq r$ we have that $\Gamma\gamma\Gamma_\infty = \Gamma\gamma_i\Gamma_\infty$, and hence $\gamma = \alpha\gamma_i\beta$ for some $\alpha \in \Gamma$ and $\beta \in \Gamma_\infty$. This implies that

$$(f\big|_k\gamma)(\tau) = (f\big|_k\gamma_i\beta)(\tau) = (c(\gamma_i\beta)\tau + d(\gamma_i\beta))^{-k} f(\gamma_i\beta\tau). \tag{2.8}$$

Since $\beta(i\infty) = i\infty$, it is clear that

$$\lim_{\mathrm{Im}(\tau)\to\infty} |f(\gamma_i\beta\tau)| = \lim_{\mathrm{Im}(\tau)\to\infty} |f(\gamma_i\tau)|. \tag{2.9}$$

We also have that $|c(\gamma_i\beta)\tau + d(\gamma_i\beta)|^{-k} = O(\mathrm{Im}(\tau)^{-k})$ as $\mathrm{Im}(\tau) \to \infty$, uniformly in $\mathrm{Re}(\tau)$, and thus it follows that f satisfies (2.2). □

Remark 2.1. By Proposition 2.1, we have that $\{\gamma_j(i\infty)\}_{j=1}^{r}$ forms a complete set of representatives for the cusp classes $\Gamma\backslash\mathbb{Q} \cup \{i\infty\}$. For this reason, the growth condition is often summarized as f having *moderate growth at the cusps*. We will use this terminology throughout the rest of this thesis.

Note that if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty = \{\Gamma\, 1\, \Gamma_\infty\}$. This shows how the condition (2.2) generalizes (1.21).

We have yet to explain why we need the growth condition at all, even for the basic case of M_k. This is a question that is more suited for an introductory textbook on modular forms, but we remark that it is needed for $\mathrm{M}_k(\Gamma, \nu)$ to be finite-dimensional.

For completeness, we give an exact formula for $\dim(\mathrm{M}_k(\Gamma, \nu))$, but before we state it we recall that an elliptic point on $\mathbb{H}$ is a point $\tau \in \mathbb{H}$ satisfying that

$$\Gamma_\tau = \{\gamma \in \Gamma : \gamma\tau = \tau\}, \tag{2.10}$$

is non-trivial. See [14, Chapter 2.3]. The following formula for the dimension is due to Borcherds [11], who bases his presentation on Fischer [7].

Proposition 2.3. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup and let $\nu : \Gamma \to \mathbb{C}^\times$ be a multiplicative character. Let also:

- E be the set of all elliptic points in $\mathbb{H}$ with respect to the action from Γ,
- $\{e_1, \ldots, e_P\} \subseteq \mathbb{H}$, $P \in \mathbb{Z}_{\geq 1}$, be a complete set of representatives for the orbit space $\Gamma\backslash E$,
- $P_j = |\Gamma_{e_j}|/2$, for each $1 \leq j \leq P$,
- R_j is a generator for Γ_{e_j} conjugate to the element $\begin{pmatrix} \cos(\pi/P_j) & -\sin(\pi/P_j) \\ \sin(\pi/P_j) & \cos(\pi/P_j) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$,
- g be the genus[1] of $\Gamma\backslash\mathbb{H}^*$,
- $\{c_1, \ldots, c_C\} \subseteq \mathbb{Q} \cup \{i\infty\}$, $C \in \mathbb{Z}_{\geq 1}$, be a set of representatives of $\Gamma\backslash\mathbb{Q} \cup \{i\infty\}$,

1 The genus is a topological invariant which abstractly counts the number of "holes" of a surface. It makes sense for $\Gamma\backslash\mathbb{H}^*$ when viewed as a compact Riemann surface, and can be computed explicitly with the Riemann-Hurwitz formula. See [14, Chapter 3.1], for details.

- T_j be an element conjugate in $\mathrm{SL}_2(\mathbb{R})$ to T^{-1} satisfying that $\Gamma_{c_j} = \langle T_j, S^2 \rangle$,
- and finally

$$\omega = 2\pi\Big(2g - 2 + C + \sum_{j=1}^{P}(1 - 1/\nu_j)\Big), \tag{2.11}$$

be the area of $\Gamma\backslash\mathbb{H}^*$ with respect to the Γ-invariant measure $\mathrm{d}x\mathrm{d}y/y^2$.

Furthermore, if $z \in \mathbb{C}^\times$ is an Nth root of unity and $\gamma \in \Gamma$, we write

$$\delta_{\nu,N}(z,\gamma) = \frac{1}{N}\sum_{j=1}^{N-1}\frac{z\cdot\nu(\gamma)}{1-e(j/N)} \quad \text{and} \quad \delta_{\nu,\infty}(z,\gamma) = \delta_{\nu,N}(z,\gamma) + \frac{\nu(\gamma)}{2N}. \tag{2.12}$$

Then

$$\dim \mathrm{M}_k(\Gamma,\nu) = \frac{1}{2}(\psi(I) + i\psi(-I)) \tag{2.13}$$

where

$$\psi(\gamma) = \frac{(k-1)\omega}{4\pi}\nu(\gamma) + \sum_{j=1}^{P}\delta_{\nu,P_j}(e(\frac{k}{2P_j})\nu(R_j),\gamma) + \sum_{j=1}^{C}\delta_{\nu,\infty}(\nu(T_j),\gamma), \tag{2.14}$$

for $\gamma \in \{I, -I\}$.

Proof. See [11]. □

We also have a Sturm bound for $\mathrm{M}_k(\Gamma,\nu)$ when its elements may be expanded into Fourier series.

Proposition 2.4. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be an index $m \in \mathbb{Z}_{\geq 1}$ subgroup with $\pm T^N \in \Gamma$ for some $N \in \mathbb{Z}_{\geq 1}$. Let also $\nu : \Gamma \to \mathbb{C}^\times$ be a multiplicative character, and $k \in \mathbb{Z}$. Then if a modular form $f \in \mathrm{M}_k(\Gamma,\nu)$ satisfies that

$$c(f;n) = 0 \quad \text{for all } n \in \mathbb{Q} \text{ with } \quad 0 \leq n \leq \Big\lfloor \frac{km}{12} \Big\rfloor, \tag{2.15}$$

it holds that $f = 0$.

Proof. See [18]. □

2.1.1 *Modular Forms of Higher Level*

When it comes to modular forms for subgroups of $\mathrm{SL}_2(\mathbb{Z})$, there is one family of subgroups of particular interest – the congruence subgroups. Let us start with the principal congruence subgroup. For any integer $N \in \mathbb{Z}_{\geq 1}$, we have a homomorphism $\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ given by

$$\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + N\mathbb{Z} & b + N\mathbb{Z} \\ c + N\mathbb{Z} & d + N\mathbb{Z} \end{pmatrix} \tag{2.16}$$

We see that the normal subgroup $\ker(\psi) \subseteq \mathrm{SL}_2(\mathbb{Z})$ is given by

$$\ker(\psi) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv_N 1_{2\times 2}\}, \tag{2.17}$$

where the congruence is regarded component-wise. This is the principal congruence subgroup, which we denote by $\Gamma(N)$.

We say that a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup if for some integer $N \in \mathbb{Z}_{\geq 1}$ it holds that $\Gamma(N) \subseteq \Gamma$. The smallest such N is called the level of Γ, and we write

$$\mathrm{level}(\Gamma) = \min\{N \in \mathbb{Z}_{\geq 1} : \Gamma(N) \subseteq \Gamma\}. \tag{2.18}$$

We see for example that $\mathrm{level}(\Gamma(N)) = N$ and that $\mathrm{level}(\mathrm{SL}_2(\mathbb{Z})) = 1$.

Similarly, if Γ is a congruence subgroup of level $N \in \mathbb{Z}_{\geq 1}$, $\nu : \Gamma \to \mathbb{C}^\times$ is a multiplicative character, and $k \in \mathbb{Z}$ is an integer, we say that the modular forms in $\mathrm{M}_k(\Gamma, \nu)$ have level N.

In addition to the principal congruence subgroup, there are two other congruence subgroups that are particularly important, $\Gamma_0(N)$ and $\Gamma_1(N)$. They are defined by

$$\begin{aligned} \Gamma_1(N) &= \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv_N \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \\ \Gamma_0(N) &= \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv_N \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}, \end{aligned} \tag{2.19}$$

where $*$ denotes an arbitrary integer. Recall now that a Dirichlet character of modulus $N \in \mathbb{Z}_{\geq 1}$ is a completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ such that for any $a \in \mathbb{Z}$, we have that

$$\chi(a) = \begin{cases} 0 & \text{if } \gcd(a, N) > 1 \\ 1 & \text{if } \gcd(a, N) = 1, \end{cases} \tag{2.20}$$

and $\chi(a+N) = \chi(a)$. Note that for an integer $a \in \mathbb{Z}$ satisfying $\gcd(a,N) = 1$, we have that $\chi(a)^{\phi(N)} = \chi(a^{\phi(N)}) = \chi(1) = 1$. This implies that the non-zero values of a Dirichlet character lie on the unit circle.

The set of Dirichlet characters modulo $N \in \mathbb{Z}_{\geq 1}$ forms a group with respect to multiplication, denoted by $D(N)$. Given a Dirichlet character $\chi \in D(N)$, we can construct a multiplicative character $\widetilde{\chi} : \Gamma_0(N) \to \mathbb{C}^\times$ by setting $\widetilde{\chi}(\gamma) = \chi(d(\gamma))$. By slight abuse of notation, we identify every element $\chi \in D(N)$ with the multiplicative character $\Gamma_0(N) \to \mathbb{C}^\times$ constructed from χ.

The reason for the importance of $\Gamma_0(N)$ and $\Gamma_1(N)$ are their corresponding spaces of modular forms $\mathrm{M}_k(\Gamma_1(N))$ and $\mathrm{M}_k(\Gamma_0(N),\chi)$ where χ is a Dirichlet character for $\Gamma_0(N)$, and where $N \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}_{\geq 2}$. These spaces are the topic of the celebrated Modularity Theorem, which famously implies Fermat's Last Theorem.

Just as with M_k, we can construct Eisenstein series for $\mathrm{M}_k(\Gamma_1(N))$ and $\mathrm{M}_k(\Gamma_0(N),\chi)$, the span of which is complementary to the spaces of cusp forms $\mathrm{S}_k(\Gamma_1(N))$ and $\mathrm{S}_k(\Gamma_0(N),\chi)$.

These Eisenstein series are constructed as linear combinations of the Eisenstein series for $\Gamma(N)$, which are defined as follows. Let $k \in \mathbb{Z}_{\geq 3}$ and $N \in \mathbb{Z}_{\geq 1}$. Then for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we let

$$G_{k,N,\Gamma_1(N)\gamma}(\tau) = \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (c,d)\equiv_N (c(\gamma),d(\gamma))}} (c\tau+d)^{-k}, \quad \tau \in \mathbb{H}. \tag{2.21}$$

We note that we have a bijection

$$\begin{aligned} \Gamma_1(N)\backslash\mathrm{SL}_2(\mathbb{Z}) &\to \{([c],[d]) \in (\mathbb{Z}/N\mathbb{Z})^2 : \gcd(c,d,N) = 1\} \\ \Gamma_1(N)\gamma &\mapsto ([c(\gamma)],[d(\gamma)]), \end{aligned} \tag{2.22}$$

showing that $G_{k,N,\Gamma_1(N)\gamma}$ is well-defined as a formal power series. After some work, see for example [16] or [14], one verifies that $G_{k,N,\Gamma_1(N)\gamma}$ converges absolutely and locally uniformly on $\mathbb{H}$, and hence it is a holomorphic function. It is also of moderate growth at the cusps. Furthermore, for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathbb{H}$, and $\delta = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we find that

$$\begin{aligned} (G_{k,N,\Gamma_1(N)\gamma}\big|_k\delta)(\tau) &= \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (c,d)\equiv_N (c(\gamma),d(\gamma))}} (\tau(Ac+Cd)+Bc+Dd)^{-k} \\ &= \sum_{\substack{(c',d')\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (c',d')\delta^{-1}\equiv_N (c(\gamma),d(\gamma))}} (c'\tau+d')^{-k} \\ &= \sum_{\substack{(c',d')\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (c',d')\equiv_N (c(\gamma),d(\gamma))\delta}} (c'\tau+d')^{-k} \\ &= G_{k,N,\Gamma_1(N)\gamma\delta}(\tau). \end{aligned} \tag{2.23}$$

Since $\Gamma(N)$, being the kernel of a homomorphism, is a normal subgroup and since $\Gamma(N) \subseteq \Gamma_1(N)$, we find that $G_{k,N,\Gamma_1(N)\gamma}|_k\delta = G_{k,N,\Gamma_1(N)\gamma}$ for $\delta \in \Gamma(N)$. This means that $G_{k,N,\Gamma_1(N)\gamma} \in \mathrm{M}_k(\Gamma(N))$. We let

$$\mathrm{E}_k(\Gamma(N)) = \mathrm{span}_{\mathbb{C}}\{G_{k,N,\Gamma_1(N)\gamma} : \gamma \in \mathrm{SL}_2(\mathbb{Z})\}. \tag{2.24}$$

Using some additional machinery one can show that

$$\mathrm{M}_k(\Gamma(N)) = \mathrm{E}_k(\Gamma(N)) \oplus \mathrm{S}_k(\Gamma(N)), \tag{2.25}$$

see for example [16]. This is the generalization to $\mathrm{M}_k(\Gamma(N))$ of Proposition 1.4. Though not very insightful to state here, we record that the Fourier series expansions of $G_{k,N,\Gamma_1(N)\gamma}$ are readily available, see for example [16] or [14].

To construct Eisenstein series for $\Gamma_0(N)$ and $\Gamma_1(N)$ we first need to talk some more about Dirichlet characters. Given integers $M, N \in \mathbb{Z}_{\geq 1}$ with $M \mid N$, and Dirichlet characters $\chi_1 \in D(N)$, $\chi_2 \in D(M)$, we say that χ_1 is *induced* by χ_2 if

$$\chi_1(a) = \chi_2(a) \quad \text{for all} \quad a \in \mathbb{Z} \quad \text{with} \quad \gcd(a, N) = 1. \tag{2.26}$$

The smallest modulus M for which there exists a character χ_2 inducing χ_1, is called the *conductor* of χ_1, and is denoted by $\mathrm{cond}(\chi_1)$. If $\mathrm{cond}(\chi_1) = N$, we say that χ_1 is *primitive*.

Let now $k \in \mathbb{Z}_{\geq 3}$ and $u, v \in \mathbb{Z}_{\geq 1}$ be integers, and let $\chi_1 \in D(u)$ and $\chi_2 \in D(v)$ be Dirichlet characters satisfying that[2] $(\chi_1\chi_2)(-1) = (-1)^k$ and that χ_2 is primitive. For $\tau \in \mathbb{H}$, we then let

$$G_{k,\chi_1,\chi_2}(\tau) = \sum_{\substack{0\leq c<u\\0\leq d<v\\0\leq e<u}} \chi_1(c)\overline{\chi_2(d)}G_{k,N,F^{-1}([cv],[d+ev])}(\tau) \tag{2.27}$$

where F denotes the bijection (2.22). We find that for $\gamma \in \Gamma_0(N)$ it holds that

$$G_{k,\chi_1,\chi_2}|_k\gamma = \sum_{\substack{0\leq c<u\\0\leq d<v\\0\leq e<u}} \chi_1(c)\overline{\chi_2(d)}G_{k,N,F^{-1}([cv],[d+ev])\gamma}(\tau). \tag{2.28}$$

Through a clever change of variables, see for example [14], one finds that

$$\sum_{\substack{0\leq c<u\\0\leq d<v\\0\leq e<u}} \chi_1(c)\overline{\chi_2(d)}G_{k,N,F^{-1}([cv],[d+ev])\gamma}(\tau) = (\chi_1\chi_2)(d(\gamma))G_{k,\chi_1,\chi_2}(\tau). \tag{2.29}$$

This means that $G_{k,\chi_1,\chi_2} \in \mathrm{M}_k(\Gamma_0(N), \chi_1\chi_2)$. Since $(\chi_1\chi_2)(1) = 1$, it also holds that $G_{k,\chi_1,\chi_2} \in \mathrm{M}_k(\Gamma_1(N))$.

2 Here juxtaposition denotes multiplication, so $(\chi_1\chi_2)(a) = \chi_1(a)\chi_2(a)$.

We now have all the ingredients to describe all the Eisenstein series in $\mathrm{M}_k(\Gamma_1(N))$ and $\mathrm{M}_k(\Gamma_0(N),\chi)$ where $\chi \in D(N)$. For $N \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}_{\geq 3}$, we let

$$A_{N,k} = \{(\chi_1,\chi_2,t) : \exists u_1,u_2 \in \mathbb{Z}_{\geq 1}.\, tu_1u_2 \mid N, \\ \chi_i \in D(u_i),\, \chi_i \text{ primitive}, (\chi_1\chi_2)(-1) = (-1)^k\}. \quad (2.30)$$

For $(\chi_1,\chi_2,t) \in A_{N,k}$ and $\tau \in \mathbb{H}$, we let

$$G_{k,\chi_1,\chi_2,t}(\tau) = G_{k,\chi_1,\chi_2}(t\tau). \quad (2.31)$$

Since $G_{k,\chi_1,\chi_2} \in \mathrm{M}_k(\Gamma_1(u_1u_2))$, where u_i is the modulus of χ_i, we find that $G_{k,\chi_1,\chi_2,t} \in \mathrm{M}_k(\Gamma_1(tu_1u_2))$. However, since $tu_1u_2 \mid N$, we have that $\mathrm{M}_k(\Gamma_1(N))$. For $\chi \in D(N)$, we now let

$$\begin{aligned} \mathrm{E}_k(\Gamma_1(N)) &= \operatorname{span}_{\mathbb{C}}\{G_{k,\chi_1,\chi_2,t} : (\chi_1,\chi_2,t) \in A_{N,k}\} \\ \mathrm{E}_k(\Gamma_0(N),\chi) &= \operatorname{span}_{\mathbb{C}}\{G_{k,\chi_1,\chi_2,t} : (\chi_1,\chi_2,t) \in A_{N,k},\, \chi_1\chi_2 = \chi\}. \end{aligned} \quad (2.32)$$

These spaces contain all the Eisenstein series, in the sense that

$$\begin{aligned} \mathrm{M}_k(\Gamma_1(N)) &= \mathrm{E}_k(\Gamma_1(N)) \oplus \mathrm{S}_k(\Gamma_1(N)) \\ \mathrm{M}_k(\Gamma_0(N),\chi) &= \mathrm{E}_k(\Gamma_0(N),\chi) \oplus \mathrm{S}_k(\Gamma_0(N),\chi). \end{aligned} \quad (2.33)$$

Hence, to compute $\mathrm{M}_k(\Gamma_1(N))$ and $\mathrm{M}_k(\Gamma_0(N),\chi)$, it remains to compute the spaces of cusp forms $\mathrm{S}_k(\Gamma_1(N))$ and $\mathrm{S}_k(\Gamma_0(N),\chi)$. Thankfully, due to a theorem by Raum and Xià, it turns out that the spaces of cusp forms are spanned by products of Eisenstein series in $\mathrm{E}_k(\Gamma(N))$. We will describe this theorem in detail in the summary of paper I.

THE VECTOR-VALUED CONNECTION In the next chapter, we will see that the spaces $\mathrm{M}_k(\Gamma,\nu)$ are isomorphic to weight k vector-valued modular forms associated to the induced representation $\operatorname{Ind}_{\Gamma}^{\mathrm{SL}_2(\mathbb{Z})}\nu$. This allows to study $\mathrm{M}_k(\Gamma,\nu)$ as a special case of vector-valued modular forms of *congruence type*.

2.2 HIGHER ORDER MODULAR FORMS

In the previous section, the major change from the classical setting was in modifying the group on which in the invariance condition is satisfied. We only made a minor change to the invariance condition itself. In this section, we will make a more substantial change to the invariance condition as well.

Our starting point can somewhat facetiously be stated as the following question:

> what if we require our functions not to be invariant after one, but after two (or more) slashes?

This question was considered in the context of what are known as *modular symbols* by Goldfeld, in his paper [10]. For clarity, let us bring in a definition.

Definition 2.2. Let $k \in \mathbb{Z}$ and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup containing $\pm T^N$ for some $N \in \mathbb{Z}_{\geq 1}$. Then a modular form of order 1 and weight k for Γ, is an element of $\mathrm{M}_k(\Gamma)$. A modular form of order $n \in \mathbb{Z}_{\geq 2}$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ with moderate growth at the cusps, satisfying that $f\big|_k(\gamma - 1) = 0$ for all $\Gamma \in \Gamma_\infty \cap \Gamma$, and that for every $\gamma \in \Gamma$, the functions

$$f\big|_k(\gamma - 1), \tag{2.34}$$

are modular forms of order $n-1$. We write $\mathrm{M}_k^{[n]}(\Gamma)$ for the $\mathbb{C}$-vector space of modular forms.

Goldfeld studied the distribution of modular symbols by relating it to the behavior of a certain ζ-function, that itself was constructed from Eisenstein series belonging to $\mathrm{M}_0^{[2]}(\Gamma_0(N))$ where $N \in \mathbb{Z}_{\geq 1}$.

We shall now look at a slightly more general example, namely that of Eisenstein series belonging to $\mathrm{M}_k^{[2]}(\Gamma)$ constructed from modular symbols, where $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a finite index subgroup containing Γ_∞ and $k \in \mathbb{Z}_{\geq 2}$.

For $\gamma \in \Gamma$ and $f \in \mathrm{S}_2(\Gamma)$ we define the modular symbol $\langle \gamma, f \rangle$ by

$$\langle \gamma, f \rangle = \int_z^{\gamma z} f(w)\mathrm{d}w, \tag{2.35}$$

where $z \in \mathbb{H}^*$ is arbitrary. Since f is a weight 2 cusp form, the integral is independent of the choice of z, so $\langle \gamma, f \rangle$ is well-defined. This also implies that for $\gamma_1, \gamma_2 \in \Gamma$ we have

$$\begin{aligned} \langle \gamma_1\gamma_2, f \rangle &= \int_z^{\gamma_1\gamma_2 z} f(w)\mathrm{d}w \\ &= \int_z^{\gamma_2 z} f(w)\mathrm{d}w + \int_{\gamma_2 z}^{\gamma_1\gamma_2 z} f(w)\mathrm{d}w \\ &= \langle \gamma_2, f \rangle + \langle \gamma_1, f \rangle, \end{aligned} \tag{2.36}$$

so that $\langle \cdot, f \rangle$ is a homomorphism. We also see that $\langle \gamma, f \rangle = 0$ for $\gamma \in \Gamma_\infty$. We now let for $k \in \mathbb{Z}_{\geq 3}$ and $\tau \in \mathbb{H}$

$$E_k^{[2]}(\tau; \langle \cdot, f \rangle) = \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma} \langle \gamma^{-1}, f \rangle (1\big|_k \gamma)(\tau). \tag{2.37}$$

Goldfeld showed that for a fixed cusp form $f \in \mathrm{S}_k$ and an arbitrary positive real number $\epsilon \in \mathbb{R}_{>0}$, it holds that

$$\langle \gamma, f \rangle \ll_\epsilon |c(\gamma)|^{\frac{1}{2}+\epsilon}. \tag{2.38}$$

Combining this with the argument (1.36), we find that $E_k^{[2]}(\tau;\langle\cdot,f\rangle)$ converges absolutely and locally uniformly on $\mathbb{H}$ for $k > \frac{5}{2}$. This implies that for $\delta \in \Gamma$, we have

$$\begin{aligned}(E_k^{[2]}(\cdot;\langle\cdot,f\rangle)\big|_k\delta)(\tau) &= \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma}\langle\gamma^{-1},f\rangle(1\big|_k\gamma\delta)(\tau)\\ &= \sum_{[\gamma']\in\Gamma_\infty\backslash\Gamma}\langle\delta(\gamma')^{-1},f\rangle(1\big|_k\gamma')(\tau)\\ &= \langle\delta,f\rangle E_{k,\Gamma}(\tau)+E_k^{[1]}(\tau;\langle\cdot,f\rangle),\end{aligned} \tag{2.39}$$

where $E_{k,\Gamma} = \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma} 1\big|_k\gamma$. By the usual arguments, we have that $E_{k,\Gamma} \in \mathrm{M}_k(\Gamma)$ for $k > 2$. Altogether, we have that

$$E_k^{[2]}(\cdot;\langle\cdot,f\rangle) \in \mathrm{M}_k^{[1]}(\Gamma). \tag{2.40}$$

Using the same methodology as Goldfeld, we can also obtain Fourier series expansions for $E_k^{[2]}(\Gamma)$.

We note that $E_k^{[2]}(\cdot;\langle\cdot,f\rangle)\big|_k(\gamma-1) = \langle\gamma,f\rangle E_{k,\Gamma}$, that is, the modular deficit at γ is equal to a homomorphism $\Gamma\to\mathbb{C}$ evaluated at γ multiplied with a modular form in $\mathrm{M}_k(\Gamma)$. This turns out to always be the case for second order modular forms.

Proposition 2.5. Let $k\in\mathbb{Z}$, let $\Gamma\subseteq\mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup containing Γ_∞, and let $f:\mathbb{H}\to\mathbb{C}$ be a function. Then $f\in\mathrm{M}_k^{[2]}(\Gamma)$ if and only if f is holomorphic and has moderate growth at the cusps, $f\big|_k(\gamma-1)=0$ for $\gamma\in\Gamma_\infty\cap\Gamma$, and there exists a finite set of homomorphisms $A=\{\phi_i\}_{i=1}^m$, $m\geq 1$, $\phi_i:\Gamma\to\mathbb{C}$, and a corresponding set of modular forms $\{g_\phi\}_{\phi\in A}\subseteq\mathrm{M}_k(\Gamma)$ such that for all $\gamma\in\Gamma$, we have that

$$f\big|_k(\gamma-1) = \sum_{\phi\in A}\phi(\gamma)\cdot g_\phi. \tag{2.41}$$

Proof. If f satisfies the conditions of the proposition and A and $\{g_\phi\}_{\phi\in A}\subseteq\mathrm{M}_k(\Gamma)$ are corresponding finite sets of homomorphisms $\Gamma\to\mathbb{C}$ and modular forms, then since $\mathrm{M}_k(\Gamma)$ is a vector space, we have that

$$f\big|_k(\gamma-1) = \sum_{\phi\in A}\phi(\gamma)\cdot g_\phi \in \mathrm{M}_k(\Gamma), \tag{2.42}$$

showing that $f\in\mathrm{M}_k^{[2]}(\Gamma)$.

Conversely, let us now suppose that $f\in\mathrm{M}_k^{[2]}(\Gamma)$. Let $\{g_i\}_{i=1}^d$, $d\in\mathbb{Z}_{\geq 1}$, be a basis for $\mathrm{M}_k(\Gamma)$. For a given $\gamma\in\Gamma$, there are then coefficients $c_i^\gamma\in\mathbb{C}$ such that

$$f\big|_k(\gamma-1) = \sum_{i=1}^d c_i^\gamma g_i. \tag{2.43}$$

For $\gamma_1, \gamma_2 \in \Gamma$ we thus obtain

$$\begin{aligned}\sum_{i=1}^{d} c_i^{\gamma_1\gamma_2} g_i &= f\big|_k(\gamma_1\gamma_2 - 1) \\ &= f\big|_k(\gamma_1 - 1)\big|_k\gamma_2 + f\big|_k(\gamma_2 - 1) \\ &= \sum_{i=1}^{d}(c_i^{\gamma_1}(g_i\big|_k\gamma_2) + c_i^{\gamma_2} g_i) \\ &= \sum_{i=1}^{d}(c_i^{\gamma_1} + c_i^{\gamma_2}) g_i.\end{aligned} \tag{2.44}$$

This shows that $\phi_i(\gamma) = c_i^{\gamma}$ is a homomorphism, so letting $A = \{\phi_i\}_{i=1}^{d}$ and $g_{\phi_i} = g_i$ will do. □

Building on Goldfeld's work, Diamantis and O'Sullivan studied the structure of $\mathrm{M}_k^{[2]}(\Gamma)$, $k \in \mathbb{Z}_{\geq 3}$, and computed its dimension [20]. Finally, in [21], Diamantis and Sim provided a complete classification of $\mathrm{M}_k^{[n]}(\Gamma)$, for $k \in \mathbb{Z}_{\geq 3}$ and $n \in \mathbb{Z}_{\geq 2}$.

THE VECTOR-VALUED CONNECTION While not immediately apparent, Proposition 2.5 provides us with a connection to group cohomology. Indeed, homomorphisms from Γ to $\mathbb{C}$ are nothing but 1-cocycles when Γ acts trivially on $\mathbb{C}$. Since the first cohomology group is isomorphic to the group of extension classes in $\mathbb{C}[\Gamma]$, we also have a connection to representation theory. Using this, we can view the elements of $\mathrm{M}_k^{[n]}(\Gamma)$ as components of certain vector-valued modular forms. Furthermore, in this setting it is very easy to generalize higher order modular forms, essentially by changing the action to something non-trivial. In paper II, we show that this allows us to view Eichler integrals as generalized second order modular forms. Building on this, we show in paper III that iterated Eichler-Shimura integrals of depth 2 can be viewed as generalized third order modular forms – this construction generalizes to arbitrary depth, but in depth 2 we were able to construct a novel scalar-valued depth two Eichler integral.

2.3 THE PRODUCTS OF EISENSTEIN SERIES PHILOSOPHY

As we have indicated in the previous section, modular forms for subgroups with character, and higher order modular forms can both be studied from a perspective of vector-valued modular forms. In the beginning of this chapter, we also mentioned that this allows us to use a unified framework to compute them. In this section, we describe what we mean by this.

To start with, let us state our intent in abstract terms.

GOAL

For a space M of general modular forms whose domain is $\mathbb{H}$, find a basis for M in terms of products of Eisenstein series-type objects whose Fourier series expansions converge rapidly. Use this basis to develop algorithms for expressing any element $f \in M$ in terms of products of Eisenstein series-type objects, and for evaluating $f(\tau)$ to arbitrary precision at any $\tau \in \mathbb{H}$.

To accomplish the GOAL we follow what the author of this thesis has taken to calling the *products of Eisenstein series philosophy*. It can be stated as follows.

THE PRODUCTS OF EISENSTEIN SERIES PHILOSOPHY

Let M be a space of general modular forms. To achieve the GOAL for M, we:

1. Find a space $\widetilde{M}$ of vector-valued modular forms, whose components contain the elements of M.
2. Obtain a Sturm-type bound for elements in $\widetilde{M}$, and an exact formula for $\dim(\widetilde{M})$.
3. Find Eisenstein series in $\widetilde{M}$, and compute their Fourier series expansions.
4. Obtain a theorem that says that $\widetilde{M}$ is spanned by products of the Eisenstein series, or objects derived from them.
5. Compute a basis for $\widetilde{M}$ in terms of products of Eisenstein series (or derived objects), by computing involved the Fourier series expansions, truncating at the Sturm bound, and performing row-reduction.

We remark that this philosophy bears a very close resemblance to the way we showed how M_k could be computed in Chapter 1. In the rest of this thesis, we will:

1. Introduce vector-valued modular forms, and show how they encapsulate the higher order modular forms, modular forms for subgroups, and considerable generalizations thereof.
2. In all cases under consideration, show how we have developed the PHILOSOPHY to an effective tool that we have used to solve the GOAL.

Part II

A CRASH COURSE IN VECTOR-VALUED MODULAR FORMS

3 ARITHMETIC TYPES AND VECTOR-VALUED MODULAR FORMS

In this chapter, we present a condensed introduction to vector-valued modular forms and explain how they relate to the general modular forms that were covered in the previous chapter.

Vector-valued modular forms are a natural generalization of modular forms for subgroups of $\mathrm{SL}_2(\mathbb{Z})$ with respect to a multiplicative character, to modular forms taking values in an n-dimensional complex vector space V, $n \in \mathbb{Z}_{\geq 2}$. One then needs to replace the multiplicative character with a complex finite-dimensional representation of a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We call these *arithmetic types*.

Remark 3.1. The presentation that follows builds largely on the papers [32] by Raum, [38] by Mertens and Raum, and [11]. An alternative approach, based on Poincaré series is provided by Knopp and Mason in [13] and [12].

3.1 ARITHMETIC TYPES

Let us get straight to the definition.

Definition 3.1 (Arithmetic type)**.** Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be finite index subgroup. Then an arithmetic type for Γ is a finite-dimensional complex representation[1]. We write $V(\rho)$ for the representation space of ρ.

If in addition $\ker(\rho)$ is a congruence subgroup, we call ρ a congruence type of level equal to $\mathrm{level}(\ker(\rho))$.

Remark 3.2. If $\ker(\rho)$ has finite index in Γ, we can use Weyl's unitarity trick to construct an inner product of $V(\rho)$ for which ρ is unitary. This implies that ρ is semi-simple.

Note that an arithmetic type ρ for a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is fully determined by its value at the generators of Γ.

We also remark that the slash action $\big|_k$ extends to an action on functions from $\mathbb{H}$ to V, where V is any finite-dimensional complex vector space. If in addition, ρ is an arithmetic type for a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, $k \in \mathbb{Z}$ is an integer, $f : \mathbb{H} \to V(\rho)$ is a function, and $\gamma \in \Gamma$, we let

$$(f\big|_{k,\rho}\gamma)(\tau) = \rho(\gamma^{-1})(f\big|_k\gamma)(\tau) \quad \tau \in \mathbb{H}. \tag{3.1}$$

1 Which we recall is a group homomorphism $\rho : \Gamma \to \mathrm{GL}(V)$, where V is a finite-dimensional complex vector space.

The slash action is linear, in the sense that if V is a finite-dimensional complex vector space, then

$$(c_1 f_1 + c_2 f_2)\big|_k \gamma = c_1 (f_1\big|_k \gamma) + c_2 (f_2\big|_k \gamma), \qquad c_1, c_2 \in \mathbb{C}, \quad f_1, f_2 : \mathbb{H} \to V. \tag{3.2}$$

It also extends linearly to a right-action of the group ring $\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$ on the set of functions from $\mathbb{H} \to V$ by setting

$$f\big|_{k,\rho}(c_1\gamma_1 + c_2\gamma_2) = c_1 (f\big|_{k,\rho}\gamma_1) + c_2 (f\big|_{k,\rho}\gamma_2), \qquad c_1, c_2 \in \mathbb{C}, \quad \gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z}). \tag{3.3}$$

Note that this is addition[2] in $\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$. It should **not** be confused with element-wise addition of matrices. The analogous statements hold for the action $\big|_{k,\rho}$.

We can now define vector-valued modular forms.

Definition 3.2 (Vector-valued modular forms)**.** Let $k \in \mathbb{Z}$ be an integer, let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let ρ be an arithmetic type for Γ. Let also $f : \mathbb{H} \to V(\rho)$ be a function. Then we say that f is a vector-valued modular form of type ρ if

1. for all $\gamma \in \Gamma$, it holds that
$$f\big|_{k,\rho}\gamma = f, \tag{3.4}$$
2. f is holomorphic, and
3. for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ it holds that there exists a number $a \in \mathbb{R}$ such that
$$\|(f\big|_k \gamma)(\tau)\|_2 = O(\mathrm{Im}(\tau)^a) \quad \text{as} \quad \mathrm{Im}(\tau) \to \infty, \tag{3.5}$$
uniformly in $\mathrm{Re}(\tau)$.

We write $\mathrm{M}_k(\rho)$ for the $\mathbb{C}$-vector space of vector-valued modular forms of weight k and type ρ. If in addition, it holds that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ that

$$\|(f\big|_k \gamma)(\tau)\|_2 \to 0 \quad \text{as} \quad \mathrm{Im}(\tau) \to \infty, \tag{3.6}$$

uniformly in $\mathrm{Re}(\tau)$, we say that f is a vector-valued cusp form of type ρ. We write $\mathrm{S}_k(\rho)$ for the $\mathbb{C}$-vector space of vector-valued cusp forms of weight k and type ρ.

2 In other words, it is the group operation on the free abelian group generated by the symbols $\{\mathfrak{e}_\gamma : \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$. To simplify notation, we write γ for $\mathfrak{e}_\gamma$. This may cause some confusion at first, but we have found that this choice of notation decreases cognitive clutter.

We record that if ρ is an arithmetic type such that for some $N \in \mathbb{Z}_{\geq 1}$ we have that $T^N \in \ker(\rho)$, then elements $f \in \mathrm{M}_k(\rho)$ have a Fourier series expansion on the form

$$f(\tau) = \sum_{n \geq 0} c(f;n) e^{2\pi i n \tau / N}, \quad \tau \in \mathbb{H}, \tag{3.7}$$

where $c(f;n) \in V(\rho)$ are the Fourier series coefficients.

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup and let $\nu : \Gamma \to \mathbb{C}^\times$ be a multiplicative character. We then see that

$$\mathrm{M}_k(\nu) = \mathrm{M}_k(\Gamma, \nu), \tag{3.8}$$

where the right-hand side was defined in Definition 2.1. In particular, if $\mathbf{1} : \Gamma \to \mathbb{C}^\times$ is the trivial character, then

$$\mathrm{M}_k(\mathbf{1}) = \mathrm{M}_k(\Gamma). \tag{3.9}$$

In line with the *products of Eisenstein series philosophy*, we now present a Sturm bound for $\mathrm{M}_k(\rho)$ and a formula for $\dim \mathrm{M}_k(\rho)$, where ρ is an arithmetic type, and where $k \in \mathbb{Z}_{\geq 3}$. Let us start with the Sturm bound.

Proposition 3.1 (Sturm bound for $\mathrm{M}_k(\rho)$)**.** Let $k \in \mathbb{Z}$ and let ρ be an arithmetic type such that $T^N \in \ker(\rho)$ for some $N \in \mathbb{Z}_{\geq 1}$. Let $f \in \mathrm{M}_k(\rho)$. Then if

$$c(f;n) = 0 \text{ for all } n \in \mathbb{Q} \text{ with } 0 \leq n \leq \frac{k}{12}, \tag{3.10}$$

it holds that $f = 0$.

Proof. See [27, Theorem 1.2]. □

As for the dimension formula, we have the following proposition, due in this formulation to Borcherds and Fischer.

Remark 3.3. As we will see when we define *induced types*, we can without loss of generality restrict ourselves to arithmetic types for $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 3.2 (Dimension of $\mathrm{M}_k(\rho)$)**.** Let ρ be arithmetic type for $\mathrm{SL}_2(\mathbb{Z})$ satisfying that $T^M \in \ker(\rho)$ for some $M \in \mathbb{Z}_{\geq 1}$, and let $k \in \mathbb{Z}_{\geq 3}$. Then

$$\dim \mathrm{M}_k(\rho) = \frac{1}{2} \sum_{j=0}^{3} e(jk/2) \psi(k, \rho, j), \tag{3.11}$$

where

$$\begin{aligned} \psi(k, \rho, j) = \frac{k-1}{12} \mathrm{Tr}(\rho(Z^j)) + \delta_3(e(1/6)\rho(R), \rho(Z^j)) \\ + \delta_2(e(1/4)\rho(S), \rho(Z^j)) + \delta_0(\rho(T^{-1}), \rho(Z^j)), \end{aligned} \tag{3.12}$$

where $R = TS$, and

$$\delta_N(X, g) = \frac{1}{N} \sum_{j=1}^{N-1} \frac{\mathrm{Tr}(X^j g)}{1 - e(j/N)}, \tag{3.13}$$

for $N \in \mathbb{Z}_{\geq 1}$, and

$$\delta_0(X, g) = \frac{\mathrm{Tr}(g)}{2M} + \frac{1}{M} \sum_{j=1}^{M-1} \frac{\mathrm{Tr}(X^j g)}{1 - e(j/M)}, \tag{3.14}$$

where $M = \min\{M \in \mathbb{Z}_{\geq 1} : T^M \in \ker(\rho)\}$.

Proof. See [11] or [7, Corollary 2.5.5]. □

3.2 EXAMPLES OF ARITHMETIC TYPES

Let ρ be an arithmetic type for a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. We can then view $V(\rho)$ as $\mathbb{C}[\Gamma]$-module, by setting

$$\gamma.v = \rho(\gamma)v, \quad \gamma \in \Gamma \text{ and } v \in V(\rho). \tag{3.15}$$

It is a standard result in representation theory [1, 8] that the category of complex representations of Γ is isomorphic to the category of $\mathbb{C}[\Gamma]$-modules, and thus we henceforth identify ρ with the $\mathbb{C}[\Gamma]$-module $V(\rho)$ unless otherwise indicated.

Before looking at some concrete examples, we recall that given a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and a $\mathbb{C}[\Gamma]$-module V, the space of invariants $\mathrm{H}^0_\Gamma(V)$ is defined by

$$\mathrm{H}^0_\Gamma(V) = \{v \in V : \gamma.v = v\}. \tag{3.16}$$

The subscript is often omitted if it is understood from the context.

3.2.1 *Induced types*

We now show how one can extend an arithmetic type for a proper subgroup of $\mathrm{SL}_2(\mathbb{Z})$ to an arithmetic type for $\mathrm{SL}_2(\mathbb{Z})$ itself. For subgroups $\Gamma \subseteq \Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$ of finite index, and an arithmetic type ρ of Γ, we let the induced type of ρ for Γ' be given by

$$\mathrm{Ind}_\Gamma^{\Gamma'} \rho = \mathbb{C}[\Gamma'] \otimes_{\mathbb{C}[\Gamma]} \rho. \tag{3.17}$$

Recall that the action of $\mathbb{C}[\Gamma']$ on the tensor product is defined by

$$\gamma(\delta \otimes v) = (\gamma\delta) \otimes v, \tag{3.18}$$

for $v \in V(\rho)$.

The induced type allows us to always work with arithmetic types for $\mathrm{SL}_2(\mathbb{Z})$, rather than for proper subgroups, simply let $\Gamma' = \mathrm{SL}_2(\mathbb{Z})$ in the following proposition.

Proposition 3.3. Let $\Gamma \subseteq \Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$ be finite index subgroups, $k \in \mathbb{Z}$ be an integer, and σ be an arithmetic type for Γ. Then

$$\begin{aligned} \mathrm{Ind} : \mathrm{M}_k(\sigma) &\to \mathrm{M}_k(\mathrm{Ind}_\Gamma^{\Gamma'}\sigma) \\ f &\mapsto \Big(\tau \mapsto \sum_{[\gamma]\in\Gamma'/\Gamma} \gamma \otimes (f\big|_k\gamma^{-1})(\tau)\Big), \end{aligned} \tag{3.19}$$

is an isomorphism of $\mathbb{C}$-vector spaces.

Proof. If $\gamma \in \Gamma'$ and $\alpha \in \Gamma$, we have that

$$\begin{aligned} \gamma\alpha \otimes (f\big|_k\alpha^{-1}\gamma^{-1})(\tau) &= \gamma\alpha \otimes \sigma(\alpha^{-1})(f\big|_k\gamma^{-1})(\tau) \\ &= \gamma(\alpha \otimes \sigma(\alpha^{-1})(f\big|_k\gamma^{-1})(\tau) \\ &= \gamma(1 \otimes \sigma(\alpha)\sigma(\alpha^{-1})(f\big|_k\gamma^{-1})(\tau) \\ &= \gamma \otimes (f\big|_k\gamma^{-1})(\tau). \end{aligned} \tag{3.20}$$

This shows that the sum (3.19) is well-defined. Let now $\delta \in \Gamma'$. Then

$$\begin{aligned} (\mathrm{Ind}(f)\big|_{k,\rho}\delta)(\tau) &= \rho(\delta^{-1}) \sum_{[\gamma]\in\Gamma'/\Gamma} \gamma \otimes (f\big|_k\gamma^{-1}\delta)(\tau) \\ &= \sum_{[\gamma]\in\Gamma'/\Gamma} \delta^{-1}\gamma \otimes (f\big|_k\gamma^{-1}\delta)(\tau) \\ &= \sum_{[\beta]\in\Gamma'/\Gamma} \beta \otimes (f\big|_k\beta^{-1})(\tau), \end{aligned} \tag{3.21}$$

where the last equality follows since $\gamma\Gamma \mapsto \delta^{-1}\gamma\Gamma$ is a permutation of Γ'/Γ. This shows that Ind is a well-defined map. It is also clear that it is linear. It remains to show that it is bijective.

Let $\{\gamma_i\}_{i=1}^r \subseteq \Gamma'$ be a complete set of representatives for Γ'/Γ where $\gamma_1 = 1$. For brevity, let us write $\rho = \mathrm{Ind}_\Gamma^{\Gamma'}\sigma$. Given a modular form $f \in \mathrm{M}_k(\rho)$, there are unique functions $f_i : \mathbb{H} \to V(\sigma)$, $1 \le i \le r$, such that

$$f(\tau) = \sum_{i=1}^r \gamma_i \otimes f_i(\tau). \tag{3.22}$$

For a given element $\delta \in \Gamma'$, there is a unique permutation $\pi_\delta \in \mathrm{S}_r$ and a unique function $f_\delta : \{1, \dots, r\} \to \Gamma$, defined by

$$\delta^{-1}\gamma_i = \gamma_{\pi_\delta(i)} f_\delta(i). \tag{3.23}$$

The relation $f\big|_{k,\rho}\delta = f$ then implies that

$$f(\tau) = (f\big|_{k,\rho}\delta)(\tau) = \sum_{i=1}^{r} \gamma_{\pi_\delta(i)} \otimes \sigma(f_\delta(i))(f_i\big|_k\delta)(\tau), \tag{3.24}$$

and so $f_{\pi_\delta(i)} = \sigma(f_\delta(i))f_i\big|_k\delta$. In particular, if $\delta \in \Gamma$ and $i = 1$, we obtain that $\pi_\delta(1) = 1$ and $f_\delta(1) = \delta^{-1}$, and therefore $f_1 = \sigma(\delta^{-1})f_1\big|_k\delta = f\big|_{k,\sigma}\delta$. This means that $f_1 \in \mathrm{M}_k(\delta)$. We also have that $\pi_{\gamma_j}(j) = 1$ and $f_{\gamma_j}(1) = 1$, implying that $f_1 = f_j\big|_k\gamma_j$ and thus $f_1\big|_k\gamma_j^{-1} = f_j$.

We can now define the linear function $F : \mathrm{M}_k(\rho) \to \mathrm{M}_k(\sigma)$ by $F(f) = f_1$. Let $f \in \mathrm{M}_k(\rho)$. We then obtain

$$\mathrm{Ind}(F(f)) = \mathrm{Ind}(f_1) = \sum_{i=1}^{r} \gamma_i \otimes f_1\big|_k\gamma_i^{-1} = \sum_{i=1}^{r} \gamma_i \otimes f_i = f. \tag{3.25}$$

Similarly, if $g \in \mathrm{M}_k(\sigma)$, we have that

$$F(\mathrm{Ind}(g)) = F(\sum_{i=1}^{r} \gamma_i \otimes g\big|_k\gamma_i^{-1}) = g\big|_k\gamma_1^{-1} = g, \tag{3.26}$$

since $\gamma_1 = 1$. This shows that Ind is a bijective linear function, and so we are done. □

In particular, we let for a positive integer $N \in \mathbb{Z}_{\geq 1}$ and a multiplicative character $\chi : \Gamma_0(N) \to \mathbb{C}^\times$, the induced types ρ_N and ρ_χ be given by

$$\rho_N = \mathrm{Ind}_{\Gamma_1(N)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1} \quad \text{and} \quad \rho_\chi = \mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_2(\mathbb{Z})}\chi. \tag{3.27}$$

To understand how ρ_N and ρ_χ look like, we next provide two simple examples. Namely those of ρ_3 and ρ_ψ where $\psi \in D(3)$ is given by $\psi(2) = -1$.

Let start by considering ρ_3. We have that

$$\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{i=1}^{8} \gamma_i\Gamma_1(3), \tag{3.28}$$

where

$$\begin{aligned} &\gamma_1 = I, \quad &&\gamma_2 = -I, \quad &&\gamma_3 = S, \quad &&\gamma_4 = -S, \\ &\gamma_5 = STS, \quad &&\gamma_6 = -STS, \quad &&\gamma_7 = T^{-2}S, \quad &&\gamma_8 = -T^{-2}S. \end{aligned} \tag{3.29}$$

Given an arbitrary element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, there exists a unique number $1 \leq i \leq 8$ and a unique element $\gamma' \in \Gamma_1(3)$ such that

$$\gamma = \gamma_i\gamma'. \tag{3.30}$$

This implies that

$$\mathcal{B} = \{\gamma_1 \otimes 1, \gamma_2 \otimes 1, \ldots, \gamma_8 \otimes 1\}, \tag{3.31}$$

is a basis for $V(\rho_3)$. To determine $\rho_3(S)$ and $\rho_3(T)$, we have to express $S(\gamma_i \otimes 1)$ and $T(\gamma_i \otimes 1)$ as linear combinations of elements of $\mathcal{B}$. We note that if

$$S\gamma_i = \gamma_j \gamma', \tag{3.32}$$

then

$$\begin{aligned} S(\gamma_i \otimes 1) = \gamma_j \gamma' \otimes 1 = \gamma_j(\gamma' \otimes 1) = \gamma_j(1 \otimes \gamma'.1) = \gamma_j(1 \otimes 1) \\ = \gamma_j \otimes 1, \end{aligned} \tag{3.33}$$

and similarly for left multiplication by T. Hence, S and T act on $\mathcal{B}$ by permutations. After some work, one finds that

$$\begin{aligned} S(\gamma_i \otimes 1) &= \gamma_{\sigma_S(i)} \otimes 1, \text{ and} \\ T(\gamma_i \otimes 1) &= \gamma_{\sigma_T(i)} \otimes 1, \end{aligned} \tag{3.34}$$

where

$$\sigma_S = (3,4,2,1,8,7,5,6), \quad \sigma_T = (1,2,7,8,3,4,5,6). \tag{3.35}$$

As for ρ_χ, we first need to compute a set representatives for $\mathrm{SL}_2(\mathbb{Z})/\Gamma_0(3)$. One finds, for example, that $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{i=1}^4 \delta_i \Gamma_0(3)$, where

$$\delta_1 = I, \quad \delta_2 = -S, \quad \delta_3 = -STS, \quad \delta_4 = -T^{-2}S. \tag{3.36}$$

Consequently, $\mathcal{B}' = \{\delta_i \otimes 1\}_{i=1}^4$ is a basis for $V(\rho_\chi)$. In this case, S and T no longer act exactly like permutations, but almost – they act like permutations with multiplicative twists. For example, we have that

$$\begin{aligned} S(\delta_1 \otimes 1) = S \otimes 1 = \delta_2(-I) \otimes 1 = \delta_2(1 \otimes (-I).1) = \delta_2(1 \otimes \psi(-I)) \\ = -(\delta_2 \otimes 1). \end{aligned} \tag{3.37}$$

Continuing, we find that

$$\begin{aligned} S(\delta_i \otimes 1) &= f_S(i) \cdot (\delta_{\sigma'_S(i)} \otimes 1), \text{ and} \\ T(\delta_i \otimes 1) &= f_T(i) \cdot (\delta_{\sigma'_T(i)} \otimes 1), \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} \sigma'_S &= (2,1,4,3), \quad f_S = (-1,1,-1,1), \text{ and} \\ \sigma'_T &= (1,4,2,3), \quad f_T = (1,1,1,1). \end{aligned} \tag{3.39}$$

We notice that both ρ_3 and ρ_ψ share a similar form – they act as *twisted* permutations. This is of course true for any tensor product on the form

$$\rho_3^{\otimes n} \otimes_{\mathbb{C}} \rho_\psi^{\otimes m}, \tag{3.40}$$

where $n, m \in \mathbb{Z}_{\geq 1}$ are integers, and the n-fold and m-fold tensor products are taken over $\mathbb{C}$; as well. By allowing twists of higher dimension than 1, we are naturally lead to what we call *twisted permutation types*.

3.2.2 *Twisted permutation types*

Recall that for a group G and an integer $n \in \mathbb{Z}_{\geq 1}$, the wreath product of G with the permutation group S_n, denoted by $G \wr \mathrm{S}_n$, is given by

$$\begin{aligned} G \wr \mathrm{S}_n &= \{(f, \pi) \mid f : \{1, \dots, n\} \to G, \pi \in \mathrm{S}_n\}, \text{ and} \\ (f', \pi')(f, \pi) &= (i \mapsto f'(\pi(i)) f(i), \pi' \pi). \end{aligned} \tag{3.41}$$

Furthermore, given an arithmetic type ρ for G, we define the representation $\rho \wr \mathrm{S}_n : G \wr \mathrm{S}_n \to \mathrm{GL}(\mathbb{C}^n \otimes V(\sigma))$ by

$$(\rho \wr \mathrm{S}_n)(f, \pi)(e_i \otimes w) = e_{\pi(i)} \otimes \rho(f(\pi(i)))w, \tag{3.42}$$

where e_i denotes the canonical basis of $\mathbb{C}^n$.

We now arrive at the following definition.

Definition 3.3 (Twisted permutation type)**.** Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup, $n \in \mathbb{Z}_{\geq 1}$ be an integer, and σ be a finite-dimensional complex representation of some group G. Then we say that an arithmetic type ρ for Γ is a twisted permutation type of order n and with twist representation σ, if there exists a group homomorphism $\rho^\wr$ such that ρ factors as

$$\rho : \Gamma \xrightarrow{\rho^\wr} G \wr \mathrm{S}_n \xrightarrow{\sigma \wr \mathrm{S}^n} \mathrm{GL}(\mathbb{C}^n \otimes V(\sigma)). \tag{3.43}$$

We call $V(\sigma)$ the twist space of ρ and $\dim(V(\sigma))$ the twist dimension of ρ.

By slight abuse of terminology, we refer to an arithmetic type as a twisted permutation type even if it is only isomorphic to one.

To get a feeling for the definition, let us show that $\rho_3 \otimes_{\mathbb{C}} \rho_\psi$ is a twisted permutation type of order 32 with twist representation $\mathrm{std}(\mathbb{C})$.[3] Let

$$\mathfrak{e}_{i,j} = (\gamma_i \otimes 1) \otimes_{\mathbb{C}} (\delta_j \otimes 1). \tag{3.44}$$

3 Recall that for a finite-dimensional complex vector space V, the representation $\mathrm{std}(V) : \mathrm{GL}(V) \to \mathrm{GL}(V)$ is given by $\mathrm{std}(V)(\gamma)(v) = \gamma(v)$.

where $1 \leq i \leq 8$ and $1 \leq j \leq 4$. We then find that

$$S\mathfrak{e}_{i,j} = f_S(j) \cdot \mathfrak{e}_{\sigma_S(i),\sigma'_S(j)} \quad \text{and} \quad T\mathfrak{e}_{i,j} = f_T(j) \cdot \mathfrak{e}_{\sigma_T(i),\sigma'_T(j)} \tag{3.45}$$

where σ_S and σ_T are given as in (3.35), and σ'_S, σ'_T, f_S, and f_T are given as in (3.39). Let now the permutation $\Sigma_S, \Sigma_T \in \mathrm{S}_{32}$ be given by

$$\begin{aligned} \Sigma_S(4(i-1)+j) &= 4(\sigma_S(i)-1) + \sigma'_S(j) \\ \Sigma_T(4(i-1)+j) &= 4(\sigma_T(i)-1) + \sigma'_T(j), \end{aligned} \tag{3.46}$$

where $1 \leq i \leq 8$ and $1 \leq j \leq 4$. Let also the functions $F_S, F_T : \{1, \ldots, 32\} \to \mathbb{C}$ be given by

$$F_S(\Sigma_S(4(i-1)+j)) = f_S(j) \quad \text{and} \quad F_T(\Sigma_T(4(i-1)+j)) = f_T(j), \tag{3.47}$$

with i and j as above. We now let ρ be given by $V(\rho) = \mathbb{C}^{32}$ and

$$\rho(S)e_i = F_S(\Sigma_S(i))e_{\Sigma_S(i)} \quad \text{and} \quad \rho(T)e_i = F_T(\Sigma_T(i))e_{\Sigma_T(i)}, \tag{3.48}$$

and $\rho^\wr : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^\times \wr \mathrm{S}_{32}$ be given by

$$\rho^\wr(S) = (F_S, \Sigma_S) \quad \text{and} \quad \rho^\wr(T) = (F_T, \Sigma_T). \tag{3.49}$$

One verifies that $\rho^\wr(S)^4 = (\rho^\wr(S)\rho^\wr(T))^6 = 1$, implying that $\rho^\wr$ is a group homomorphism. By construction, we have that

$$\rho = \mathrm{std}(\mathbb{C}) \wr \mathrm{S}_n \circ \rho^\wr, \tag{3.50}$$

and it is clear that $F : V(\rho_3 \otimes_{\mathbb{C}} \rho_\psi) \to V(\rho)$ given by $F(\mathfrak{e}_{i,j}) = e_{4(i-1)+j}$ is an isomorphism of representations.

WHY TWISTED PERMUTATION TYPES? As we will re-tell below, in [36] Raum and Xià showed that vector-valued modular forms of congruence type can be expressed in terms of components of certain vector-valued Eisenstein series. This accomplishes steps 3 and 4 of the PHILOSOPHY for modular forms that occur as components of vector-valued modular forms for congruence type. To further accomplish step 5, we show in paper I that the result of Raum-Xià can be recast in terms of invariants of tensor products of induced types, that is, invariants of twisted permutation types.

3.2.3 *Extension types*

We now describe a different way of constructing new arithmetic types from existing ones. In constract to the induction construction, we create a new arithmetic type of a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ from two given arithmetic types of the same group

Γ. This construction originates in group cohomology, and the explanation we give here follows the one presented in [38] and papers II and III.

Given arithmetic types ρ and σ for a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, we let the space of 1-cocycles, and the space of parabolic 1-cocycles, of type (ρ, σ), be given by

$$\begin{aligned} \mathrm{Z}^1(\rho,\sigma) &= \{f : \Gamma \to \mathrm{Hom}(V(\rho), V(\sigma)) \mid \\ &\qquad f(\gamma_1\gamma_2) = \sigma(\gamma_1)f(\gamma_2) + f(\gamma_1)\rho(\gamma_2)\}, \text{ and} \\ \mathrm{Z}^1_{\mathrm{pb}}(\rho,\sigma) &= \{f \in \mathrm{Z}^1(\rho,\sigma) : \forall \gamma \in \Gamma_\infty \cap \Gamma.\, f(\gamma) = 0\}. \end{aligned} \tag{3.51}$$

We also let the corresponding spaces of 1-coboundaries and parabolic 1-coboundaries of type (ρ, σ), be given by

$$\begin{aligned} \mathrm{B}^1(\rho,\sigma) &= \{f \in \mathrm{Z}^1_{\mathrm{pb}}(\rho,\sigma) : \exists h \in \mathrm{Hom}(V(\rho), V(\sigma)). \\ &\qquad \forall \gamma \in \Gamma.\, f(\gamma) = \sigma(\gamma)h - h\rho(\gamma)\}, \text{ and} \\ \mathrm{B}^1_{\mathrm{pb}}(\rho,\sigma) &= \mathrm{B}^1(\rho,\sigma) \cap \mathrm{Z}^1_{\mathrm{pb}}(\rho,\sigma), \end{aligned} \tag{3.52}$$

and define $\mathrm{H}^1(\rho,\sigma)$ and $\mathrm{H}^1_{\mathrm{pb}}(\rho,\sigma)$ by

$$\mathrm{H}^1(\rho,\sigma) = \frac{\mathrm{Z}^1(\rho,\sigma)}{\mathrm{B}^1(\rho,\sigma)} \quad \text{and} \quad \mathrm{H}^1_{\mathrm{pb}}(\rho,\sigma) = \frac{\mathrm{Z}^1_{\mathrm{pb}}(\rho,\sigma)}{\mathrm{B}^1_{\mathrm{pb}}(\rho,\sigma)}. \tag{3.53}$$

We also fix, once and for all, an injection $\nu : \mathrm{H}^1(\rho,\sigma) \to \mathrm{Z}^1(\rho,\sigma)$.

For[4] $\phi \in \mathrm{Z}^1(\sigma,\rho)$, we let the extension of type (ρ,σ), denoted by $\rho \boxplus_\phi \sigma$, be given by

$$\begin{aligned} V(\rho \boxplus_\phi \sigma) &= V(\rho) \oplus V(\sigma) \\ (\rho \boxplus_\phi \sigma)(\gamma)(v,w) &= (\rho(\gamma)v + \phi(\gamma)w, \sigma(\gamma)w). \end{aligned} \tag{3.54}$$

Let us briefly explain the justification for its name. Note that $\rho \boxplus_\phi \sigma$ fits into the following extension of σ by ρ

$$0 \to \rho \xrightarrow{v \mapsto (v,0)} \rho \boxplus_\phi \sigma \xrightarrow{(v,w) \mapsto w} \sigma \to 0. \tag{3.55}$$

We say that two extensions $\rho \hookrightarrow \xi_1 \twoheadrightarrow \sigma$ and $\rho \hookrightarrow \xi_2 \twoheadrightarrow \sigma$ are equivalent if and only if there exists a map f such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \rho & \longrightarrow & \xi_1 & \longrightarrow & \sigma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \rho & \longrightarrow & \xi_2 & \longrightarrow & \sigma & \longrightarrow & 0. \end{array} \tag{3.56}$$

4 Note the order of ρ and σ.

We denote the set of extension classes of σ by ρ by $\mathrm{Ext}(\sigma,\rho)$. We also define the corresponding set of parabolic extension classes by

$$\begin{aligned}\mathrm{Ext}_{\mathrm{pb}}(\sigma,\rho) = \{[\rho \hookrightarrow \xi \twoheadrightarrow \sigma] \in \mathrm{Ext}(\sigma,\rho) : \\ \forall \gamma \in \Gamma_\infty \cap \Gamma.\, \xi(\gamma) = 0\}.\end{aligned} \tag{3.57}$$

The sets $\mathrm{Ext}(\sigma,\rho)$ and $\mathrm{Ext}_{\mathrm{pb}}(\sigma,\rho)$ have the additional structure of abelian groups through the Baer sum. For its precise definition, we refer to [4]. Given an extension class $[\rho \hookrightarrow \xi \twoheadrightarrow \sigma] \in \mathrm{Ext}(\sigma,\rho)$ it turns out that $\xi \cong \rho \boxplus_\phi \sigma$ for some $\phi \in Z^1(\sigma,\rho)$, unique up to the addition of a coboundary. More precisely, we have the following isomorphism

$$\mathrm{Ext}(\sigma,\rho) \ni [\rho \hookrightarrow \rho \boxplus_\phi \sigma \twoheadrightarrow \sigma] \mapsto \phi + B^1(\sigma,\rho) \in H^1(\sigma,\rho), \tag{3.58}$$

descending to an isomorphism $\mathrm{Ext}_{\mathrm{pb}}(\sigma,\rho) \cong H^1_{\mathrm{pb}}(\sigma,\rho)$ between parabolic extension classes and parabolic cohomology.

We also let the universal parabolic extension $\rho \boxplus_{\mathrm{pb}} \sigma$ of type (ρ,σ) be defined by $V(\rho \boxplus_{\mathrm{pb}} \sigma) = V(\rho) \oplus V(\sigma) \otimes H^1_{\mathrm{pb}}(\sigma,\rho)$, and

$$(\rho \boxplus_{\mathrm{pb}} \sigma)(\gamma)(v_1, v_2 \otimes \phi) = (\rho(\gamma)v_1 + \nu(\phi)(\gamma)v_2, \sigma(\gamma)v_2 \otimes \phi). \tag{3.59}$$

WHY EXTENSION TYPES? In paper II and III, we show, building upon the theoretical framework developed by Mertens and Raum [38], that vector-valued modular forms of extension type capture both modular forms of higher order, and iterated Eichler-Shimura integrals. For modular forms of higher order, the connection is provided by Proposition 2.5. We will describe in this detail in the next section.

3.3 EXAMPLES OF VECTOR-VALUED MODULAR FORMS AND THEIR COMPONENTS

It is finally time to see what we can do with all of the theory that we have developed so far. In this section, we present an assortment of vector-valued modular forms for the different types of arithmetic types we have presented so far. We will also connect them back to what we saw in Chapter 2.

3.3.1 *Modular forms of induced type*

Proposition 3.3 implies that for a finite index subgroup Γ, a multiplicative character $\nu : \Gamma \to \mathbb{C}^\times$, and $k \in \mathbb{Z}$, we have that

$$M_k(\Gamma,\nu) \cong M_k(\mathrm{Ind}_\Gamma^{\mathrm{SL}_2(\mathbb{Z})}\nu). \tag{3.60}$$

This accomplishes step 1 of the PHILOSOPHY for $M_k(\Gamma, \nu)$. As a special case, we find that

$$M_k(\rho_N) \cong M_k(\Gamma_1(N)) \quad \text{and} \quad M_k(\rho_\chi) \cong M_k(\Gamma_0(N), \chi). \tag{3.61}$$

In particular, if $g \in M_k(\mathrm{Ind}_\Gamma^{\mathrm{SL}_2(\mathbb{Z})}\nu)$, then there exists a unique modular form $f \in M_k(\Gamma, \nu)$ such that

$$g = \sum_{[\gamma] \in \mathrm{SL}_2(\mathbb{Z})/\Gamma} \gamma \otimes f\big|_k \gamma^{-1}. \tag{3.62}$$

Hence if g admits a Fourier series expansion, then

$$c(f; n) = \sum_{[\gamma] \in \mathrm{SL}_2(\mathbb{Z})/\Gamma} \gamma \otimes c(f\big|_k \gamma^{-1}; n), \tag{3.63}$$

so f contains the Fourier series expansion at every cusp of f. This is a key advantage of recasting $M_k(\Gamma, \nu)$ as $M_k(\mathrm{Ind}_\Gamma^{\mathrm{SL}_2(\mathbb{Z})}\nu)$; if one can compute a basis in terms of Fourier series expansions of the latter, then one obtains cusp expansions for the elements of $M_k(\Gamma, \nu)$ at no additional cost.

3.3.2 *Vector-valued Eisenstein series*

Let ρ be an arithmetic type for $\mathrm{SL}_2(\mathbb{Z})$ with finite index kernel (so that ρ is semi-simple), and let $k \in \mathbb{Z}_{\geq 3}$. For a vector $v \in V(\rho)$, we let $\mathrm{Stab}(v) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \rho(\gamma)v = v\}$ and $\Gamma_\infty(v) = \Gamma_\infty \cap \mathrm{Stab}(v)$. Note that $\Gamma_\infty(v)$ has finite index in Γ_∞, since $\ker(\rho)$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$. Following [32], we now define the weight k vector-valued Eisenstein series of type ρ at v, by

$$E_{k,v}(\tau) = \frac{1}{|\Gamma_\infty/\Gamma_\infty(v)|} \sum_{[\gamma] \in \Gamma_\infty(v)\backslash \mathrm{SL}_2(\mathbb{Z})} v\big|_{k,\rho}\gamma \tag{3.64}$$

One verifies that $E_{k,v}$ converges absolutely and locally uniformly on $\mathbb{H}$, and has moderate growth at $i\infty$, and hence $E_{k,v} \in M_k(\rho)$ for any $v \in V(\rho)$. For $k \in \{1, 2\}$ we apply Hecke's trick [19], and put for $v \in V(\rho)$

$$E_{k,v}(\tau) = \frac{1}{|\Gamma_\infty/\Gamma_\infty(v)|} \lim_{s \to 0} \sum_{[\gamma] \in \Gamma_\infty(v)\backslash \mathrm{SL}_2(\mathbb{Z})} y^s v\big|_{k,\rho}\gamma, \tag{3.65}$$

where $y = \mathrm{Im}(\tau)$. One can verify, see [16, Chapter 7] or [14, Chapter 4], that $E_{1,v} \in M_1(\rho)$. However, $E_{2,v}$ is not holomorphic in general. That being said, Raum showed in [32, Lemma 3.2] that $E_{2,v}$ is holomorphic, and thus $E_{2,v} \in M_2(\rho)$, when ρ does not contain an isomorphic copy of the trivial type $\mathbf{1}$. Since $M_2(\mathbf{1}) = M_2 = \{0\}$, we do not have to worry about the "trivial part" of ρ.

We are thus led to define

$$\mathrm{E}_k(\rho) = \mathrm{span}_{\mathbb{C}}\{E_{k,v} : v \in V(\rho)\}, \tag{3.66}$$

for $k \in \{1\} \cup \mathbb{Z}_{\geq 3}$ or $k = 2$ if ρ does not contain an isomorphic copy of $\mathbf{1}$. If there exists a subrepresentation $\mathbf{1}' \subseteq \rho$ such that $\mathbf{1}' \cong \mathbf{1}$, we write[5] $\rho = \mathbf{1}' \oplus \rho'$ and define

$$\mathrm{E}_2(\rho) = \mathrm{span}_{\mathbb{C}}\{E_{2,v} : v \in V(\rho')\}. \tag{3.67}$$

We have, though this is non-trivial to prove, for $k \in \mathbb{Z}_{\geq 1}$ that

$$\mathrm{M}_k(\rho) = \mathrm{E}_k(\rho) \oplus \mathrm{S}_k(\rho), \tag{3.68}$$

see the remark following Proposition 1.4 in [32].

To see how the Eisenstein series in $\mathrm{E}_k(\rho)$ relate to the Eisenstein series of Chapter 2, we let

$$\mathcal{E}_k[\rho] = \mathrm{span}_{\mathbb{C}}\{v \circ E : v \in V(\rho)^\vee, E \in \mathrm{E}_k(\rho)\}. \tag{3.69}$$

be the space of components of vector-valued Eisenstein series. One finds, see for example [35], that for $k \in \mathbb{Z}_{\geq 3}$ and $N \in \mathbb{Z}_{\geq 1}$ it holds that

$$\mathcal{E}_k[\rho_N] = \mathrm{E}_k(\Gamma(N)). \tag{3.70}$$

Hence, we define $\mathcal{E}_1(\Gamma(N))$ and $\mathcal{E}_2(\Gamma(N))$ as $\mathcal{E}_1[\rho_N]$ and $\mathcal{E}_2[\rho_N]$.[6]

The following proposition shows how we can recover vector-valued modular forms from a space of scalar-valued modular forms that is invariant under the slash action.

Proposition 3.4. Let ρ be an arithmetic type for $\mathrm{SL}_2(\mathbb{Z})$, let $k \in \mathbb{Z}$ be an integer, and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup. Let $W \subseteq \mathrm{M}_k(\Gamma)$ be a subspace of modular forms invariant under weight k slash action of $\mathrm{SL}_2(\mathbb{Z})$, viewed as a $\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$-module. Then we have the map

$$\begin{aligned} &\mathrm{H}^0(W \otimes \rho) \to \mathrm{M}_k(\rho) \\ &\sum_i f_i \otimes v_i \mapsto \Big(\tau \mapsto \sum_i f_i(\tau) v_i\Big). \end{aligned} \tag{3.72}$$

Proof. This is Lemma 1.3 of Paper I. □

5 This is possible because ρ is semi-simple.

6 This coincides with the usual definition of holomorphic Eisenstein series of level N and weight 1 and 2. The usual definition uses the extended Eisenstein series

$$G_{k,N,\Gamma\gamma}(\tau, s) = \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (c,d)\equiv_N (c(\gamma),d(\gamma))}} y^s |c\tau + d|^{-2s} (c\tau + d)^{-k}, \tag{3.71}$$

where $y = \mathrm{Im}(\tau)$, and puts $G_{k,N,\Gamma\gamma}(\tau) = \lim_{s\to 0} G_{k,N\Gamma\gamma}(\tau, s)$. Some modification, analogous to the definition of $\mathrm{E}_2(\rho_N)$, needs to be done in the case of $k = 2$. See [16, Chapter 7], for details. We opted for the alternative definition for the sake of brevity.

Note that by (2.23), we have that $\mathrm{E}_k(\Gamma(N))$ is invariant under the slash action. This means that for $k, N \in \mathbb{Z}_{\geq 1}$, we have that

$$\mathrm{H}^0(\mathrm{E}_k(\Gamma(N)) \otimes \rho) \subseteq \mathrm{M}_k(\rho), \tag{3.73}$$

under the map (3.72).

The following theorem, due to Raum and Xià, is a vector-valued analogue of Proposition 1.9.

Theorem 3.1 (Raum and Xià)**.** Let $k, l \in \mathbb{Z}$ with $k \geq 2$ and $1 \leq l \leq k-1$, let $N \in \mathbb{Z}_{\geq 1}$, and let ρ be a congruence type of level N. Then there exists an integer $N_0 \in \mathbb{Z}_{\geq 1}$, with $N \mid N_0$ such that under the map (3.72), we have that

$$\mathrm{M}_k(\rho) = \mathrm{H}^0(\mathcal{E}_k[\rho_N] \otimes \rho) + \mathrm{H}^0((\mathcal{E}_l[\rho_{N_0}] \cdot \mathcal{E}_{k-l}[\rho_{N_0}]) \otimes \rho). \tag{3.74}$$

Proof. See [36, Theorem 4.4]. □

Propositions 3.1 and 3.2, Theorem 3.1, and equation (3.70), settles steps 2, 3, and 4, for modular forms that occur as components of vector-valued modular forms of congruence type. This the starting point for paper I.

3.3.3 *Modular forms of higher order*

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup, let $k \in \mathbb{Z}$, and let $\{\phi_i\}_{i=1}^d$, $d \in \mathbb{Z}_{\geq 1}$, be a basis for $\mathrm{H}^1_{\mathrm{pb}}(\mathbf{1}_\Gamma, \mathbf{1}_\Gamma)$. Let now $f \in \mathrm{M}_k(\mathbf{1}_\Gamma \boxplus_{\mathrm{pb}} \mathbf{1}_\Gamma)$. Then we may write

$$f = (f_1, \sum_{i=1}^{d} g_i \otimes \phi_i), \quad \tau \in \mathbb{H}, \tag{3.75}$$

where f_1 and $\{g_i\}_{i=1}^d$ are functions from $\mathbb{H}$ to $\mathbb{C}$. For brevity, we now write $\rho = \mathbf{1}_\Gamma \boxplus_{\mathrm{pb}} \mathbf{1}_\Gamma$. Then for $\gamma \in \Gamma$, we have that

$$\begin{aligned} f|_{k,\rho}\gamma &= \rho(\gamma^{-1})\Big[\begin{pmatrix} f_1|_k\gamma \\ 0 \end{pmatrix} + \sum_{i=1}^{d} \begin{pmatrix} 0 \\ g_i|_k\gamma \otimes \phi_i \end{pmatrix}\Big] \\ &= \begin{pmatrix} f_1|_k\gamma \\ 0 \end{pmatrix} + \sum_{i=1}^{d} \begin{pmatrix} \phi_i(\gamma^{-1})g_i|_k\gamma \\ g_i|_k\gamma \otimes \phi_i \end{pmatrix} \end{aligned} \tag{3.76}$$

Since f is modular, we have that

$$\begin{pmatrix} f_1 \\ \sum_{i=1}^d g_i \otimes \phi_i \end{pmatrix} = \begin{pmatrix} f_1|_k\gamma + \sum_{i=1}^d \phi_i(\gamma^{-1})g_i|_k\gamma \\ \sum_{i=1}^d g_i|_k\gamma \otimes \phi_i \end{pmatrix} \tag{3.77}$$

This implies that $g_i\big|_k\gamma = g_i$ for all i, and hence

$$f_1\big|_k(\gamma - 1) = \sum_{i=1}^{d} \phi_i(\gamma) g_i, \tag{3.78}$$

so $f_1 \in \mathrm{M}_k^{[2]}(\Gamma)$. Combining this with Proposition 2.5, we conclude that the map

$$\begin{aligned} F_2 : \mathrm{M}_k(\mathbf{1}_\Gamma \boxplus_{\mathrm{pb}} \mathbf{1}_\Gamma) &\to \mathrm{M}_k^{[2]}(\Gamma) \\ (f, *) &\mapsto f, \end{aligned} \tag{3.79}$$

is well-defined, linear, and surjective.

With an inductive argument, we can extend this to higher orders. For $n = 1$, we let $\mathbf{1}_\Gamma^{[n]} = \mathbf{1}_\Gamma$ and for $n > 1$, we let

$$\mathbf{1}_\Gamma \boxplus_{\mathrm{pb}} \mathbf{1}_\Gamma^{[n-1]}. \tag{3.80}$$

For $n \in \mathbb{Z}_{\geq 1}$, we now let F_n be given by

$$\begin{aligned} F_n : \mathrm{M}_k(\mathbf{1}_\Gamma^{[n]}) &\to \mathrm{M}_k^{[n]}(\Gamma) \\ (f, *) &\mapsto f. \end{aligned} \tag{3.81}$$

To see that F_n is well-defined, it is enough to show that for any elements $\gamma_1, \dots, \gamma_n \in \Gamma$ and any $f \in \mathrm{M}_k(\mathbf{1}_\Gamma^{[n]})$, we have that

$$f\big|_k(\gamma_1 - 1)(\gamma_2 - 1)\cdots(\gamma_n - 1) = 0. \tag{3.82}$$

Let now $n \in \mathbb{Z}_{\geq 1}$ and $f \in \mathrm{M}_k(\mathbf{1}_\Gamma^{[n]})$, and suppose that

$$f = \begin{pmatrix} g \\ \sum_{i=1}^{m} h_i \otimes \phi_i \end{pmatrix}, \tag{3.83}$$

where $\{\phi_i\}_{i=1}^m$ is a basis for $\mathrm{H}^1_{\mathrm{pb}}(\mathbf{1}_\Gamma^{[n-1]}, \mathbf{1}_\Gamma)$. Then, since f is modular we find that $h_i \in \mathrm{M}_k(\mathbf{1}_\Gamma^{[n-1]})$ for all i, and that

$$g\big|_k(\gamma - 1) = \sum_{i=1}^{m} \phi_i(\gamma) h_i, \tag{3.84}$$

for any $\gamma \in \Gamma$. As an inductive assumption, let us now assume (3.82) holds for $\mathrm{M}_k(\mathbf{1}_\Gamma^{[n-1]})$. Then for any elements $\gamma_1, \dots, \gamma_n \in \Gamma$, we have that

$$h_i\big|_k(\gamma_2 - 1)\cdots(\gamma_n - 1) = 0, \tag{3.85}$$

and thus

$$g\big|_k(\gamma_1 - 1)(\gamma_2 - 1)\dots(\gamma_n - 1) \\ = \sum_{i=1}^{m} \phi_i(\gamma_1) h_i\big|_k(\gamma_2 - 1)\cdots(\gamma_n - 1) = 0. \quad (3.86)$$

It follows that $f\big|_k(\gamma_1 - 1)\cdots(\gamma_n - 1) = 0$. By induction, we obtain that (3.82) holds for any space $\mathrm{M}_k(\mathbf{1}_\Gamma^{[n]})$. This implies that F_n is a well-defined linear map.

It is also the case that F_n is surjective for $n > 2$. However,the proof of this is beyond the scope of this thesis. We refer to [38, Proposition 3.15], for details.

3.3.4 *Generalized second order modular forms*

The fact that second order modular forms are images of modular forms of type $\mathbf{1}_\Gamma \boxplus_{\mathrm{pb}} \mathbf{1}_\Gamma$ suggests a natural generalization of second order modular forms: we may consider the "ρ component" of modular forms of type $\rho \boxplus_{\mathrm{pb}} \sigma$ where ρ and σ are general arithmetic types. This is the point of view that we take in paper II. For clarity, let us bring in another definition.

Definition 3.4. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup and $k \in \mathbb{Z}$. Let also ρ and σ be arithmetic types. Then we say that a holomorphic function $f : \mathbb{H} \to V(\rho)$ with moderate growth at the cusps is a generalized second order modular form of type (ρ, σ) and weight k if there exists a finite set of cocycles $A \subseteq \mathrm{Z}^1_{\mathrm{pb}}(\sigma, \rho)$ and a corresponding set of modular forms $\{g_\phi\}_{\phi \in A} \subseteq \mathrm{M}_k(\sigma)$, such that for every $\gamma \in \Gamma$ we have that

$$f\big|_{k,\rho}(\gamma - 1) = \sum_{\phi \in A} \rho(\gamma^{-1})\phi(\gamma) g_\phi. \quad (3.87)$$

We write $\mathrm{M}_k^{[1]}(\rho, \sigma)$ for the space of generalized second order modular forms of weight k and type (ρ, σ).

Remark 3.4. Paralleling the map F_2 from the previous section, we have the following linear and surjective map:

$$\begin{aligned} F : \mathrm{M}_k(\rho \boxplus_{\mathrm{pb}} \sigma) &\to \mathrm{M}_k^{[1]}(\rho, \sigma) \\ (f, *) &\mapsto f. \end{aligned} \quad (3.88)$$

Our initial motivation for studying generalized second order modular forms is that they contain Eichler integrals. They are defined as follows. Let $k \in \mathbb{Z}_{\geq 2}$ be an even

integer, and let $f \in S_k$ be a cusp form. Then the Eichler integral associated to f, is defined by

$$\mathcal{E}_f(\tau) = \int_\tau^{i\infty} f(z)(\tau - z)^{k-2}\,\mathrm{d}z, \quad \tau \in \mathbb{H}. \tag{3.89}$$

Note that if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then since

$$\gamma\tau - \gamma z = \frac{\tau - z}{(c\tau + d)(cz + d)}, \tag{3.90}$$

and $\mathrm{d}(\gamma z) = (c\tau + d)^{-2}\,\mathrm{d}z$, we have that

$$(\mathcal{E}_f|_{2-k}\gamma)(\tau) = \int_\tau^{\gamma^{-1}(i\infty)} f(z)(\tau - z)^{k-2}\,\mathrm{d}z. \tag{3.91}$$

This implies that

$$\phi_{\mathcal{E}_f}(\gamma;\tau) = \mathcal{E}_f|_{2-k}(1-\gamma^{-1})(\tau) = \int_{\gamma(i\infty)}^{i\infty} f(z)(\tau - z)^{k-2}\mathrm{d}z. \tag{3.92}$$

If $k = 2$, and we allow f to be a cusp form for a congruence subgroup Γ of level greater than one, we have that $\phi_{\mathcal{E}_f}(\gamma;\tau) = -\langle \gamma, f\rangle$, where $\langle \gamma, f\rangle$ is the modular symbol defined in (2.36), and so in the weight 2 case we have that $\phi_{\mathcal{E}_f} \in Z^1_{\mathrm{pb}}(\mathbf{1}_\Gamma, \mathbf{1}_\Gamma)$.

To relate $\mathcal{E}_f$ to generalized second order modular forms, we first need to define the dth symmetric power of the standard representation, denoted by $\mathrm{sym}^{\mathsf{d}}(X)$. It is the arithmetic type for $\mathrm{SL}_2(\mathbb{Z})$ given by

$$V(\mathrm{sym}^{\mathsf{d}}(X)) = \mathbb{C}[X]_{\mathsf{d}} = \{p \in \mathbb{C}[X] : \deg(p) \leq \mathsf{d}\}, \tag{3.93}$$

where X is a formal variable, and

$$\mathrm{sym}^{\mathsf{d}}(X)(\gamma)p = p|_{-\mathsf{d}}\gamma^{-1} = (-cX + a)^{\mathsf{d}} p\left(\frac{dX - b}{-cX + a}\right), \tag{3.94}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We see that $\phi_{\mathcal{E}_f}(\gamma; X) \in \mathbb{C}[X]_{k-2}$, and hence $\phi_{\mathcal{E}_f}(\cdot; X) \in Z^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$. Hence, *if* we can transform $\phi_{\mathcal{E}_f}(\cdot; X)$ to a cocycle $\psi_{\mathcal{E}_f} \in Z^1_{\mathrm{pb}}(\mathrm{sym}^{\mathsf{d}}(X), \mathbf{1})$ satisfying that

$$\psi_{\mathcal{E}_f}(\gamma)g(\tau) = -\phi_{\mathcal{E}_f}(\gamma^{-1};\tau) = \mathcal{E}_f|_{2-k}(\gamma - 1)(\tau), \quad \tau \in \mathbb{H}, \tag{3.95}$$

for some modular form $g \in M_{2-k}(\mathrm{sym}^{k-2}(X))$, we would have that $\mathcal{E}_f \in M_{2-k}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$. Thankfully, this is indeed possible. To construct $\psi_{\mathcal{E}_f}$ we make use of the following pairing on $\mathbb{C}[X]_{\mathsf{d}}$:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathbb{C}[X]_{\mathsf{d}} \times \mathbb{C}[Y]_{\mathsf{d}} &\longrightarrow \mathbb{C} \\ (p, q) &\longmapsto \sum_{i=0}^{\mathsf{d}} (-1)^i \binom{\mathsf{d}}{i}^{-1} p_i q_{\mathsf{d}-i}, \end{aligned} \tag{3.96}$$

where $p = \sum_{i=0}^{\mathsf{d}} p_i X^i$ and $q = \sum_{i=0}^{\mathsf{d}} q_i Y^i$. One can verify that for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $p \in \mathbb{C}[X]_{\mathsf{d}}$ and $q \in \mathbb{C}[Y]_{\mathsf{d}}$, we have that

$$\langle \gamma.p, \gamma.q \rangle = \langle p, q \rangle. \tag{3.97}$$

One also sees that for a polynomial $p \in \mathbb{C}[X]_{\mathsf{d}}$, we have that

$$\langle p, (X - Y)^{\mathsf{d}} \rangle = p(Y). \tag{3.98}$$

For $\phi \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$, we now let

$$\phi^\vee(\gamma)v = \langle \phi(\gamma^{-1}), v \rangle, \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}),\, v \in \mathbb{C}[X]_{\mathsf{d}}. \tag{3.99}$$

The invariance of $\langle \cdot, \cdot \rangle$ implies that $\phi^\vee \in \mathrm{Z}^1_{\mathrm{pb}}(\mathrm{sym}^{\mathsf{d}}(X), \mathbf{1})$. While we will not use it now, we record that

$$\begin{aligned} \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X)) &\longrightarrow \mathrm{Z}^1_{\mathrm{pb}}(\mathrm{sym}^{\mathsf{d}}(X), \mathbf{1}) \\ \phi &\longmapsto \phi^\vee, \end{aligned} \tag{3.100}$$

is a vector space isomorphism. We now let $\psi_{\mathcal{E}_f} = -\phi_{\mathcal{E}_f}(\cdot; X)^\vee$. Then it holds that

$$\psi_{\mathcal{E}_f}(\gamma)(X - \tau)^{k-2} = -\langle \phi_{\mathcal{E}_f}(\gamma^{-1}; X), (X - \tau)^{k-2} \rangle = -\phi_{\mathcal{E}_f}(\gamma^{-1}; \tau), \tag{3.101}$$

and since $(X - \tau)^{k-2} \in \mathrm{M}_{2-k}(\mathrm{sym}^{k-2}(X))$, we have arrived at what we wanted. We conclude that $\mathcal{E}_f \in \mathrm{M}_{2-k}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$, and that

$$\begin{pmatrix} \mathcal{E}_f \\ (X - \cdot)^{k-2} \otimes \psi_{\mathcal{E}_f} \end{pmatrix} \in \mathrm{M}_{2-k}(\mathbf{1} \boxplus_{\mathrm{pb}} \mathrm{sym}^{k-2}(X)). \tag{3.102}$$

The above discussion expands on the condensed proof we gave of Proposition 2.2 in Paper II.

3.3.5 *Iterated Eichler-Shimura integrals*

The Eichler integral $\mathcal{E}_f$ that we considered in the previous section can readily be generalized to a multivariate setting, essentially by *iterating* the definition of $\mathcal{E}_f$ a number of times. We then arrive at iterated Eichler-Shimura integrals. They also turn out to correspond to components of vector-valued modular forms of extension type.

The first step is to decouple the τ-dependence of the integrand in $\mathcal{E}_f$, and consider a polynomial-valued analogue of $\mathcal{E}_f$. Specifically, for a cusp form $f \in \mathrm{S}_k$ of even weight $k \in \mathbb{Z}_{\geq 2}$ we let the polynomial-valued Eichler integral associated to f be the function $I_f : \mathbb{H} \to \mathbb{C}[X]_{\mathsf{d}}$, be defined by

$$I_f(\tau; X) = \int_\tau^{i\infty} f(z)(X - z)^{k-2} \mathrm{d}z, \quad \tau \in \mathbb{H}. \tag{3.103}$$

Since f is a cusp form, it is clear that I_f is holomorphic and has moderate growth at the cusp. We also record that $I_f(\tau;\tau) = \mathcal{E}_f(\tau)$.

We now let

$$\phi_{I_f} = I_f(\cdot;X)\big|_{0,\mathrm{sym}^{k-2}(X)}(1-\gamma^{-1}). \tag{3.104}$$

Using the same technique as when we computed $\phi_{\mathcal{E}_f}$ in the previous section, we find that $\phi_{I_f} \in Z^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$. Let now $\rho_f = \mathrm{sym}^{k-2}(X) \boxplus_{\phi_{I_f}} \mathbf{1}$. Then for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned}\begin{pmatrix} I_f(\cdot;X) \\ 1 \end{pmatrix}\big|_{0,\rho_f}\gamma &= \rho_f(\gamma^{-1})\begin{pmatrix} I_f(\cdot;X)\big|_0\gamma^{-1} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{sym}^{k-2}(X)(\gamma^{-1})I_f(\cdot;X)\big|_0\gamma^{-1} + \phi_{I_f}(\gamma^{-1}) \\ 1 \end{pmatrix} = \begin{pmatrix} I_f(\cdot;X) \\ 1 \end{pmatrix}, \end{aligned} \tag{3.105}$$

showing that $(I_f, 1)^T \in \mathrm{M}_0(\rho_f)$.

We can now consider the iterated version of I_f. These are known as iterated Eichler-Shimura integrals.

Definition 3.5. Let $k \in \mathbb{Z}_{\geq 2}$ be an even integer, let $n \in \mathbb{Z}_{\geq 1}$, and let $(f_1, \dots, f_n) \in \mathrm{S}_k^n$ be an n-tuple of weight k cusp forms. Then we let the depth n iterated Eichler-Shimura integral associated to $(f_1, \dots, f_n)$ be the function $I_{f_1,\dots,f_n} : \mathbb{H} \to \mathbb{C}[X_1, \dots, X_n]_{\mathsf{d}}$ recursively defined by

$$I_{f_1,\dots,f_n}(\tau; X_1, \dots, X_n) = \int_\tau^{i\infty} f(z)(X_1 - z)^{k-2} I_g(z; X_2, \dots, X_n)\mathrm{d}z. \tag{3.106}$$

Hence, with the terminology of Definition 3.5, we can say that I_f is the same as the depth 1 iterated Eichler-Shimura integral associated to the 1-tuple (f).

Iterated Eichler-Shimura integrals have been studied extensively for a variety of deep reasons. We shall explain in this in greater detail in the summary of papers II and III, but suffice it to say that it would be greatly useful to be able to study them with another set of tools. By relating them to vector-valued modular forms of extension type, we accomplish exactly this.

Put concretely, in paper III, we provide the following theorem, characterizing iterated Eichler-Shimura integrals of depth 2.

Theorem (Theorem 2.3 of paper III)**.** Let $k \in \mathbb{Z}_{\geq 2}$ be an even integer, let $n \in \mathbb{Z}_{\geq 1}$, and let $f, g \in \mathrm{S}_k$ be cusp forms of weight k. Let also

$$\psi_{f,g}(\gamma) = \int_{\gamma(i\infty)}^{i\infty} f(z)(X - z)^{k-2} I_g(z; Y)\mathrm{d}z, \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}), \tag{3.107}$$

and let

$$(\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X))(\gamma)p = \phi_{I_g}(\gamma) \cdot \mathrm{sym}^{k-2}(X)(\gamma)p, \tag{3.108}$$

so that $\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X)$ takes values in $\mathbb{C}[X,Y]_{k-2}$. Let now $\widetilde{\rho}_{f,g}$ be the extension type[7]

$$\widetilde{\rho}_{f,g} = \mathrm{sym}^{k-2}(X,Y) \boxplus_{(\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X), \psi_{f,g})} (\mathrm{sym}^{k-2}(X) \boxplus_{\phi_{I_f}} \mathbf{1}), \tag{3.109}$$

where $\mathrm{sym}^{k-2}(X,Y) = \mathrm{sym}^{k-2}(X) \otimes \mathrm{sym}^{k-2}(Y)$. Then

$$I_{f,g} \in \mathrm{M}_0(\widetilde{\rho}_{f,g}). \tag{3.110}$$

Remark 3.5. The extension type $\widetilde{\rho}_{f,g}$ can be realized as the block matrix representation given by

$$\widetilde{\rho}_{f,g}(\gamma) = \begin{pmatrix} \mathrm{sym}^{k-2}(X,Y)(\gamma) & (\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X))(\gamma) & \psi_{f,g}(\gamma) \\ 0 & \mathrm{sym}^{k-2}(X)(\gamma) & \phi_{I_f}(\gamma) \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.111}$$

where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This is of course equivalent to its description as an extension type.

In paper III, we show that $I_{f,g}$ is related to a scalar-valued Eichler-Shimura type integral, denoted by $\mathcal{E}_{f,g}$, in a way that is parallel to how I_f relates to $\mathcal{E}_f$. We also show that this relation provides an alternative explanation of some properties of Eichler cohomology.

At this point, the reader should have a basic understanding of the utility of vector-valued modular forms. Therefore, we now deem it appropriate to give a summary of the papers contained in this thesis.

7 The reason for the occurence of tilde in the notation will be made clear in Chapter 5.

Part III

SUMMARIES OF PAPERS

4 PAPER I

4.1 INTRODUCTION

As we have stated before, the goal of paper I is to develop an algorithm for computing bases for $\mathrm{M}_k(\rho)$, where $k \in \mathbb{Z}_{\geq 2}$ and ρ is a congruence type of level $N \in \mathbb{Z}_{\geq 1}$, in terms of products of components of vector-valued Eisenstein series of type ρ_N and their Fourier series expansions, using the Raum-Xià theorem (that is, Theorem 3.1). Together with the paper, we also provide an implementation of the algorithm (and adjacent tooling) as a `Julia` [28] package, fitting within the `Nemo.jl` [30] ecosystem.

Recall that the Raum-Xià theorem implies that for integers $k \in \mathbb{Z}_{\geq 2}$ and $1 \leq l \leq k-1$, and ρ a congruence type of level N, there exists an integer $N \mid N_0 \in \mathbb{Z}_{\geq 1}$ such that

$$\mathrm{M}_k(\rho) = \mathrm{H}^0(\mathcal{E}_k[\rho_N] \otimes \rho) + \mathrm{H}^0((\mathcal{E}_l[\rho_{N_0}] \cdot \mathrm{E}_{k-l}[\rho_{N_0}]) \otimes \rho) \tag{4.1}$$

Here that we use the map from Proposition 3.4 to view the invariant spaces as subspaces of $\mathrm{M}_k(\rho)$.

To use this theorem effectively, we need some way of enumerating a basis for the elements of

$$\mathrm{H}^0(\mathcal{E}_k[\rho_N] \otimes \rho), \tag{4.2}$$

and

$$\mathrm{H}^0((\mathcal{E}_l[\rho_{N_0}] \cdot \mathcal{E}_{k-l}[\rho_{N_0}]) \otimes \rho). \tag{4.3}$$

We do this by relating (4.2) and (4.3) to spaces that are easier to represent on a computer – namely invariant spaces of twisted permutation types. This can be seen as representing spaces of analytic objects in terms of abstract algebraic objects, and thus we think of it as a process of *algebraization*.

4.2 ALGEBRAIZATION

In the previous chapter, we saw that for $k, N \in \mathbb{Z}_{\geq 1}$, it holds that

$$\mathcal{E}_k[\rho_N] = \mathrm{E}_k(\Gamma(N)) = \mathrm{span}_{\mathbb{C}}\{G_{k,N,\Gamma_1(N)\gamma} : \gamma \in \mathrm{SL}_2(\mathbb{Z})\}. \tag{4.4}$$

In order to not have to work with right actions, we define

$$G_{k,N,\gamma\Gamma_1(N)} = G_{k,N,\Gamma_1(N)\gamma^{-1}}. \tag{4.5}$$

Recall that we may view $\mathcal{E}_k[\rho_N]$ as an $\mathrm{SL}_2(\mathbb{Z})$-representation through the weight k slash action. In fact, $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{E}_k[\rho_N]$ in the same way as $\mathrm{SL}_2(\mathbb{Z})$ acts on $V(\rho_N)$. To be precise, let $\{\gamma_i\}_{i=1}^n$ be a complete set of representatives for $\mathrm{SL}_2(\mathbb{Z})/\Gamma_1(N)$. Then for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $1 \leq i \leq n$, we have that

$$\gamma.G_{k,N,\gamma_i\Gamma_1(N)} = G_{k,N,\Gamma_1(N)\gamma_i^{-1}\gamma^{-1}} = G_{k,N,\gamma_{\pi_\gamma(i)}\Gamma_1(N)}, \tag{4.6}$$

and

$$\gamma.(\gamma_i \otimes 1) = \gamma_{\pi_\gamma(i)}\gamma' \otimes 1 = \gamma_{\pi_\gamma(i)} \otimes 1, \tag{4.7}$$

for some permutation $\pi_\gamma \in \mathrm{S}_n$ and an element $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ depending on γ. This implies that for $k, N \in \mathbb{Z}_{\geq 1}$ we have a surjection of representations

$$f_{k,N} : \rho_N^\vee \longrightarrow\!\!\!\!\!\rightarrow \mathcal{E}_k[\rho_N] \quad \text{given by} \quad \gamma \otimes 1 \longmapsto G_{k,N,\gamma\Gamma_1(N)}. \tag{4.8}$$

This result is Proposition 1.2 of the paper. It is a standard fact representation theory, that if we have a morphism $f : \rho \to \sigma$ of semi-simple finite-dimensional complex representations, then the induced map

$$\begin{aligned} \widetilde{f} : \mathrm{H}^0(\rho) &\longrightarrow \mathrm{H}^0(\sigma) \\ v &\longmapsto f(v), \end{aligned} \tag{4.9}$$

is surjective, if f is surjective. The representations ρ_N and $\mathcal{E}_k[\rho_N]$ are both semi-simple, in that they have finite index kernels and are therefore unitarizable through Weyl's trick. Taken together, this implies that the maps

$$\begin{aligned} \mathrm{H}^0(\rho_N^\vee \otimes \rho) &\longrightarrow \mathrm{H}^0(\mathcal{E}_k[\rho_N] \otimes \rho) \\ v \otimes w &\longmapsto f_{k,N}(v) \otimes w, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho) &\longrightarrow \mathrm{H}^0(\mathcal{E}_l[\rho_{N_0}] \otimes \mathcal{E}_{k-l}[\rho_{N_0}] \otimes \rho) \\ v_1 \otimes v_2 \otimes w &\longmapsto f_{l,N_0}(v_1) \otimes f_{k-l,N_0}(v_2) \otimes w, \end{aligned} \tag{4.11}$$

are surjective. This means that the direct sum of (4.10) and (4.11) is a surjection on the form

$$\begin{aligned} \Phi_{\mathcal{E}} : \mathrm{H}^0(\rho_N^\vee \otimes \rho) \oplus \mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho) \\ \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^0(\mathcal{E}_k[\rho_N] \otimes \rho) \oplus \mathrm{H}^0(\mathcal{E}_l[\rho_{N_0}] \otimes \mathcal{E}_{k-l}[\rho_{N_0}] \otimes \rho), \end{aligned} \tag{4.12}$$

For subrepresentations ρ_1 and ρ_2 of a representation ρ, it is clear that the map $\rho_1 \oplus \rho_2 \to \rho_1 + \rho_2$ given by $(v, w) \mapsto v + w$ is surjective. It is also clear that the map

$$\begin{aligned} &\mathcal{E}_l[\rho_{N_0}] \otimes \mathcal{E}_{k-l}[\rho_{N_0}] \to \mathcal{E}_l[\rho_{N_0}] \cdot \mathcal{E}_{k-l}[\rho_{N_0}] \\ &\quad \text{given by} \quad f \otimes g \mapsto f \cdot g, \end{aligned} \tag{4.13}$$

is surjective, for any $k \in \mathbb{Z}_{\geq 2}$, $1 \leq l \leq k-1$ and $N_0 \in \mathbb{Z}_{\geq 1}$. Notice also that the space $\mathcal{E}_l[\rho_{N_0}] \cdot \mathcal{E}_{k-l}[\rho_{N_0}]$ is finite-dimensional in that it is a subspace of $\mathrm{M}_k(\Gamma(N_0))$ and that it is semi-simple as a representation with respect to the weight k slash action since its kernel contains $\Gamma(N_0)$, and so has finite index.

This means that we have surjections on the form

$$\begin{aligned}\Phi_\times : \mathrm{H}^0(\mathcal{E}_k[\rho_N] \otimes \rho) \oplus \mathrm{H}^0(\mathcal{E}_l[\rho_{N_0}] \otimes \mathcal{E}_{k-l}[\rho_{N_0}] \otimes \rho) \\ \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^0(\mathcal{E}_k[\rho_N] \otimes \rho) \oplus \mathrm{H}^0((\mathcal{E}_l[\rho_{N_0}] \cdot \mathcal{E}_{k-l}[\rho_{N_0}]) \otimes \rho),\end{aligned} \tag{4.14}$$

given by $(v, f_1 \otimes f_2 \otimes w) \mapsto (v, (f_1 \cdot f_2) \otimes w)$, and

$$\begin{aligned}\Phi_\Sigma : \mathrm{H}^0(\mathcal{E}_k[\rho_N] \otimes \rho) \oplus \mathrm{H}^0((\mathcal{E}_l[\rho_{N_0}] \cdot \mathcal{E}_{k-l}[\rho_{N_0}]) \otimes \rho) \\ \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^0(\mathcal{E}_k[\rho_N] \otimes \rho) + \mathrm{H}^0((\mathcal{E}_l[\rho_{N_0}] \cdot \mathcal{E}_{k-l}[\rho_{N_0}]) \otimes \rho),\end{aligned} \tag{4.15}$$

given by $(v, w) \mapsto v + w$. In conclusion, we have that

$$\mathrm{M}_k(\rho) = \mathrm{im}(\Phi_\Sigma \circ \Phi_\times \circ \Phi_{\mathcal{E}}). \tag{4.16}$$

Hence, if we can compute a basis $\mathcal{B}$ for

$$\mathrm{H}^0(\rho_N^\vee \otimes \rho) \oplus \mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho) \tag{4.17}$$

then $\mathcal{B}' = \Phi_\Sigma(\Phi_\times(\Phi_{\mathcal{E}}(\mathcal{B})))$ is a spanning set for $\mathrm{M}_k(\rho)$. To reduce it to a basis, we employ the Fourier series expansions of the Eisenstein series in combination with the Sturm bound and row reduction, in a way that parallels Algorithm 1. However, this is somewhat subtle, so let us describe it in a new section.

4.3 FOURIER SERIES EXPANSIONS

For a ring A and a rational number $B \in \mathbb{Q}$, we let the ring of Puiseux series with coefficients in A, denoted by $\mathrm{FE}(A)$, and the ring of Puiseux series with coefficients in A truncated at B, denoted by $\mathrm{FE}_B(A)$, be given by

$$\begin{aligned}\mathrm{FE}(A) &= \bigcup_{N \in \mathbb{Z}_{\geq 1}} A[\![q^{1/N}]\!][q^{-1}], \text{ and} \\ \mathrm{FE}_B(A) &= \bigcup_{N \in \mathbb{Z}_{\geq 1}} A[\![q^{1/N}]\!][q^{-1}]/q^B \bigcup_{N \in \mathbb{Z}_{\geq 1}} A[\![q^{1/N}]\!].\end{aligned} \tag{4.18}$$

If $\rho(T)$ is diagonalizable, we obtain maps[1]

$$\begin{aligned}\mathrm{fe} : \mathrm{M}_k(\rho) &\longrightarrow \mathrm{FE}(\mathbb{C}) \otimes V(\rho) \\ f &\longmapsto \sum_{n \in \mathbb{Q}} c(f; n) q^n,\end{aligned} \tag{4.19}$$

1 Here we make the identification $v \otimes \sum_{n \in \mathbb{Q}} c_n q^n = \sum_{n \in \mathbb{Q}} (c_n v) q^n$, where $c_n \in \mathbb{C}$ and $v \in V(\rho)$.

and

$$\begin{aligned} \mathrm{fe}_P : \mathrm{M}_k(\rho) &\longrightarrow \mathrm{FE}_B(\mathbb{C}) \otimes V(\rho) \\ f &\longmapsto \sum_{\substack{n \in \mathbb{Q} \\ n < P}} c(f;n) q^n \end{aligned} \tag{4.20}$$

by computing the Fourier series expansions of the elements of $\mathrm{M}_k(\rho)$. Note also that if P is the Sturm bound for $\mathrm{M}_k(\rho)$, then fe_P is injective.

In Section 2.1 of the paper, we show that the Fourier series expansion map fe_B, $B \in \mathbb{Q}$, commutes with the maps $\Phi_{\mathcal{E}}$, $\Phi_\times$, and Φ_Σ, in the sense that for $B \in \mathbb{Q}$, we have

$$\mathrm{fe}_B \circ \Phi_\Sigma \circ \Phi_\times \circ \Phi_{\mathcal{E}} = \Phi_\Sigma \circ \Phi_\times \circ (\mathrm{fe}_B \otimes \mathrm{id}_\rho \oplus \mathrm{fe}_P^{\otimes 2} \otimes \mathrm{id}_\rho) \circ \Phi_{\mathcal{E}} \tag{4.21}$$

As we shall see in the next section, this can have a considerable impact on performance. Instead of having to enumerate the complete spanning set $\mathcal{B}'$ (referred to in the previous section) at once, and then perform row reduction on the corresponding matrix of Fourier series coefficients, we can construct $\mathcal{B}'$ iteratively, and perform row reduction on the matrix of coefficients as we go along. If, as experimental data indicates usually is the case, $|\mathcal{B}'|$ is much larger than $\dim(\mathrm{M}_k(\rho))$, this allows us to abort execution as soon as the rank of matrix of coefficients equals $\dim(\mathrm{M}_k(\rho))$.

4.4 THE ALGORITHM

If we have an efficient means to compute a basis for the space

$$\mathrm{H}^0(\rho_N^\vee \otimes \rho) \oplus \mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho), \tag{4.22}$$

then the above discussion leads to an algorithm for computing a corresponding basis for $\mathrm{M}_k(\rho)$, in terms of Fourier series expansions. We state it as a theorem below.

Theorem (Theorem 2.2 of Paper I)**.** Let $k \in \mathbb{Z}_{\geq 2}$ and let ρ be a congruence type of level N. Assume that S is greater than or equal to the Sturm bound for k and ρ. Then Algorithm 2 computes a basis for $\mathrm{fe}_S(\mathrm{M}_k(\rho)) \cong \mathrm{M}_k(\rho)$.

Algorithm 4.1: Computing a basis for $\mathrm{M}_k(\rho)$

let N_0 be chosen as in Theorem 3.1;
$P \leftarrow \lceil SN \rceil / N$;
let v_i, $1 \leq i \leq \dim(\rho)$ be a basis of $V(\rho)$;
let M be the empty matrix of size $0 \times PN \dim(\rho)$ over $\mathbb{C}$;
let $\mathcal{B} = \bigcup_{i=1}^{I} \mathcal{B}_i$ be a disjoint decomposition of a basis of $\mathrm{H}^0(\rho_N^\vee \otimes \rho) \oplus \mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho)$;
for $1 \leq i \leq I$ **do**
  for $b \in \mathcal{B}_i$ **do**
    let $f = \sum_i f_i v_i \leftarrow \Phi_\Sigma \, \Phi_\times \, \mathrm{fe}_P \, \Phi_{\mathcal{E}}(b)$;
    append to M the row with entries $r_{i,n} \leftarrow c(f_i; n/P)$, for $0 \leq n < NP$ and $1 \leq i \leq \dim(\rho)$;
  end
  replace M by its reduced row echelon form;
  if $\operatorname{rank} M = \dim \mathrm{M}_k(\rho)$ **then**
    for each row r of M, output the truncated Fourier series expansion f with coefficient $c(f_i; n) = r_{i,n}$;
    return;
  end
end

Remark 4.1. In the paper, we provide a modified version of the above algorithm that works with coefficients in the universal cyclotomic field[2] $\mathbb{Q}^{\mathrm{ab}}$. This allows for exact arithmetic, but requires some understanding of what are known as $\mathbb{Q}^{\mathrm{ab}}$-structures on the relevant spaces of modular forms. For brevity we omit this from this summary, and we refer to Section 2.2 of the paper for a detailed discussion.

As we remark in the paper, we can also obtain an expression for every element in the computed basis in terms of products of Eisenstein series, by keeping track of all the formal linear combinations of the elements in $\mathcal{B}$ occur in every row of M.

We provide a formal proof of the correctness of Algorithm 2 in the paper. For brevity, we do not include it here, but suffice it to say that it follows by a straightforward induction argument relying on the intertwining property of the truncated Fourier series expansion map fe_P that we described in (4.21), together with the fact that fe_P is injective when P is at least the Sturm bound.

However, to be able to make effective use of Algorithm 2, we need:

- an efficient way of computing a basis for

$$\mathrm{H}^0(\rho_N^\vee \otimes \rho) \oplus \mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho),$$

2 The smallest subfield of $\mathbb{C}$ containing all roots of unity.

- an estimate of the time and space complexity, and
- a comparison to other available algorithms.

That it is the purpose of the rest of the paper, which we summarize below.

4.5 ORBIT-STABILIZER DECOMPOSITION

Given an arithmetic type σ for a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, we can compute a basis for $\mathrm{H}^0(\sigma)$ naïvely by computing a basis for

$$\ker \underbrace{\begin{pmatrix} \sigma(\gamma_1) - 1 \\ \sigma(\gamma_2) - 1 \\ \vdots \\ \sigma(\gamma_m) - 1 \end{pmatrix}}_{:= T \in \mathrm{Hom}(V(\sigma), V(\sigma)^m)}, \quad \text{where } \{\gamma_i\}_{i=1}^m,\ m \in \mathbb{Z}_{\geq 1}, \text{ is a generating set for } \Gamma. \tag{4.23}$$

This can be accomplished with row reduction on a matrix representation of T. If we have no additional structural information for σ, this is the best we can do. However, in our case, σ will be on the form $\rho_N^\vee \otimes \rho$ or $\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho$ and these arithmetic types certainly do have additional structural information.

Indeed, since ρ_N acts by permutation, we have that $\rho_N^\vee \cong \rho_N$. This means that we may equivalently compute the invariants for

$$\rho_N \otimes \rho \quad \text{and} \quad \rho_{N_0} \otimes \rho_{N_0} \otimes \rho. \tag{4.24}$$

However, they are evidently (in the terminology of Section 3.2.2) twisted permutation types of order

$$|\mathrm{SL}_2(\mathbb{Z})/\Gamma_1(N)| \quad \text{and} \quad |\mathrm{SL}_2(\mathbb{Z})/\Gamma_1(N_0)|^2, \tag{4.25}$$

respectively. Twisted permutation types (of order greater than 1) have the advantage that they split into direct sums of smaller (induced) representations. The associated invariant space will therefore also split into a direct sum of smaller invariant spaces.

To state this result, we need some additional terminology. Let ρ be a twisted permutation type of order $n \in \mathbb{Z}_{\geq 1}$ of a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ with twist representation σ of a group G. If $I \subseteq \{1, \ldots, n\}$ is a subset, we let

$$V(\rho)_I = \mathrm{span}_{\mathbb{C}}\{e_i \otimes v : i \in I,\ v \in V(\sigma)\} \subseteq V(\rho), \tag{4.26}$$

where $\{e_i\}_{i=1}^n$ is the canonical basis for $\mathbb{C}^n$. We also let the stabilizer of $V(\rho)_I$ be the subgroup given by

$$\mathrm{Stab}(V(\rho)_I) = \{\gamma \in \Gamma : \rho(\gamma)V(\rho)_I \subseteq V(\rho)_I\} \subseteq \Gamma. \tag{4.27}$$

Restricting ρ to $\mathrm{Stab}(V(\rho)_I)$ yields a representation

$$\rho_I : \mathrm{Stab}(V(\rho)_I) \to \mathrm{GL}(V(\rho)_I). \tag{4.28}$$

Let now $\pi : G \wr S_n \to S_n$ be the projection $\pi(f, \alpha) = \alpha$, let $\rho_\pi^\wr = \pi \circ \rho_\pi^\wr$, and let $R \subseteq \{1, \dots, n\}$ be a set of representatives for $\rho_\pi^\wr(\Gamma)\backslash\{1, \dots, n\}$. That is, we have

$$\{1, \dots, n\} = \bigsqcup_{i \in R} \rho_\pi^\wr(\Gamma)\{i\}. \tag{4.29}$$

Proposition 3.5 of the paper states that

$$\rho \cong \bigoplus_{i \in R} \mathrm{Ind}_{\mathrm{Stab}(V(\rho)_i)}^{\Gamma} \rho_i, \tag{4.30}$$

where $V(\rho)_i = V(\rho)_{\{i\}}$ and $\rho_i = \rho_{\{i\}}$. To obtain R and generating sets for the $\mathrm{Stab}(V(\rho)_{\{i\}})$ we apply the orbit-stabilizer algorithm [15, Chapter 4.1]. In our implementation this is performed via the computer algebra system GAP.

Hence, with ρ as above, we obtain

$$\mathrm{H}^0(\rho) \cong \bigoplus_{i \in R} \mathrm{H}^0(\mathrm{Ind}_{\mathrm{Stab}(V(\rho)_i)}^{\Gamma} \rho_i). \tag{4.31}$$

There is an additional benefit to considering invariant spaces of induced types. Namely, we can employ the following result, which historically goes under the name *Frobenius reciprocity*.

Proposition (Frobenius reciprocity)**.** Let $H \subseteq G$ be groups, with H a subgroup of G. Then, if ρ is a complex representation of H and σ is a representation of G, we have

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G \rho, \sigma) \cong \mathrm{Hom}_H(\rho, \mathrm{Res}_H^G \sigma). \tag{4.32}$$

Similarly, if ρ is a representation of G and σ is a representation of H, we have that

$$\mathrm{Hom}_G(\rho, \mathrm{Ind}_H^G \sigma) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G \rho, \sigma). \tag{4.33}$$

Proof. See any standard book on representation theory, such as [3] or [1]. □

By Frobenius reciprocity, we have that

$$\begin{aligned} \mathrm{H}^0(\mathrm{Ind}_{\mathrm{Stab}(V(\rho)_i)}^{\Gamma} \rho_i) &= \mathrm{Hom}_\Gamma(\mathbf{1}, \mathrm{Ind}_{\mathrm{Stab}(V(\rho)_i)}^{\Gamma} \rho_i) \\ &\cong \mathrm{Hom}_{\mathrm{Stab}(V(\rho)_i)}(\mathbf{1}, \rho_i) \cong \mathrm{H}^0_{\mathrm{Stab}(V(\rho)_i)}(\rho_i). \end{aligned} \tag{4.34}$$

However, for $\gamma \in \mathrm{Stab}(V(\rho)_i)$ it holds that

$$\rho(\gamma)(e_i \otimes w) = e_i \otimes \sigma(\rho^\wr(\gamma)_1(i))w, \tag{4.35}$$

implying that

$$\mathrm{H}^0_{\mathrm{Stab}(V(\rho)_i)}(\rho_i) \cong \{w \in V(\sigma) : \forall \gamma \in \mathrm{Stab}(V(\rho)_i).\, \sigma(\rho^\flat(\gamma)_1(i))w = w\}. \quad (4.36)$$

Hence, if $\{\gamma_1, \ldots, \gamma_m\}$ is a set of generators for $\mathrm{Stab}(V(\rho)_i)$, we have that

$$\mathrm{H}^0_{\mathrm{Stab}(V(\rho)_i)}(\rho_i) \cong \ker \begin{pmatrix} \sigma(\rho^\flat(\gamma_1)_1(i)) - 1 \\ \sigma(\rho^\flat(\gamma_2)_1(i)) - 1 \\ \vdots \\ \sigma(\rho^\flat(\gamma_m)_1(i)) - 1 \end{pmatrix}. \quad (4.37)$$

As mentioned earlier, we can compute a basis for (4.37) using row reduction.

We conclude that if ρ is a twisted permutation type, then we can compute a basis for the $\mathrm{H}^0(\rho)$ by first decomposing ρ into a direct sum of smaller induced types, and then computing bases for their invariant spaces by computing bases for the kernels (4.37). As we shall see in Section 4.8, this gives a significant reduction in time complexity in comparison to computing $\mathrm{H}^0(\rho)$ through the naïve method mentioned at the outset of this section.

4.6 ISOTYPIC AND DOUBLE-COSET DECOMPOSITIONS

To be able to estimate the time complexity of Algorithm 2, we need to have a good estimate of

$$h^0(\rho_N^\vee) + h^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho), \quad (4.38)$$

where $h^0 = \dim \mathrm{H}^0$, preferably in terms of $\dim(\rho)$, N, and N_0. For this, we use another decomposition that leverages:

(i) Frobenius reciprocity,

(ii) the $\Gamma_1(N)$-isotypic decomposition of $\mathrm{Res}_{\Gamma_1(N)}\rho$, and

(iii) a result often referred to as *Mackey's double coset theorem*, or *Mackey's formula*.

We first describe the isotypic decomposition. Let ρ be a congruence type of level $N \in \mathbb{Z}_{\geq 1}$ for $\Gamma_1(N)$. Then since

$$\Gamma_1(N)/\Gamma(N) = \{T^k \Gamma(N) : k \in \mathbb{Z}\}, \quad (4.39)$$

we have that ρ is determined by $\rho(T)$. We furthermore have that $\rho(T^N) = \rho(T)^N = 1$, and so $\rho(T)$ is diagonalizable with eigenvalues equal to Nth roots of unity. This implies that

$$\rho \cong \bigoplus_{n=0}^{N-1} \rho[e(\frac{\cdot n}{N})], \tag{4.40}$$

where the so-called *isotypic component* $\rho[e(\frac{\cdot n}{N})]$ equals the direct sum of all irreducible subrepresentations of ρ isomorphic to the representation

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^\times \quad \text{given by} \quad \gamma \mapsto e(\frac{nb(\gamma)}{N}). \tag{4.41}$$

The decomposition (4.40) is called the *isotypic decomposition* or the *canonical decomposition*, see [1, Chapter 2.6]

Mackey's formula, on the other hand, describes how the restrictions of induced types split into direct sums indexed by double cosets.

Proposition (Mackey's double coset theorem)**.** Let H and K be subgroups of a group G, and let ρ be representation of K. For $[g] \in H\backslash G/K$, let $K_g = H \cap gKg^{-1}$. Let also $\rho_g : K_g \to \mathrm{GL}(V(\rho))$ be the representation given by $\rho_g(k)w = \rho(g^{-1}kg)w$. Then it holds that

$$\mathrm{Res}_H^G \mathrm{Ind}_K^G \rho = \bigoplus_{[g]\in H\backslash G/K} \mathrm{Ind}_{K_g}^H \rho_g. \tag{4.42}$$

Proof. See [1, Chapter 7.3]. □

We now write, as in Section 3.1 of the paper

$$\pi_g = \mathrm{Ind}_{\Gamma_1(N_0)\cap g^{-1}\Gamma_1(N_0)g}^{\Gamma_1(N_0)} \mathbf{1}, \quad [g] \in \Gamma_1(N_0)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_1(N_0). \tag{4.43}$$

We record here, and this is shown in Lemma 3.2 of the paper, that $\pi_g(T)$ corresponds to a transitive permutation. This implies that $\pi_g(T)$ has distinct eigenvalues, and hence the isotypic components of π_g are at most one-dimensional.

Combining the above, we obtain Proposition 3.1 of the paper.

Proposition (Proposition 3.1 of Paper I)**.** Let ρ be a congruence type of $\mathrm{SL}_2(\mathbb{Z})$ of level $N \in \mathbb{Z}_{\geq 1}$ and let N_0 be chosen as in Theorem 3.1. Then it holds that

$$\mathrm{H}^0(\rho_N^\vee \otimes \rho) \cong (\mathrm{Res}_{\Gamma_1(N)} \rho)[\mathbf{1}], \tag{4.44}$$

and

$$\mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho) \cong \bigoplus_{\substack{[g]\in\Gamma_1(N_0)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_1(N_0) \\ 0\le m_0<N_0 \\ 0\le m<N \\ m_0\equiv_{N_0} -\frac{mN_0}{N}}} \pi_g[e(\frac{\cdot m_0}{N_0})] \otimes \mathrm{Res}_{\Gamma_1(N_0)}\Big((\mathrm{Res}_{\Gamma_1(N)}\rho)[e(\frac{\cdot m}{N})]\Big) \tag{4.45}$$

Sketch of proof. By Frobenius reciprocity, we have that

$$\mathrm{H}^0(\rho_N^\vee \otimes \rho) \cong \mathrm{Hom}(\rho_N, \rho) \cong \mathrm{H}^0_{\Gamma_1(N)}(\mathrm{Res}_{\Gamma_1(N)}\rho), \tag{4.46}$$

and

$$\mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho) \cong \mathrm{H}^0_{\Gamma_1(N)}(\mathrm{Res}_{\Gamma_1(N_0)}(\rho_{N_0}^\vee) \otimes \mathrm{Res}_{\Gamma_1(N_0)}(\rho)). \tag{4.47}$$

However, $\mathrm{Res}_{\Gamma_1(N_0)}\rho = \mathrm{Res}_{\Gamma_1(N_0)}\mathrm{Res}_{\Gamma_1(N)}\rho$, and since ρ_M, $M \in \mathbb{Z}_{\ge 1}$, is self-dual, we also have that

$$\mathrm{Res}_{\Gamma_1(N_0)}\rho_{N_0}^\vee \cong \mathrm{Res}_{\Gamma_1(N_0)}\rho_{N_0} = \mathrm{Res}_{\Gamma_1(N_0)}\rho_{N_0}\mathrm{Ind}_{\Gamma_1(N_0)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1}. \tag{4.48}$$

The result now follows by using Mackey's formula on (4.48) and noting that for arithmetic types ρ_1, ρ_2, for $\Gamma_1(M)$ and $M \in \mathbb{Z}_{\ge 1}$, we have that

$$\mathrm{H}^0_{\Gamma_1(M)}\Big(\rho_1[e(\frac{\cdot m_1}{M})] \otimes \rho_2[e(\frac{\cdot m_2}{M})]\Big) = \{0\}, \tag{4.49}$$

unless $m_1 + m_2 \equiv_N 0$. □

The above considerations now leads to Proposition 3.3, which presents a bound for (4.38) on the form we desire.

Proposition (Proposition 3.3 of Paper I)**.** Let ρ be a congruence type of level $N \in \mathbb{Z}_{\ge 1}$ and let N_0 be chosen as in Theorem 3.1. Then it holds that

$$h^0(\rho_N^\vee \otimes \rho) \ll \dim(\rho), \tag{4.50}$$

and

$$h^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho) \ll |\Gamma_1(N_0)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_1(N_0)| \dim(\rho) \ll_\epsilon N_0^{1+\epsilon} \dim(\rho), \tag{4.51}$$

where $\epsilon \in \mathbb{R}_{>0}$ is arbitrary.

Sketch of proof. Since the isotypic components of π_g are at most one-dimensional, the dimension of $\mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho)$ reduces to a sum over $\Gamma_1(N_0)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_1(N_0)$ and $0 \leq m < N$ of the isotypic components of $\mathrm{Res}_{\Gamma_1(N)}$, with some terms possibly equal to zero. This is less than $|\Gamma_1(N_0)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_1(N_0)| \dim(\rho)$.

The last bound follows by comparing the double coset to $\Gamma_1(N_0)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$ which is in bijection with the cusps of $\mathbb{H}^*/\Gamma_1(N_0)$. □

Remark 4.2. It is possible to compute a basis for $\mathrm{H}^0(\rho_N^\vee \otimes \rho)$ and $\mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho)$ by using Proposition 3.1 instead of the orbit-stabilizer decomposition. In our implementation, we opted not to do so and instead rely on the aforementioned computer algebra system GAP.

4.7 T-ORBIT DECOMPOSITION

Before we give a summary of the time-complexity, we want to mention a technique that we use in our implementation to drastically reduce the number of columns of the matrix M on lines 3 and 8 of Algorithm 2.

Let ρ be an arithmetic type for $\mathrm{SL}_2(\mathbb{Z})$, satisfying that $\rho(T)$ is diagonalizable. Then T acts on $\mathrm{FE}(\mathbb{C}) \otimes V(\rho)$ by

$$T(v \otimes \sum_{n\in\mathbb{Q}} c_n q^n) = \rho(T^{-1})v \otimes \sum_{n\in\mathbb{Q}} e(n)c_n q^n. \tag{4.52}$$

If $f \in \mathrm{M}_k(\rho)$, where $k \in \mathbb{Z}$, then $\mathrm{fe}(f)$ is invariant with respect to this action, since

$$\mathrm{fe}(f) = \mathrm{fe}(f|_{k,\rho} T) = \mathrm{fe}(\rho(T^{-1}) f|_k T) = T\,\mathrm{fe}(T). \tag{4.53}$$

Now, if ρ is a twisted permutation type of order n, and R is a set of representatives for $\rho_\pi^\iota(T^{\mathbb{Z}})\backslash\{1,\dots,n\}$, then we have *deflation* and *inflation* maps, given by

$$\begin{aligned} \texttt{defl} : \mathrm{FE}(\mathbb{C}) \otimes \mathbb{C}^n \otimes V(\sigma) &\longrightarrow \mathrm{FE}(\mathbb{C}) \otimes \mathbb{C}^{|R|} \otimes V(\sigma) \\ \sum_{i=1}^{n} f_i &\longmapsto \sum_{i\in R} f_i, \end{aligned} \tag{4.54}$$

and

$$\begin{aligned} \texttt{infl} : \mathrm{FE}(\mathbb{C}) \otimes \mathbb{C}^{|R|} \otimes V(\sigma) &\longrightarrow \mathrm{FE}(\mathbb{C}) \otimes \mathbb{C}^n \otimes V(\sigma) \\ \sum_{i\in R} f_i &\longmapsto \sum_{i\in R} \sum_{0\leq h<|\rho_\pi^\iota(T^{\mathbb{Z}})\{i\}|} T^h f_i. \end{aligned} \tag{4.55}$$

where $f_i \in \mathrm{FE}(\mathbb{C}) \otimes \mathbb{C}e_i \otimes V(\sigma)$. The T-invariance property (4.53) implies that `defl` is injective on $\mathrm{fe}(\mathrm{M}_k(\rho))$, with inverse `infl`.

The consequence of this is that the numbers of columns required in the matrix M in Algorithm 2 can be reduced from $PN\dim(\rho)$ to $P\dim(\rho)$, by working with $\texttt{defl} \circ \mathrm{fe}(\mathrm{M}_k(\rho))$ instead of $\mathrm{fe}(\mathrm{M}_k(\rho))$.

4.8 COMPLEXITY, EXAMPLES, AND COMPARISON

Our last, and arguably most important, contribution in the paper is a comparison between Algorithm 2 (assuming that we compute $\mathrm{H}^0(\rho_N^\vee \otimes \rho)$ and $\mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho)$ as described above) with existing algorithms, in terms of time complexity.

In Section 4 of the paper, we provide a heuristic analysis of the time complexity of our method by analyzing the time complexity of its constituent parts. To recapitulate, these are:

(i) the computation of the invariant spaces $\mathrm{H}^0(\rho_N^\vee \otimes \rho)$ and $\mathrm{H}^0(\rho_{N_0}^\vee \otimes \rho_{N_0}^\vee \otimes \rho)$, as described in Section 4.5,

(ii) the computation of the T-orbits, as described in Section 4.7,

(iii) the computation of the Fourier series expansions up to the Sturm bound, and

(iv) the row reduction of the matrix M in Algorithm 2.

In Table 4.1, we summarize the time-complexities for these steps in the case of $\mathrm{M}_k(\rho)$ where $k \in \mathbb{Z}_{\geq 2}$ and ρ is a congruence type of level N for a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma_1(N) \subseteq \Gamma$, that is also a twisted permutation type of order n and twist dimension d.

(i)	$\mathcal{O}_\epsilon(N_0^{3+\epsilon}d^2 + d^\kappa + n^2 N_0^{4+\epsilon})$ with the orbit-stabilizer algorithm if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then $\mathcal{O}_\epsilon(N_0^{3+\epsilon}(nd)^2)$ with Proposition 3.1
(ii)	$\mathcal{O}(\dim(\rho)(d^{\kappa-1} + n))$
(iii)	$\mathcal{O}_\epsilon((kN_0^2)^{1+\epsilon}\dim(\rho))$
(iv)	$\mathcal{O}_\epsilon(\max\{N_0^{1+\epsilon}\dim(\rho), k\lvert\mathrm{SL}_2(\mathbb{Z})/\Gamma\rvert\dim(\rho)\}^\kappa)$

Table 4.1: Time-complexities for the different steps in our method – here κ denotes the exponent of n in the time-complexity for multiplication of $n \times n$ matrices. In our setting we have $\kappa \approx 2.807$, though for extremely (indeed, prohibitively) large values of n there exists algorithms with $\kappa \lessapprox 2.373$. We refer to Section 4 of the paper for an extensive discussion.

In our analysis we assume that fundamental arithmetic operations have cost $\mathcal{O}(1)$. This assumption is true over $\mathbb{Q}$ and fixed precision arithmetic in $\mathbb{C}$, but not over $\mathbb{Q}^{\mathrm{ab}}$ or arbitrary precision arithmetic in $\mathbb{C}$. We provide a discussion on what consequences this has in Section 4.3 of the paper.

In Section 4.5 of the paper, we provide example code for how to use our package in `Julia`, and in particular we provide a concrete example of how it can be used to compute bases for

$$\mathrm{M}_6(\mathrm{Ind}_{\Gamma_{\mathrm{ns}}(7)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1}) \quad \text{and} \quad \mathrm{M}_4(\rho),$$

where $\Gamma_{\mathrm{ns}}(N)$, $N \in \mathbb{Z}_{\geq 1}$ is the non-split Cartan subgroup of level N, and where ρ is a Moonshine-like arithmetic type that we specify further on page 24 of the paper. These examples showcase the generality of our algorithm. Indeed, existing methods to compute these bases are ad-hoc in nature, whereas we can compute them merely as special cases of spaces of vector-valued modular of congruence type.

4.8.1 *Comparison*

Finally, in Section 5 of the paper, we provide a comparison with other available algorithms. At the time of writing, there were no other algorithms available that target vector-valued modular forms for congruence types as a whole. Instead, the existing algorithms target the special cases of:

(i) scalar-valued modular forms for a congruence subgroup of level $N \in \mathbb{Z}_{\geq 1}$ and a character, that is the spaces $\mathrm{M}_k(\Gamma, \nu)$ where $k \in \mathbb{Z}_{\geq 2}$, $\nu \in D(N)$, and Γ is a congruence subgroup of level N,

(ii) vector-valued modular forms of type ρ_χ where $\chi \in D(N)$ and $N \in \mathbb{Z}_{\geq 1}$, and

(iii) vector-valued modular forms of Weil type.

For (i), there are two well-established methods: modular symbols and the Eichler-Selberg trace formula. (See [37] for an extensive summary with a view towards time-complexity.) By default, they both provide bases for $\mathrm{M}_k(\Gamma, \nu)$ in terms of Fourier series expansions at the cusp $i\infty$, truncated at a precision $P \in \mathbb{Q}_{>0}$ where $P \gg N$ is at least the Sturm bound. The time-complexity is

$$\begin{aligned} &\mathcal{O}_\epsilon(N^{1+\epsilon}P^2), \text{ using modular symbols, and} \\ &\mathcal{O}_\epsilon(N^{\frac{3}{2}+\epsilon}P^{\frac{3}{2}}), \text{ using the trace formula.} \end{aligned} \tag{4.56}$$

Unfortunately, our method is at best on par with this. If $N_0 \approx N$ (which experimental data suggests) we can achieve a total time-complexity of $\mathcal{O}_\epsilon(N^{3+\epsilon})$. We remark that this assumes that we do not employ the isomorphism $\mathrm{M}_k(\Gamma, \nu) \cong \mathrm{M}_k(\mathrm{Ind}_\Gamma^{\mathrm{SL}_2(\mathbb{Z})}\nu)$. This would yield Fourier series expansions at all cusp classes, but with higher time-complexity.

For (ii), Cohen and Belabas [33] developed as part of an algorithm to compute Petersson scalar-products of modular forms in $\mathrm{M}_k(\Gamma_0(N), \chi)$, a method that can be used to compute a basis of $\mathrm{M}_k(\rho_\chi)$ in terms of cusp expansions of elements in $\mathrm{M}_k(\Gamma_0(N), \chi)$. In the paper, we show that Cohen and Belabas' method has the same general form as ours, but differs in which invariants are computed. Instead of relying on an invariant space on the form

$$\mathrm{H}^0(\mathrm{E}_k(\Gamma(N)) \otimes \rho_\chi) + \mathrm{H}^0((\mathrm{E}_l(\Gamma(N_0)) \cdot \mathrm{E}_{k-l}(\Gamma(N_0))) \otimes \rho_\chi), \tag{4.57}$$

it relies on an invariant space of the form

$$\mathrm{E}_k(\Gamma_0(N),\chi) + \sum_{l=1}^{k-1} \mathrm{H}^0((\mathrm{E}_l(\Gamma_1(N)) \cdot \mathrm{E}_{k-l}(\Gamma_1(N))) \otimes \chi). \tag{4.58}$$

We recall (see Section 2.1.1) that

$$\mathrm{E}_k(\Gamma_1(N)) = \bigoplus_{\chi \in D(N)} \mathrm{E}_k(\Gamma_0(N),\chi),$$

and since $\mathrm{H}^0((\mathrm{E}_l(\Gamma_0(N),\chi_1) \cdot \mathrm{E}_{k-l}(\Gamma_0(N),\chi_2)) \otimes \chi)$ is equal to zero unless $\chi_1\chi_2 = \chi$, the invariant space (4.58) reduces to

$$\mathrm{E}_k(\Gamma_0(N),\chi) + \sum_{\substack{1 \le l < k \\ \chi_1\chi_2 = \chi}} \mathrm{H}^0((\mathrm{E}_l(\Gamma_0(N),\chi_1) \cdot \mathrm{E}_{k-l}(\Gamma_1(N),\chi_2)) \otimes \chi). \tag{4.59}$$

In other words, the computation of the invariants is reduced to factorization in $D(N)$, having time complexity $\mathcal{O}(N)$. As we can see from Table 4.1, Cohen's algorithm is thus significantly faster in the case of $\rho = \rho_\chi$.

Finally for (iii), Brandon Williams [34] obtained a result that can be used to compute vector-valued modular forms of Weil type. A result of Skoruppa [22] states that the space $\mathrm{M}_k(\rho)$ where $k \in \mathbb{Z}_{\ge 2}$ and ρ is a congruence type is contained in some space $\mathrm{M}_k(\sigma)$ where σ is a Weil type. Williams' results can also be reframed in terms of invariants, analogous to Theorem 3.1. However, in contrast to our method, his algorithm avoids the computation of bases of these invariant spaces altogether, and instead generates a basis $\mathrm{M}_k(\rho)$ based on certain canonical elements of the invariants. In this way, the most costly step in our method is avoided, leading to improved performance in the case of ρ equal to a Weil type.

5 PAPERS II AND III

5.1 INTRODUCTION

The goal of papers II and III is to show how Eichler integrals and iterated Eichler-Shimura integrals can be studied using of vector-valued modular forms of extension type, and to show how the corresponding spaces of modular forms of extension type can be effectively computed in terms of (vector-valued) Eisenstein series.

As mentioned before, paper II focuses on the base case of scalar-valued Eichler integrals of depth one. That is, integrals of the form

$$\mathcal{E}_f(\tau) = \int_\tau^{i\infty} f(z)(\tau - z)^{k-2}\mathrm{d}z, \quad \text{where } \tau \in \mathbb{H},\ f \in \mathrm{S}_k, \text{ and } k \in \mathbb{Z}_{\geq 2}. \tag{5.1}$$

As described in Section 3.3.4, we show that in Proposition 2.2 of paper II, that

$$\begin{pmatrix} \mathcal{E}_f \\ (X-\cdot)^{k-2} \otimes \psi_{\mathcal{E}_f} \end{pmatrix} \in \mathrm{M}_{2-k}(\mathbf{1} \boxplus_{\mathrm{pb}} \mathrm{sym}^{k-2}(X)), \tag{5.2}$$

where $\psi_{\mathcal{E}_f} = -\phi_{\mathcal{E}_f}(\cdot; X)^\vee$. In a similar fashion, one also obtains that

$$\begin{pmatrix} I_f \\ 1 \end{pmatrix} \in \mathrm{M}_0(\mathrm{sym}^{k-2}(X) \boxplus_{\phi_{I_f}} \mathbf{1}). \tag{5.3}$$

Paper III instead focuses on the case of depth two, and in particular on the iterated Eichler-Shimura integral $I_{f,g}$ and on the hitherto unexamined *scalar-valued* Eichler-Shimura integral of depth two $\mathcal{E}_{f,g}$, where $f, g \in \mathrm{S}_k$ and $k \in \mathbb{Z}_{\geq 2}$. The latter is defined by

$$\mathcal{E}_{f,g}(\tau) = \int_\tau^{i\infty} f(z)\mathcal{E}_g(z)\mathrm{d}z, \quad \tau \in \mathbb{H}. \tag{5.4}$$

We want to emphasize that the construction of $\mathcal{E}_{f,g}$ is intrinsic to depth two. Indeed, its existence stems from the pairing

$$\begin{aligned} \mathrm{sym}^{k-2}(X) \otimes \mathrm{sym}^{k-2}(Y) &\longrightarrow \mathbb{C} \\ p \otimes q &\longmapsto \sum_{i=0}^{k-2} (-1)^i \binom{k-2}{i}^{-1} p_i q_{k-2-i}, \end{aligned} \tag{5.5}$$

defined in Section 3.3.4. This pairing can be applied to the "top" component of the extension type $\widetilde{\rho}_{f,g}$ which we recall is given by

$$\widetilde{\rho}_{f,g}(\gamma) = \begin{pmatrix} \mathrm{sym}^{k-2}(X,Y)(\gamma) & (\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X))(\gamma) & \psi_{f,g}(\gamma) \\ 0 & \mathrm{sym}^{k-2}(X)(\gamma) & \phi_{I_f}(\gamma) \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.6}$$

where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, see also Remark 3.5.

In Theorem 2.4 of paper III, we show by direct computation that this leads to an arithmetic type $\rho_{f,g}$ of depth two (in the sense of [38, Section 1.6]), defined by

$$\begin{aligned} \rho_{f,g} : \mathrm{SL}_2(\mathbb{Z}) &\longrightarrow \mathrm{GL}(\mathbb{C} \oplus \mathbb{C}[X]_{k-2} \oplus \mathbb{C}) \\ \gamma &\longmapsto \begin{pmatrix} 1 & -\phi_{I_g}^{\vee}(\gamma) & \psi_{f,g}(\gamma) \\ 0 & \mathrm{sym}^{k-2}(X)(\gamma) & \phi_{I_f}(\gamma) \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{5.7}$$

where $f, g \in \mathrm{S}_k$, $k \in \mathbb{Z}_{\geq 2}$, and

$$\psi_{f,g}(\gamma) = \int_{\gamma(i\infty)}^{i\infty} f(z)\mathcal{E}_g(z)\mathrm{d}z \tag{5.8}$$

for $h \in \mathrm{S}_k$, and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. In particular, Theorem 2.4 of paper III states that

$$\begin{pmatrix} \mathcal{E}_{f,g} \\ I_f \\ 1 \end{pmatrix} \in \mathrm{M}_0(\rho_{f,g}). \tag{5.9}$$

As we described in Section 3.3.4, we also show that

$$\begin{pmatrix} I_{f,g} \\ I_f \\ 1 \end{pmatrix} \in \mathrm{M}_0(\rho_{f,g}). \tag{5.10}$$

Furthermore, as we alluded to earlier, the relation between $\widetilde{\rho}_{f,g}$ and $\rho_{f,g}$, provides an alternative explanation of an identity in Eichler cohomology that is originally due to Paşol and Popa [26]. As we shall see later, the extension type $\rho_{f,g}$ belongs to a family of extension types ρ_{ϕ_1,ϕ_2} parameterized by pairs of cocycles $\phi_1, \phi_2 \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))$, $\mathrm{d} \in 2\mathbb{Z}_{\geq 0}$ satisfying a certain orthogonality criterion.

We shall now summarize the remaining parts of paper II and III, namely the the representation theory that underlies the cohomological interpretation of the relation between $\widetilde{\rho}_{f,g}$ and $\rho_{f,g}$, and the computational aspects (including the definition of the relevant Eisenstein series).

5.2 ON THE TYPES $\widetilde{\rho}_{f,g}$ AND $\rho_{f,g}$

In this section, we provide a brief outline of the results obtained in Section 2 of paper III.

To begin, we recall that we have the following antisymmetric bilinear form

$$\begin{aligned} \langle\!\langle \cdot, \cdot \rangle\!\rangle : \mathbb{C}[X]_{\mathsf{d}}^2 &\longrightarrow \mathbb{C} \\ \langle\!\langle p, q \rangle\!\rangle &= \langle T^{-1}.p - T.p, q \rangle \end{aligned} \tag{5.11}$$

For a cusp form $f \in \mathrm{S}_k$, $k \in \mathbb{Z}_{\geq 2}$, we also let $r_f = \phi_{I_f}(S)$. This is usually referred to as the *period polynomial* associated to f, and is used to define the celebrated Eichler-Shimura isomorphism; which provides a vector-space isomorphism between $\mathrm{S}_k \oplus \overline{\mathrm{S}}_k$ and $\mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{k-2}(X))$. We refer Section 1.4 of paper III for a brief recapitulation of the basic theory.

In [26], Paşol and Popa showed that

$$\langle\!\langle r_f, r_g \rangle\!\rangle = 0, \quad \text{where} \quad f, g \in \mathrm{S}_k,\, k \in \mathbb{Z}_{\geq 2}. \tag{5.12}$$

This is complementary to Haberland's formula [5], which states that

$$\langle\!\langle r_f, \overline{r_g} \rangle\!\rangle = -6(2i)^{k-1}(f, g), \quad \text{where} \quad f, g \in \mathrm{S}_k,\, k \in \mathbb{Z}_{\geq 2}, \tag{5.13}$$

and where (f, g) denotes the Petersson inner product [14, Chapter 5.4].

Let now $\mathsf{d} \in 2\mathbb{Z}_{\geq 0}$ be an even integer. For cocycles $\phi_1, \phi_2 \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$ and a function $\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$, we let $\rho_{\phi_1,\phi_2,\psi} : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C} \oplus \mathbb{C}[X]_{\mathsf{d}} \oplus \mathbb{C})$ be defined by

$$\rho_{\phi_1,\phi_2,\psi}(\gamma) = \begin{pmatrix} 1 & \phi_2^\vee(\gamma) & \psi(\gamma) \\ 0 & \mathrm{sym}^{\mathsf{d}}(X)(\gamma) & \phi_1(\gamma) \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.14}$$

Proposition 2.1 of paper III tells us when $\rho_{\phi_1,\phi_2,\psi}$ is a representation.

Proposition (Proposition 2.1 of paper III)**.** Let $\mathsf{d} \in 2\mathbb{Z}_{\geq 0}$ and let $\phi_1, \phi_2 \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$ be cocycles. Let also $\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ be a function. Then the following are equivalent

(i) $\rho_{\phi_1,\phi_2,\psi}$ is a representation,

(ii) for all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ it holds that $\psi(\gamma_1\gamma_2) = \psi(\gamma_1) + \psi(\gamma_2) + \phi_2^\vee(\gamma_1)\phi_1(\gamma_2)$,

(iii) $(\phi_2^\vee, \psi) \in \mathrm{Z}^1_{\mathrm{pb}}\big(\mathrm{sym}^{\mathsf{d}}(X) \boxplus_{\phi_1} \mathbf{1}, \mathbf{1}\big)$,

(iv) $(\psi, \phi_1)^T \in \mathrm{Z}^1_{\mathrm{pb}}\big(\mathbf{1}, \mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X)\big)$.

Furthermore, if any of these conditions hold, then

$$\rho_{\phi_1,\phi_2,\psi} = \mathbf{1} \boxplus_{(\phi_2^\vee,\psi)} \left(\mathrm{sym}^{\mathsf{d}}(X) \boxplus_{\phi_1} \mathbf{1}\right) = \left(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X)\right) \boxplus_{\binom{\psi}{\phi_1}} \mathbf{1}.$$

Furthermore, Theorem 2.2 of paper III gives a necessary and sufficient condition for these criteria to hold.

Theorem (Theorem 2.2 of paper III)**.** Let $\mathsf{d} \in 2\mathbb{Z}_{\geq 0}$ and let $\phi_1, \phi_2 \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$ be cocycles. Then there exists a function $\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ making $\rho_{\phi_1,\phi_2,\psi}$ a representation if and only if

$$\langle\!\langle \phi_1(S), \phi_2(S) \rangle\!\rangle = 0. \tag{5.15}$$

If such a function exists, it is unique, and is given by

$$\psi(S) = -\frac{1}{2}\phi_2^\vee(S)\phi_1(S). \tag{5.16}$$

As we stated before, we show in Theorem 2.4 of paper III that $\rho_{f,g}$ is a representation. However, we also see that

$$\rho_{f,g} = \rho_{\phi_{I_f},-\phi_{I_g},\psi_{f,g}}, \tag{5.17}$$

showing that $\langle\!\langle \phi_{I_f}(S), \phi_{I_g}(S) \rangle\!\rangle = 0$, by Theorem 2.2. This is precisely the identity obtained by Paşol and Popa.

Furthermore, we refer to pairs of cocycles $(\phi_1, \phi_2) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$ satisfying $\langle\!\langle \phi_1(S), \phi_2(S) \rangle\!\rangle = 0$ as admissible.

5.3 EISENSTEIN SERIES AND SATURATION

In paper II, we work with three types of Eisenstein series, namely:

- vector-valued Eisenstein series of type $\mathrm{sym}^{\mathsf{d}}(X)$,
- generalized second order Eisenstein series of type $(\mathrm{sym}^{\mathsf{d}}(X), \mathbf{1})$, and
- generalized second order Eisenstein series of type $(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$.

The latter two correspond to the "top" component of vector-valued Eisenstein series of type

$$\mathrm{sym}^{\mathsf{d}}(X) \boxplus_{\mathrm{pb}} \mathbf{1} \quad \text{and} \quad \mathbf{1} \boxplus_{\mathrm{pb}} \mathrm{sym}^{\mathsf{d}}(X), \tag{5.18}$$

respectively. In paper III, we work with the above Eisenstein series, but also with vector-valued Eisenstein series of type ρ_{ϕ_1,ϕ_2} where $(\phi_1, \phi_2) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))^2$ is a pair of admissible cocycles.

We will provide their definitions here, but we refer to paper II and III for proof of their convergence and computation of their Fourier series expansions. In particular, see Proposition 3.4 and Theorem 3.8 of paper II, and Theorem 3.1 of paper III.

Let $\mathsf{d} \in 2\mathbb{Z}_{\geq 0}$, $k > 2 + \mathsf{d}$, and $0 \leq j \leq \mathsf{d}$ be integers, with k and d even. Then we let the vector-valued Eisenstein series of type $\mathrm{sym}^{\mathsf{d}}(X)$ be defined by

$$E_k(\tau; \mathsf{d}, j) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (X - \tau)^j \big|_{k, \mathrm{sym}^{\mathsf{d}}(X)} \gamma, \quad \tau \in \mathbb{H}. \tag{5.19}$$

As for generalized second order Eisenstein series, let $\mathsf{d} \in 2\mathbb{Z}_{\geq 0}$, $k > 2 + \mathsf{d}$, and $0 \leq j \leq \mathsf{d}$ be integers, with k and d even. If $\phi \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$ is a cocycle, we let the weight k generalized second order Eisenstein series of type $(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$ associated to $(\phi^\vee, (X - \tau)^j)$, denoted by $E_k^{[1]}(\cdot; \phi^\vee, j)$ be given by

$$\begin{pmatrix} E_k^{[1]}(\tau; \phi^\vee, j) \\ E_k(\tau; \mathsf{d}, j) \end{pmatrix} = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \left[\begin{pmatrix} 0 \\ (X - \cdot)^j \end{pmatrix} \big|_{k, \mathbf{1} \boxplus_{\phi^\vee} \mathrm{sym}^{\mathsf{d}}(X)} \gamma \right] (\tau), \quad \tau \in \mathbb{H}. \tag{5.20}$$

Similarly, we let the weight k generalized second order Eisenstein series of type $(\mathrm{sym}^{\mathsf{d}}(X), \mathbf{1})$ associated to ϕ, denoted by $E_k^{[1]}(\cdot; \phi)$, be given by

$$\begin{pmatrix} E_k^{[1]}(\tau; \phi) \\ E_k \end{pmatrix} = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \big|_{k, \mathrm{sym}^{\mathsf{d}}(X) \boxplus_\phi \mathbf{1}} \gamma \right] (\tau), \quad \tau \in \mathbb{H}. \tag{5.21}$$

We also define the jth Eisenstein series of type $\mathbf{1} \boxplus_{\phi^\vee} \mathrm{sym}^{\mathsf{d}}(X)$ and the Eisenstein series of type $\mathrm{sym}^{\mathsf{d}}(X) \boxplus_\phi \mathbf{1}$ by

$$\begin{aligned} E_k(\tau; \mathbf{1} \boxplus_{\phi^\vee} \mathrm{sym}^{\mathsf{d}}(X), j) &= \begin{pmatrix} E_k^{[1]}(\tau; \phi, j) \\ E_k(\tau; \mathsf{d}, j) \end{pmatrix}, \text{ and} \\ E_k(\tau; \mathrm{sym}^{\mathsf{d}}(X) \boxplus_\phi \mathbf{1}) &= \begin{pmatrix} E_k^{[1]}(\tau; \phi) \\ E_k \end{pmatrix}, \quad \tau \in \mathbb{H}. \end{aligned} \tag{5.22}$$

Finally, for a pair of admissible cocycles $(\phi_1, \phi_2) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))^2$ we let the weight k Eisenstein series of type ρ_{ϕ_1, ϕ_2}, denoted by $E_k(\cdot; \phi_1, \phi_2)$, be given by

$$E_k(\tau; \phi_1, \phi_2) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \big|_{k, \rho_{\phi_1, \phi_2}} \gamma \right] (\tau), \quad \tau \in \mathbb{H}. \tag{5.23}$$

Paralleling the terminology used in paper II for the generalized second order Eisenstein series, we define in paper III the weight k generalized third order Eisenstein series of type (ϕ_1, ϕ_2), denoted by $E_k^{[2]}(\cdot; \phi_1, \phi_2)$, as follows

$$\begin{pmatrix} E_k^{[2]}(\tau; \phi_1, \phi_2) \\ E_k^{[1]}(\tau; \phi_1) \\ E_k \end{pmatrix} = E_k(\tau; \phi_1, \phi_2), \quad \tau \in \mathbb{H}. \tag{5.24}$$

Since we have convergence and moderate growth at cusps, it is clear from the definitions that

$$\begin{aligned} E_k(\cdot; \mathbf{1} \boxplus_{\phi^\vee} \operatorname{sym}^{\mathsf{d}}(X), j) &\in \mathrm{M}_k(\mathbf{1} \boxplus_{\phi^\vee} \operatorname{sym}^{\mathsf{d}}(X)), \\ E_k(\cdot; \operatorname{sym}^{\mathsf{d}}(X) \boxplus_{\phi} \mathbf{1}) &\in \mathrm{M}_k(\operatorname{sym}^{\mathsf{d}}(X) \boxplus_{\phi} \mathbf{1}), \text{ and} \\ E_k(\cdot; \phi_1, \phi_2) &\in \mathrm{M}_k(\rho_{\phi_1, \phi_2}). \end{aligned} \tag{5.25}$$

To see how we can compute modular forms in $\mathrm{M}_k(\mathbf{1} \boxplus_{\phi^\vee} \operatorname{sym}^{\mathsf{d}}(X))$, $\mathrm{M}_k(\operatorname{sym}^{\mathsf{d}}(X) \boxplus_{\phi} \mathbf{1})$, and $\mathrm{M}_k(\rho_{\phi_1, \phi_2})$; in terms of these Eisenstein series, we first need to recall the definition of saturated ideals. If R is a ring, M is an R-module, $I \subseteq M$ is a submodule, and $f \in M$, then the saturation of I at f is the R-module

$$(I : f^\infty) = \{g \in M : \exists n \in \mathbb{Z}_{\geq 0}. f^n g \in I\}. \tag{5.26}$$

Let now $\mathrm{M}_\bullet = \bigoplus_{k \in \mathbb{Z}} \mathrm{M}_k$ be the graded ring of modular forms. Then for an arithmetic type ρ, we have that

$$\mathrm{M}_\bullet(\rho) = \bigoplus_{k \in \mathbb{Z}} \mathrm{M}_k(\rho), \tag{5.27}$$

is an $\mathrm{M}_\bullet$-module. Let furthermore $\mathsf{d} \in 2\mathbb{Z}_{\geq 0}$ and $k_0 > 2 + \mathsf{d}$ be integers, and let $(\phi_1, \phi_2) \in Z^1_{\mathrm{pb}}(\mathbf{1}, \operatorname{sym}^{\mathsf{d}}(X))^2$ be admissible. Then we have the following submodules of Eisenstein series of $\mathrm{M}_\bullet(\mathbf{1} \boxplus_{\phi_1^\vee} \operatorname{sym}^{\mathsf{d}}(X))$, $\mathrm{M}_\bullet(\operatorname{sym}^{\mathsf{d}}(X) \boxplus_{\phi_1} \mathbf{1})$, and $\mathrm{M}_\bullet(\rho_{\phi_1, \phi_2})$:

$$\mathrm{E}_{\geq k_0}(\mathbf{1} \boxplus_{\phi_1^\vee} \operatorname{sym}^{\mathsf{d}}(X)) = \operatorname{span}_{\mathrm{M}_\bullet}\{E_k(\cdot; \mathbf{1} \boxplus_{\phi_1^\vee} \operatorname{sym}^{\mathsf{d}}(X), j) : k \geq k_0, 0 \leq j \leq \mathsf{d}\} \tag{5.28}$$

$$\mathrm{E}_{\geq k_0}(\operatorname{sym}^{\mathsf{d}}(X) \boxplus_{\phi_1} \mathbf{1}) = \operatorname{span}_{\mathrm{M}_\bullet}\{E_k(\cdot; \operatorname{sym}^{\mathsf{d}}(X) \boxplus_{\phi_1} \mathbf{1}) : k \geq k_0\} \tag{5.29}$$

$$\mathrm{E}_{\geq k_0}(\rho_{\phi_1, \phi_2}) = \operatorname{span}_{\mathrm{M}_\bullet}\{E_k(\cdot; \phi_1, \phi_2) : k \geq k_0\}. \tag{5.30}$$

We can now state Theorem 4.3 of paper II in terms of extension types.

Theorem (Theorem 4.3 of paper II)**.** Let $\mathsf{d} \in 2\mathbb{Z}_{\geq 0}$, $k_0 > 2 + \mathsf{d}$ be integers, and let $\phi \in Z^1_{\mathrm{pb}}(\mathbf{1}, \operatorname{sym}^{\mathsf{d}}(X))$. Let also ι be given by $\iota(f) = (f, 0)^T$. Then

$$\begin{aligned} (\mathrm{E}_{\geq k_0}(\mathbf{1} \boxplus_{\phi^\vee} \operatorname{sym}^{\mathsf{d}}(X)) + \iota(\mathrm{M}_\bullet) : \Delta^\infty) &= \mathrm{M}_\bullet(\mathbf{1} \boxplus_{\phi^\vee} \operatorname{sym}^{\mathsf{d}}(X)), \text{ and} \\ (\mathrm{E}_{\geq k_0}(\operatorname{sym}^{\mathsf{d}}(X) \boxplus_{\phi} \mathbf{1}) + \iota(\mathrm{M}_\bullet) : \Delta^\infty) &= \mathrm{M}_\bullet(\operatorname{sym}^{\mathsf{d}}(X) \boxplus_{\phi} \mathbf{1}). \end{aligned} \tag{5.31}$$

Stated more plainly, Theorem 4.3 implies that any modular form of type ρ is, after multiplication with a suitable power of Δ, equal to a sum of scalar-valued Eisenstein series and products of scalar-valued Eisenstein series with vector-valued Eisenstein series of type ρ, where $\rho = \mathbf{1} \boxplus_{\phi_1^\vee} \mathrm{sym}^{\mathsf{d}}(X)$ or $\rho = \mathrm{sym}^{\mathsf{d}}(X) \boxplus_{\phi_1} \mathbf{1}$.

This allows for numerical evaluation of vector-valued modular forms of type $\mathbf{1} \boxplus_{\mathrm{pb}} \mathrm{sym}^{\mathsf{d}}(X)$ and $\mathrm{sym}^{\mathsf{d}}(X) \boxplus_{\mathrm{pb}} \mathbf{1}$ to high precision, given that one can identify the contribution from $\iota(\mathrm{M}_\bullet)$. We provide a concrete example of this in Section 5 of paper II, applied to the special case of the Eichler integral $\mathcal{E}_\Delta \in \mathrm{M}_{-10}(\mathbf{1} \boxplus_{\mathrm{pb}} \mathrm{sym}^{10}(X))$.

In paper III, we show that Theorem 4.3 generalizes to the case of ρ_{ϕ_1,ϕ_2}, and thus of scalar-valued Eichler-Shimura integrals of depth two. Namely, we provide the following theorem.

Theorem (Theorem 3.1 of paper III)**.** Let $\mathsf{d} \in 2\mathbb{Z}_{\geq 0}$ and $k_0 > 2 + \mathsf{d}$ be integers, and let $(\phi_1, \phi_2) \in Z^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))^2$ be admissible. Then we have

$$\mathrm{M}_\bullet(\rho_{\phi_1,\phi_2}) = \big(\mathrm{E}_{\geq k_0}(\rho_{\phi_1,\phi_2}) + \iota_1\big(\mathrm{E}_{\geq k_0}(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X))\big) + \iota_2(\mathrm{M}_\bullet) \colon \Delta^\infty\big), \quad (5.32)$$

where $\iota_1(f,g) = (f,g,0)^T$ and $\iota_2(f) = (f,0,0)^T$.

We hope that papers II and III convinces the reader of the utility of extension types. Furthermore, it is our belief that the methods we have developed can with further research be generalized to handle higher depth cases, and that Theorem 4.3 of paper II and Theorem 3.1 of paper III can be developed as to fit within a general computational framework for modular forms of extension types.

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