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CRITICAL BRANCHING AS A PURE DEATH PROCESS COMING DOWN FROM INFINITY

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Abstract

We consider the critical Galton–Watson process with overlapping generations stemming from a single founder. Assuming that both the variance of the offspring number and the average generation length are finite, we establish the convergence of the finite-dimensional distributions, conditioned on non-extinction at a remote time of observation. The limiting process is identified as a pure death process coming down from infinity. This result brings a new perspective on Vatutin’s dichotomy, claiming that in the critical regime of age-dependent reproduction, an extant population either contains a large number of short-living individuals or consists of few long-living individuals.

Keywords: Galton–Watson process with overlapping generations; Bellman–Harris process; Sevastyanov process; Crump–Mode–Jagers process; convergence of finite-dimensional distributions; Vatutin’s dichotomy

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1. Introduction

Consider a self-replicating system evolving in the discrete-time setting according to the following rules:

Rule 1: The system is founded by a single individual, the founder, born at time 0.

Rule 2: The founder dies at a random age L and gives a random number N of births at random ages τ_j satisfying $1 \leq \tau_1 \leq \dots \leq \tau_N \leq L$.

Rule 3: Each new individual lives independently from others according to the same life law as the founder.

An individual that was born at time t_1 and dies at time t_2 is considered to be alive during the time interval $[t_1, t_2 - 1]$. Letting $Z(t)$ stand for the number of individuals alive at time t , we study the random dynamics of the sequence

$$Z(0) = 1, Z(1), Z(2), \dots,$$

which is a natural extension of the well-known Galton–Watson process, or *GW process* for short; see [13]. The process $Z(\cdot)$ is the discrete-time version of what is usually called the

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Crump–Mode–Jagers process or the general branching process; see [5]. To emphasise the discrete-time setting, we call it a GW process with overlapping generations, or *GWO process* for short.

Put $b := \frac{1}{2}\text{var}(N)$. This paper deals with the GWO processes satisfying

$$E(N) = 1, \quad 0 < b < \infty. \quad (1)$$

The condition $E(N) = 1$ says that the reproduction regime is critical, implying $E(Z(t)) \equiv 1$ and making extinction inevitable, provided $b > 0$. According to [1, Chapter I.9], given (1), the survival probability

$$Q(t) := P(Z(t) > 0)$$

of a GW process satisfies the asymptotic formula $tQ(t) \rightarrow b^{-1}$ as $t \rightarrow \infty$ (this was first proven in [6] under a third moment assumption). A direct extension of this classical result for the GWO processes,

$$tQ(ta) \rightarrow b^{-1}, \quad t \rightarrow \infty, \quad a := E(\tau_1 + \dots + \tau_N),$$

was obtained in [3, 4] under the conditions (1), $a < \infty$,

$$t^2P(L > t) \rightarrow 0, \quad t \rightarrow \infty, \quad (2)$$

plus an additional condition. (Notice that by our definition, $a \geq 1$, and $a = 1$ if and only if $L \equiv 1$, that is, when the GWO process in question is a GW process.) Treating a as the *mean generation length* (see [5, 8]), we may conclude that the asymptotic behaviour of the critical GWO process with *short-living individuals* (see the condition (2)) is similar to that of the critical GW process, provided time is counted generation-wise.

New asymptotic patterns for the critical GWO processes are found under the assumption

$$t^2P(L > t) \rightarrow d, \quad 0 \leq d < \infty, \quad t \rightarrow \infty, \quad (3)$$

which, compared to (2), allows the existence of *long-living individuals* given $d > 0$. The condition (3) was first introduced in the pioneering paper [12] dealing with the *Bellman–Harris processes*. In the current discrete-time setting, the Bellman–Harris process is a GWO process subject to two restrictions: (a) $P(\tau_1 = \dots = \tau_N = L) = 1$, so that all births occur at the moment of an individual's death, and (b) the random variables L and N are independent. For the Bellman–Harris process, the conditions (1) and (3) imply $a = E(L)$, $a < \infty$, and according to [12, Theorem 3], we get

$$tQ(t) \rightarrow h, \quad t \rightarrow \infty, \quad h := \frac{a + \sqrt{a^2 + 4bd}}{2b}. \quad (4)$$

As was shown in [11, Corollary B] (see also [7, Lemma 3.2] for an adaptation to the discrete-time setting), the relation (4) holds even for the GWO processes satisfying the conditions (1), (3), and $a < \infty$.

The main result of this paper, Theorem 1 of Section 2, considers a critical GWO process under the above-mentioned set of assumptions (1), (3), $a < \infty$, and establishes the convergence of the finite-dimensional distributions conditioned on survival at a remote time of observation. A remarkable feature of this result is that its limit process is fully described by a single parameter $c := 4bda^{-2}$, regardless of complicated mutual dependencies between the random variables τ_j, N, L .

Our proof of Theorem 1, requiring an intricate asymptotic analysis of multi-dimensional probability generating functions, is split into two sections for the sake of readability. Section 3 presents a new proof of (4) inspired by the proof of [12]. The crucial aspect of this approach, compared to the proof of [7, Lemma 3.2], is that certain essential steps do not rely on the monotonicity of the function $Q(t)$. In Section 4, the technique of Section 3 is further developed to finish the proof of Theorem 1.

We conclude this section by mentioning the illuminating family of GWO processes called the *Sevastyanov processes* [9]. The Sevastyanov process is a generalised version of the Bellman–Harris process, with possibly dependent L and N . In the critical case, the mean generation length of the Sevastyanov process, $a = E(LN)$, can be represented as

$$a = \text{cov}(L, N) + E(L).$$

Thus, if L and N are positively correlated, the average generation length a exceeds the average life length $E(L)$.

Turning to a specific example of the Sevastyanov process, take

$$P(L = t) = p_1 t^{-3} (\ln \ln t)^{-1}, \quad P(N = 0|L = t) = 1 - p_2, \quad P(N = n_t|L = t) = p_2, \quad t \geq 2,$$

where $n_t := \lfloor t(\ln t)^{-1} \rfloor$ and (p_1, p_2) are such that

$$\sum_{t=2}^{\infty} P(L = t) = p_1 \sum_{t=2}^{\infty} t^{-3} (\ln \ln t)^{-1} = 1, \quad E(N) = p_1 p_2 \sum_{t=2}^{\infty} n_t t^{-3} (\ln \ln t)^{-1} = 1.$$

In this case, for some positive constant c_1 ,

$$E(N^2) = p_1 p_2 \sum_{t=1}^{\infty} n_t^2 t^{-3} (\ln \ln t)^{-1} < c_1 \int_2^{\infty} \frac{d(\ln t)}{(\ln t)^2 \ln \ln t} < \infty,$$

implying that the condition (1) is satisfied. Clearly, the condition (3) holds with $d = 0$. At the same time,

$$a = E(NL) = p_1 p_2 \sum_{t=1}^{\infty} n_t t^{-2} (\ln \ln t)^{-1} > c_2 \int_2^{\infty} \frac{d(\ln t)}{(\ln t)(\ln \ln t)} = \infty,$$

where c_2 is a positive constant. This example demonstrates that for the GWO process, unlike for the Bellman–Harris process, the conditions (1) and (3) do not automatically imply the condition $a < \infty$.

2. The main result

Theorem 1. *For a GWO process satisfying (1), (3) and $a < \infty$, there holds a weak convergence of the finite-dimensional distributions*

$$(Z(ty), 0 < y < \infty | Z(t) > 0) \xrightarrow{\text{fdd}} (\eta(y), 0 < y < \infty), \quad t \rightarrow \infty.$$

The limiting process is a continuous-time pure death process $(\eta(y), 0 \leq y < \infty)$, whose evolution law is determined by a single compound parameter $c = 4bda^{-2}$, as specified next.

The finite-dimensional distributions of the limiting process $\eta(\cdot)$ are given below in terms of the k -dimensional probability generating functions $E(z_1^{\eta(y_1)} \dots z_k^{\eta(y_k)})$, $k \geq 1$, assuming

$$0 = y_0 < y_1 < \dots < y_j < 1 \leq y_{j+1} < \dots < y_k < y_{k+1} = \infty, \\ 0 \leq j \leq k, \quad 0 \leq z_1, \dots, z_k < 1. \quad (5)$$

Here the index j highlights the pivotal value 1 corresponding to the time of observation t of the underlying GWO process.

As will be shown in Section 4.2, if $j = 0$, then

$$E(z_1^{\eta(y_1)} \dots z_k^{\eta(y_k)}) = 1 - \frac{1 + \sqrt{1 + \sum_{i=1}^k z_1 \dots z_{i-1} (1 - z_i) \Gamma_i}}{(1 + \sqrt{1 + c}) y_1}, \quad \Gamma_i := c(y_1/y_i)^2,$$

and if $j \geq 1$,

$$E(z_1^{\eta(y_1)} \dots z_k^{\eta(y_k)}) \\ = \frac{\sqrt{1 + \sum_{i=1}^j z_1 \dots z_{i-1} (1 - z_i) \Gamma_i + cz_1 \dots z_j y_1^2} - \sqrt{1 + \sum_{i=1}^k z_1 \dots z_{i-1} (1 - z_i) \Gamma_i}}{(1 + \sqrt{1 + c}) y_1}.$$

In particular, for $k = 1$, we have

$$E(z^{\eta(y)}) = \frac{\sqrt{1 + c(1 - z) + cz y^2} - \sqrt{1 + c(1 - z)}}{(1 + \sqrt{1 + c}) y}, \quad 0 < y < 1, \\ E(z^{\eta(y)}) = 1 - \frac{1 + \sqrt{1 + c(1 - z)}}{(1 + \sqrt{1 + c}) y}, \quad y \geq 1.$$

It follows that $P(\eta(y) \geq 0) = 1$ for $y > 0$, and moreover, putting here first $z = 1$ and then $z = 0$ yields

$$P(\eta(y) < \infty) = \frac{\sqrt{1 + cy^2} - 1}{(1 + \sqrt{1 + c}) y} \cdot 1_{\{0 < y < 1\}} + \left(1 - \frac{2}{(1 + \sqrt{1 + c}) y}\right) \cdot 1_{\{y \geq 1\}}, \\ P(\eta(y) = 0) = \frac{y - 1}{y} \cdot 1_{\{y \geq 1\}},$$

implying that $P(\eta(y) = \infty) > 0$ for all $y > 0$. In fact, letting $y \rightarrow 0$, we may set $P(\eta(0) = \infty) = 1$.

To demonstrate that the process $\eta(\cdot)$ is indeed a pure death process, consider the function

$$E(z_1^{\eta(y_1) - \eta(y_2)} \dots z_{k-1}^{\eta(y_{k-1}) - \eta(y_k)} z_k^{\eta(y_k)})$$

determined by

$$E(z_1^{\eta(y_1) - \eta(y_2)} \dots z_{k-1}^{\eta(y_{k-1}) - \eta(y_k)} z_k^{\eta(y_k)}) = E(z_1^{\eta(y_1)} (z_2/z_1)^{\eta(y_2)} \dots (z_k/z_{k-1})^{\eta(y_k)}).$$

This function is given by two expressions:

$$\frac{(1 + \sqrt{1+c})y_1 - 1 - \sqrt{1 + \sum_{i=1}^k (1 - z_i)\gamma_i}}{(1 + \sqrt{1+c})y_1}, \quad \text{for } j = 0,$$

$$\frac{\sqrt{1 + \sum_{i=1}^{j-1} (1 - z_i)\gamma_i + (1 - z_j)\Gamma_j + cz_j y_1^2} - \sqrt{1 + \sum_{i=1}^k (1 - z_i)\gamma_i}}{(1 + \sqrt{1+c})y_1}, \quad \text{for } j \geq 1,$$

where $\gamma_i := \Gamma_i - \Gamma_{i+1}$ and $\Gamma_{k+1} = 0$. Setting $k = 2$, $z_1 = z$, and $z_2 = 1$, we deduce that the function

$$E(z^{\eta(y_1) - \eta(y_2)}; \eta(y_1) < \infty), \quad 0 < y_1 < y_2, \quad 0 \leq z \leq 1, \tag{6}$$

is given by one of the following three expressions, depending on whether $j = 2$, $j = 1$, or $j = 0$:

$$\frac{\sqrt{1 + cy_1^2 + c(1-z)(1 - (y_1/y_2)^2)} - \sqrt{1 + c(1-z)(1 - (y_1/y_2)^2)}}{(1 + \sqrt{1+c})y_1}, \quad y_2 < 1,$$

$$\frac{\sqrt{1 + cy_1^2 + c(1-z)(1 - y_1^2)} - \sqrt{1 + c(1-z)(1 - (y_1/y_2)^2)}}{(1 + \sqrt{1+c})y_1}, \quad y_1 < 1 \leq y_2,$$

$$1 - \frac{1 + \sqrt{1 + c(1-z)(1 - (y_1/y_2)^2)}}{(1 + \sqrt{1+c})y_1}, \quad 1 \leq y_1.$$

Since the generating function (6) is finite at $z = 0$, we conclude that

$$P(\eta(y_1) < \eta(y_2); \eta(y_1) < \infty) = 0, \quad 0 < y_1 < y_2.$$

This implies

$$P(\eta(y_2) \leq \eta(y_1)) = 1, \quad 0 < y_1 < y_2,$$

meaning that unless the process $\eta(\cdot)$ is sitting at the infinity state, it evolves by negative integer-valued jumps until it gets absorbed at zero.

Consider now the conditional probability generating function

$$E(z^{\eta(y_1) - \eta(y_2)} | \eta(y_1) < \infty), \quad 0 < y_1 < y_2, \quad 0 \leq z \leq 1. \tag{7}$$

In accordance with the three expressions given above for (6), the generating function (7) is specified by the following three expressions:

$$\frac{\sqrt{1 + cy_1^2 + c(1-z)(1 - (y_1/y_2)^2)} - \sqrt{1 + c(1-z)(1 - (y_1/y_2)^2)}}{\sqrt{1 + cy_1^2} - 1}, \quad y_2 < 1,$$

$$\frac{\sqrt{1 + cy_1^2 + c(1-z)(1 - y_1^2)} - \sqrt{1 + c(1-z)(1 - (y_1/y_2)^2)}}{\sqrt{1 + cy_1^2} - 1}, \quad y_1 < 1 \leq y_2,$$

$$1 - \frac{\sqrt{1 + c(1-z)(1 - (y_1/y_2)^2)} - 1}{(1 + \sqrt{1+c})y_1 - 2}, \quad 1 \leq y_1.$$

In particular, setting $z = 0$ here, we obtain

$$P(\eta(y_1) - \eta(y_2) = 0 | \eta(y_1) < \infty) = \begin{cases} \frac{\sqrt{1+c(1+y_1^2-(y_1/y_2)^2)} - \sqrt{1+c(1-(y_1/y_2)^2)}}{\sqrt{1+cy_1^2}-1} & \text{for } 0 < y_1 < y_2 < 1, \\ \frac{\sqrt{1+c} - \sqrt{1+c(1-(y_1/y_2)^2)}}{\sqrt{1+cy_1^2}-1} & \text{for } 0 < y_1 < 1 \leq y_2, \\ 1 - \frac{\sqrt{1+c(1-(y_1/y_2)^2)} - 1}{(1+\sqrt{1+c})y_1-2} & \text{for } 1 \leq y_1 < y_2. \end{cases}$$

Notice that given $0 < y_1 \leq 1$,

$$P(\eta(y_1) - \eta(y_2) = 0 | \eta(y_1) < \infty) \rightarrow 0, \quad y_2 \rightarrow \infty,$$

which is expected because of $\eta(y_1) \geq \eta(1) \geq 1$ and $\eta(y_2) \rightarrow 0$ as $y_2 \rightarrow \infty$.

The random times

$$T = \sup\{u : \eta(u) = \infty\}, \quad T_0 = \inf\{u : \eta(u) = 0\}$$

are major characteristics of a trajectory of the limit pure death process. Since

$$P(T \leq y) = E(z^{\eta(y)}) \Big|_{z=1}, \quad P(T_0 \leq y) = E(z^{\eta(y)}) \Big|_{z=0},$$

in accordance with the above-mentioned formulas for $E(z^{\eta(y)})$, we get the following marginal distributions:

$$P(T \leq y) = \frac{\sqrt{1+cy^2}-1}{(1+\sqrt{1+c})y} \cdot 1_{\{0 \leq y < 1\}} + \left(1 - \frac{2}{(1+\sqrt{1+c})y}\right) \cdot 1_{\{y \geq 1\}},$$

$$P(T_0 \leq y) = \frac{y-1}{y} \cdot 1_{\{y \geq 1\}}.$$

The distribution of T_0 is free from the parameter c and has the Pareto probability density function

$$f_0(y) = y^{-2} 1_{\{y > 1\}}.$$

In the special case (2), that is, when (3) holds with $d = 0$, we have $c = 0$ and $P(T = T_0) = 1$. If $d > 0$, then $T \leq T_0$, and the distribution of T has the following probability density function:

$$f(y) = \begin{cases} \frac{1}{(1+\sqrt{1+c})y^2} \left(1 - \frac{1}{\sqrt{1+cy^2}}\right) & \text{for } 0 \leq y < 1, \\ \frac{2}{(1+\sqrt{1+c})y^2} & \text{for } y \geq 1, \end{cases}$$

which has a positive jump at $y = 1$ of size $f(1) - f(1-) = (1+c)^{-1/2}$; see Figure 1. Observe that $\frac{f(1-)}{f(1)} \rightarrow \frac{1}{2}$ as $c \rightarrow \infty$.

Intuitively, the limiting pure death process counts the long-living individuals in the GWO process, that is, those individuals whose life length is of order t . These long-living individuals may have descendants, however none of them would live long enough to be detected by the

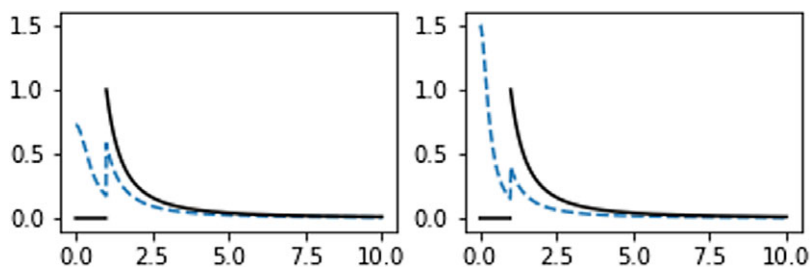


FIGURE 1. The dashed line is the probability density function of T ; the solid line is the probability density function of T_0 . The left panel illustrates the case $c = 5$, and the right panel illustrates the case $c = 15$.

finite-dimensional distributions at the relevant time scale, see Lemma 2 below. Theorem 1 suggests a new perspective on Vatutin’s dichotomy (see [12]), claiming that the long-term survival of a critical age-dependent branching process is due to either a large number of short-living individuals or a small number of long-living individuals. In terms of the random times $T \leq T_0$, Vatutin’s dichotomy discriminates between two possibilities: if $T > 1$, then $\eta(1) = \infty$, meaning that the GWO process has survived thanks to a large number of individuals, while if $T \leq 1 < T_0$, then $1 \leq \eta(1) < \infty$, meaning that the GWO process has survived thanks to a small number of individuals.

3. Proof that $tQ(t) \rightarrow h$

This section deals with the survival probability of the critical GWO process

$$Q(t) = 1 - P(t), \quad P(t) := P(Z(t) = 0).$$

By its definition, the GWO process can be represented as the sum

$$Z(t) = 1_{\{L>t\}} + \sum_{j=1}^N Z_j(t - \tau_j), \quad t = 0, 1, \dots, \tag{8}$$

involving N independent daughter processes $Z_j(\cdot)$ generated by the founder individual at the birth times $\tau_j, j = 1, \dots, N$ (here it is assumed that $Z_j(t) = 0$ for all negative t). The branching property (8) implies the relation

$$1_{\{Z(t)=0\}} = 1_{\{L \leq t\}} \prod_{j=1}^N 1_{\{Z_j(t-\tau_j)=0\}},$$

which says that the GWO process goes extinct by the time t if, on one hand, the founder is dead at time t and, on the other hand, all daughter processes are extinct by the time t . After taking expectations of both sides, we can write

$$P(t) = E\left(\prod_{j=1}^N P(t - \tau_j); L \leq t\right). \tag{9}$$

As shown next, this nonlinear equation for $P(\cdot)$ implies the asymptotic formula (4) under the conditions (1), (3), and $a < \infty$.

3.1. Outline of the proof of (4)

We start by stating four lemmas and two propositions. Let

$$\Phi(z) := E((1 - z)^N - 1 + Nz), \tag{10}$$

$$W(t) := (1 - ht^{-1})^N + Nht^{-1} - \sum_{j=1}^N Q(t - \tau_j) - \prod_{j=1}^N P(t - \tau_j), \tag{11}$$

$$D(u, t) := E\left(1 - \prod_{j=1}^N P(t - \tau_j); u < L \leq t\right) + E\left((1 - ht^{-1})^N - 1 + Nht^{-1}; L > u\right), \tag{12}$$

$$E_u(X) := E(X; L \leq u), \tag{13}$$

where $0 \leq z \leq 1, u > 0, t \geq h$, and X is an arbitrary random variable.

Lemma 1. *Given (10), (11), (12), and (13), assume that $0 < u \leq t$ and $t \geq h$. Then*

$$\Phi(ht^{-1}) = P(L > t) + E_u\left(\sum_{j=1}^N Q(t - \tau_j)\right) - Q(t) + E_u(W(t)) + D(u, t).$$

Lemma 2. *If (1) and (3) hold, then $E(N; L > ty) = o(t^{-1})$ as $t \rightarrow \infty$ for any fixed $y > 0$.*

Lemma 3. *If (1), (3), and $a < \infty$ hold, then for any fixed $0 < y < 1$,*

$$E_{ty}\left(\sum_{j=1}^N \left(\frac{1}{t - \tau_j} - \frac{1}{t}\right)\right) \sim at^{-2}, \quad t \rightarrow \infty.$$

Lemma 4. *Let $k \geq 1$. If $0 \leq f_j, g_j \leq 1$ for $j = 1, \dots, k$, then*

$$\prod_{j=1}^k (1 - g_j) - \prod_{j=1}^k (1 - f_j) = \sum_{j=1}^k (f_j - g_j)r_j,$$

where $0 \leq r_j \leq 1$ and

$$1 - r_j = \sum_{i=1}^{j-1} g_i + \sum_{i=j+1}^k f_i - R_j,$$

for some $R_j \geq 0$. If moreover $f_j \leq q$ and $g_j \leq q$ for some $q > 0$, then

$$1 - r_j \leq (k - 1)q, \quad R_j \leq kq, \quad R_j \leq k^2q^2.$$

Proposition 1. *If (1), (3), and $a < \infty$ hold, then $\limsup_{t \rightarrow \infty} tQ(t) < \infty$.*

Proposition 2. *If (1), (3), and $a < \infty$ hold, then $\liminf_{t \rightarrow \infty} tQ(t) > 0$.*

According to these two propositions, there exists a triplet of positive numbers (q_1, q_2, t_0) such that

$$q_1 \leq tQ(t) \leq q_2, \quad t \geq t_0, \quad 0 < q_1 < h < q_2 < \infty. \tag{14}$$

The claim $tQ(t) \rightarrow h$ is derived using (14) by accurately removing asymptotically negligible terms from the relation for $Q(\cdot)$ stated in Lemma 1, after setting $u = ty$ with a fixed $0 < y < 1$, and then choosing a sufficiently small y . In particular, as an intermediate step, we will show that

$$Q(t) = E_{ty}\left(\sum_{j=1}^N Q(t - \tau_j)\right) + E_{ty}(W(t)) - aht^{-2} + o(t^{-2}), \quad t \rightarrow \infty. \tag{15}$$

Then, restating our goal as $\phi(t) \rightarrow 0$ in terms of the function $\phi(t)$, defined by

$$Q(t) = \frac{h + \phi(t)}{t}, \quad t \geq 1, \tag{16}$$

we rewrite (15) as

$$\frac{h + \phi(t)}{t} = E_{ty} \left(\sum_{j=1}^N \frac{h + \phi(t - \tau_j)}{t - \tau_j} \right) + E_{ty}(W(t)) - aht^{-2} + o(t^{-2}), \quad t \rightarrow \infty. \tag{17}$$

It turns out that the three terms involving h , outside $W(t)$, effectively cancel each other, yielding

$$\frac{\phi(t)}{t} = E_{ty} \left(\sum_{j=1}^N \frac{\phi(t - \tau_j)}{t - \tau_j} + W(t) \right) + o(t^{-2}), \quad t \rightarrow \infty. \tag{18}$$

Treating $W(t)$ in terms of Lemma 4 yields

$$\phi(t) = E_{ty} \left(\sum_{j=1}^N \phi(t - \tau_j) r_j(t) \frac{t}{t - \tau_j} \right) + o(t^{-1}), \tag{19}$$

where $r_j(t)$ is a counterpart of r_j in Lemma 4. To derive from here the desired convergence $\phi(t) \rightarrow 0$, we will adapt a clever trick from Chapter 9.1 of [10], which was further developed in [12] for the Bellman–Harris process, with possibly infinite $\text{var}(N)$. Define a non-negative function $m(t)$ by

$$m(t) := |\phi(t)| \ln t, \quad t \geq 2. \tag{20}$$

Multiplying (19) by $\ln t$ and using the triangle inequality, we obtain

$$m(t) \leq E_{ty} \left(\sum_{j=1}^N m(t - \tau_j) r_j(t) \frac{t \ln t}{(t - \tau_j) \ln(t - \tau_j)} \right) + v(t), \tag{21}$$

where $v(t) \geq 0$ and $v(t) = o(t^{-1} \ln t)$ as $t \rightarrow \infty$. It will be shown that this leads to $m(t) = o(\ln t)$, thereby concluding the proof of (4).

3.2. Proof of lemmas and propositions

Proof of Lemma 1. For $0 < u \leq t$, the relations (9) and (13) give

$$P(t) = E_u \left(\prod_{j=1}^N P(t - \tau_j) \right) + E \left(\prod_{j=1}^N P(t - \tau_j); u < L \leq t \right). \tag{22}$$

On the other hand, for $t \geq h$,

$$\Phi(ht^{-1}) \stackrel{(10)}{=} E_u \left((1 - ht^{-1})^N - 1 + Nht^{-1} \right) + E \left((1 - ht^{-1})^N - 1 + Nht^{-1}; L > u \right).$$

Adding the latter relation to

$$1 = P(L \leq u) + P(L > t) + P(u < L \leq t)$$

and subtracting (22) from the sum, we get

$$\Phi(ht^{-1}) + Q(t) = E_u\left((1 - ht^{-1})^N + Nht^{-1} - \prod_{j=1}^N P(t - \tau_j) \right) + P(L > t) + D(u, t),$$

with $D(u, t)$ defined by (12). After a rearrangement, we obtain the statement of the lemma. \square

Proof of Lemma 2. For any fixed $\epsilon > 0$,

$$\begin{aligned} E(N; L > t) &= E(N; N \leq t\epsilon, L > t) + E(N; 1 < N(t\epsilon)^{-1}, L > t) \\ &\leq t\epsilon P(L > t) + (t\epsilon)^{-1} E(N^2; L > t). \end{aligned}$$

Thus, by (1) and (3),

$$\limsup_{t \rightarrow \infty} (tE(N; L > t)) \leq d\epsilon,$$

and the assertion follows as $\epsilon \rightarrow 0$. \square

Proof of Lemma 3. For $t = 1, 2, \dots$ and $y > 0$, put

$$B_t(y) := t^2 E_{ty} \left(\sum_{j=1}^N \left(\frac{1}{t - \tau_j} - \frac{1}{t} \right) \right) - a.$$

For any $0 < u < ty$, using

$$a = E_u(\tau_1 + \dots + \tau_N) + A_u, \quad A_u := E(\tau_1 + \dots + \tau_N; L > u),$$

we get

$$\begin{aligned} B_t(y) &= E_u \left(\sum_{j=1}^N \frac{t}{t - \tau_j} \tau_j \right) + E \left(\sum_{j=1}^N \frac{t}{t - \tau_j} \tau_j; u < L \leq ty \right) \\ &\quad - E_u(\tau_1 + \dots + \tau_N) - A_u \\ &= E \left(\sum_{j=1}^N \frac{\tau_j}{1 - \tau_j/t}; u < L \leq ty \right) + E_u \left(\sum_{j=1}^N \frac{\tau_j^2}{t - \tau_j} \right) - A_u. \end{aligned}$$

For the first term on the right-hand side, we have $\tau_j \leq L \leq ty$, so that

$$E \left(\sum_{j=1}^N \frac{\tau_j}{1 - \tau_j/t}; u < L \leq ty \right) \leq (1 - y)^{-1} A_u.$$

For the second term, $\tau_j \leq L \leq u$ and therefore

$$E_u \left(\sum_{j=1}^N \frac{\tau_j^2}{t - \tau_j} \right) \leq \frac{u^2}{t - u} E_u(N) \leq \frac{u^2}{t - u}.$$

This yields

$$-A_u \leq B_t(y) \leq (1 - y)^{-1} A_u + \frac{u^2}{t - u}, \quad 0 < u < ty < t,$$

implying

$$-A_u \leq \liminf_{t \rightarrow \infty} B_t(y) \leq \limsup_{t \rightarrow \infty} B_t(y) \leq (1-y)^{-1}A_u.$$

Since $A_u \rightarrow 0$ as $u \rightarrow \infty$, we conclude that $B_t(y) \rightarrow 0$ as $t \rightarrow \infty$. □

Proof of Lemma 4. Let

$$r_j := (1 - g_1) \dots (1 - g_{j-1}) (1 - f_{j+1}) \dots (1 - f_k), \quad 1 \leq j \leq k.$$

Then $0 \leq r_j \leq 1$, and the first stated equality is obtained by telescopic summation of

$$\begin{aligned} (1 - g_1) \prod_{j=2}^k (1 - f_j) - \prod_{j=1}^k (1 - f_j) &= (f_1 - g_1)r_1, \\ (1 - g_1)(1 - g_2) \prod_{j=3}^k (1 - f_j) - (1 - g_1) \prod_{j=2}^k (1 - f_j) &= (f_2 - g_2)r_2, \dots, \\ \prod_{j=1}^k (1 - g_j) - \prod_{j=1}^{k-1} (1 - g_j)(1 - f_k) &= (f_k - g_k)r_k. \end{aligned}$$

The second stated equality is obtained with

$$\begin{aligned} R_j &:= \sum_{i=j+1}^k f_i (1 - (1 - f_{j+1}) \dots (1 - f_{i-1})) \\ &\quad + \sum_{i=1}^{j-1} g_i (1 - (1 - g_1) \dots (1 - g_{i-1}) (1 - f_{j+1}) \dots (1 - f_k)), \end{aligned}$$

by performing telescopic summation of

$$\begin{aligned} 1 - (1 - f_{j+1}) &= f_{j+1}, \\ (1 - f_{j+1}) - (1 - f_{j+1})(1 - f_{j+2}) &= f_{j+2} (1 - f_{j+1}), \dots, \\ \prod_{i=j+1}^{k-1} (1 - f_i) - \prod_{i=j+1}^k (1 - f_i) &= f_k \prod_{i=j+1}^{k-1} (1 - f_i), \\ \prod_{i=j+1}^k (1 - f_i) - (1 - g_1) \prod_{i=j+1}^k (1 - f_i) &= g_1 \prod_{i=j+1}^k (1 - f_i), \dots, \\ \prod_{i=1}^{j-2} (1 - g_i) \prod_{i=j+1}^k (1 - f_i) - \prod_{i=1}^{j-1} (1 - g_i) \prod_{i=j+1}^k (1 - f_i) &= g_{j-1} \prod_{i=1}^{j-2} (1 - g_i) \prod_{i=j+1}^k (1 - f_i). \end{aligned}$$

By the above definition of R_j , we have $R_j \geq 0$. Furthermore, given $f_j \leq q$ and $g_j \leq q$, we get

$$R_j \leq \sum_{i=1}^{j-1} g_i + \sum_{i=j+1}^k f_i \leq (k-1)q.$$

It remains to observe that

$$1 - r_j \leq 1 - (1 - q)^{k-1} \leq (k-1)q,$$

and from the definition of R_j ,

$$R_j \leq q \sum_{i=1}^{k-j-1} (1 - (1 - q)^i) + q \sum_{i=1}^{j-1} (1 - (1 - q)^{k-j+i-1}) \leq q^2 \sum_{i=1}^{k-2} i \leq k^2 q^2.$$

□

Proof of Proposition 1. By the definition of $\Phi(\cdot)$, we have

$$\Phi(Q(t)) + P(t) = E_u(P(t)^N) + P(L > u) - E(1 - P(t)^N; L > u),$$

for any $0 < u < t$. This and (22) yield

$$\begin{aligned} \Phi(Q(t)) = E_u \left(P(t)^N - \prod_{j=1}^N P(t - \tau_j) \right) + P(L > u) \\ - E(1 - P(t)^N; L > u) - E \left(\prod_{j=1}^N P(t - \tau_j); u < L \leq t \right). \end{aligned} \tag{23}$$

We therefore obtain the upper bound

$$\Phi(Q(t)) \leq E_u \left(P(t)^N - \prod_{j=1}^N P(t - \tau_j) \right) + P(L > u),$$

which together with Lemma 4 and the monotonicity of $Q(\cdot)$ implies

$$\Phi(Q(t)) \leq E_u \left(\sum_{j=1}^N (Q(t - \tau_j) - Q(t)) \right) + P(L > u). \tag{24}$$

Borrowing an idea from [11], suppose to the contrary that

$$t_n := \min\{t: tQ(t) \geq n\}$$

is finite for any natural n . It follows that

$$Q(t_n) \geq \frac{n}{t_n}, \quad Q(t_n - u) < \frac{n}{t_n - u}, \quad 1 \leq u \leq t_n - 1.$$

Putting $t = t_n$ into (24) and using the monotonicity of $\Phi(\cdot)$, we find

$$\Phi(nt_n^{-1}) \leq \Phi(Q(t_n)) \leq E_u \left(\sum_{j=1}^N \left(\frac{n}{t_n - \tau_j} - \frac{n}{t_n} \right) \right) + P(L > u).$$

Setting $u = t_n/2$ here and applying Lemma 3 together with (3), we arrive at the relation

$$\Phi(nt_n^{-1}) = O(nt_n^{-2}), \quad n \rightarrow \infty.$$

Observe that under the condition (1), the L'Hospital rule gives

$$\Phi(z) \sim bz^2, \quad z \rightarrow 0. \tag{25}$$

The resulting contradiction, $n^2 t_n^{-2} = O(nt_n^{-2})$ as $n \rightarrow \infty$, finishes the proof of the proposition. \square

Proof of Proposition 2. The relation (23) implies

$$\Phi(Q(t)) \geq E_u \left(P(t)^N - \prod_{j=1}^N P(t - \tau_j) \right) - E(1 - P(t)^N; L > u).$$

By Lemma 4,

$$P(t)^N - \prod_{j=1}^N P(t - \tau_j) = \sum_{j=1}^N (Q(t - \tau_j) - Q(t))t_j^*(t),$$

where $0 \leq r_j^*(t) \leq 1$ is a counterpart of the term r_j in Lemma 4. By the monotonicity of $P(\cdot)$, we have, again referring to Lemma 4,

$$1 - r_j^*(t) \leq (N - 1)Q(t - L).$$

Thus, for $0 < y < 1$,

$$\Phi(Q(t)) \geq E_{ty} \left(\sum_{j=1}^N (Q(t - \tau_j) - Q(t))r_j^*(t) \right) - E(1 - P(t)^N; L > ty). \tag{26}$$

The assertion $\liminf_{t \rightarrow \infty} tQ(t) > 0$ is proven by contradiction. Assume that $\liminf_{t \rightarrow \infty} tQ(t) = 0$, so that

$$t_n := \min \{t: tQ(t) \leq n^{-1}\}$$

is finite for any natural n . Plugging $t = t_n$ into (26) and using

$$Q(t_n) \leq \frac{1}{nt_n}, \quad Q(t_n - u) - Q(t_n) \geq \frac{1}{n(t_n - u)} - \frac{1}{nt_n}, \quad 1 \leq u \leq t_n - 1,$$

we get

$$\Phi\left(\frac{1}{nt_n}\right) \geq n^{-1} E_{t_n y} \left(\sum_{j=1}^N \left(\frac{1}{t_n - \tau_j} - \frac{1}{t_n} \right) r_j^*(t_n) \right) - \frac{1}{nt_n} E(N; L > t_n y).$$

Given $L \leq ty$, we have

$$1 - r_j^*(t) \leq NQ(t(1 - y)) \leq N \frac{q_2}{t(1 - y)},$$

where the second inequality is based on the already proven part of (14). Therefore,

$$E_{t_n y} \left(\sum_{j=1}^N \left(\frac{1}{t_n - \tau_j} - \frac{1}{t_n} \right) (1 - r_j^*(t_n)) \right) \leq \frac{q_2 y}{t_n^2 (1 - y)^2} E(N^2),$$

and we derive

$$nt_n^2 \Phi\left(\frac{1}{nt_n}\right) \geq t_n^2 E_{t_n y} \left(\sum_{j=1}^N \left(\frac{1}{t_n - \tau_j} - \frac{1}{t_n} \right) \right) - \frac{E(N^2)q_2 y}{(1 - y)^2} - t_n E(N; L > t_n y).$$

Sending $n \rightarrow \infty$ and applying (25), Lemma 2, and Lemma 3, we arrive at the inequality

$$0 \geq a - yq_2 E(N^2)(1 - y)^{-2}, \quad 0 < y < 1,$$

which is false for sufficiently small y . □

3.3. Proof of (18) and (19)

Fix an arbitrary $0 < y < 1$. Lemma 1 with $u = ty$ gives

$$\Phi(ht^{-1}) = P(L > t) + E_{ty} \left(\sum_{j=1}^N Q(t - \tau_j) \right) - Q(t) + E_{ty}(W(t)) + D(ty, t). \tag{27}$$

Let us show that

$$D(ty, t) = o(t^{-2}), \quad t \rightarrow \infty. \tag{28}$$

Using Lemma 2 and (14), we find that for an arbitrarily small $\epsilon > 0$,

$$E\left(1 - \prod_{j=1}^N P(t - \tau_j) ; ty < L \leq t(1 - \epsilon)\right) = o(t^{-2}), \quad t \rightarrow \infty.$$

On the other hand,

$$E\left(1 - \prod_{j=1}^N P(t - \tau_j) ; t(1 - \epsilon) < L \leq t\right) \leq P(t(1 - \epsilon) < L \leq t),$$

so that in view of (3),

$$E\left(1 - \prod_{j=1}^N P(t - \tau_j) ; ty < L \leq t\right) = o(t^{-2}), \quad t \rightarrow \infty.$$

This, (12), and Lemma 2 imply (28).

Observe that

$$bh^2 = ah + d. \tag{29}$$

Combining (27), (28), and

$$P(L > t) - \Phi(ht^{-1}) \stackrel{(3)(25)}{=} dt^{-2} - bh^2t^{-2} + o(t^{-2}) \stackrel{(29)}{=} -ah t^{-2} + o(t^{-2}), \quad t \rightarrow \infty,$$

we derive (15), which in turn gives (17). The latter implies (18) since by Lemmas 2 and 4,

$$E_{ty}\left(\sum_{j=1}^N \frac{h}{t - \tau_j}\right) - \frac{h}{t} = E_{ty}\left(\sum_{j=1}^N \left(\frac{h}{t - \tau_j} - \frac{h}{t}\right)\right) - ht^{-1}E(N; L > ty) = aht^{-2} + o(t^{-2}).$$

Turning to the proof of (19), observe that the random variable

$$W(t) = (1 - ht^{-1})^N - \prod_{j=1}^N \left(1 - \frac{h + \phi(t - \tau_j)}{t - \tau_j}\right) + \sum_{j=1}^N \left(\frac{h}{t} - \frac{h + \phi(t - \tau_j)}{t - \tau_j}\right)$$

can be represented in terms of Lemma 4 as

$$\begin{aligned} W(t) &= \prod_{j=1}^N (1 - f_j(t)) - \prod_{j=1}^N (1 - g_j(t)) + \sum_{j=1}^N (f_j(t) - g_j(t)) \\ &= \sum_{j=1}^N (1 - r_j(t))(f_j(t) - g_j(t)), \end{aligned}$$

by assigning

$$f_j(t) := ht^{-1}, \quad g_j(t) := \frac{h + \phi(t - \tau_j)}{t - \tau_j}. \tag{30}$$

Here $0 \leq r_j(t) \leq 1$, and for sufficiently large t ,

$$1 - r_j(t) \stackrel{(14)}{\leq} Nq_2t^{-1}. \tag{31}$$

After plugging into (18) the expression

$$W(t) = \sum_{j=1}^N \left(\frac{h}{t} - \frac{h}{t - \tau_j} \right) (1 - r_j(t)) - \sum_{j=1}^N \frac{\phi(t - \tau_j)}{t - \tau_j} (1 - r_j(t)),$$

we get

$$\frac{\phi(t)}{t} = E_{ty} \left(\sum_{j=1}^N \frac{\phi(t - \tau_j)}{t - \tau_j} r_j(t) \right) + E_{ty} \left(\sum_{j=1}^N \left(\frac{h}{t - \tau_j} - \frac{h}{t} \right) (1 - r_j(t)) \right) + o(t^{-2}), \quad t \rightarrow \infty.$$

The latter expectation is non-negative, and for an arbitrary $\epsilon > 0$, it has the following upper bound:

$$E_{ty} \left(\sum_{j=1}^N \left(\frac{h}{t - \tau_j} - \frac{h}{t} \right) (1 - r_j(t)) \right) \stackrel{(31)}{\leq} q_2 \epsilon E_{ty} \left(\sum_{j=1}^N \left(\frac{h}{t - \tau_j} - \frac{h}{t} \right) \right) + \frac{q_2 h}{(1 - y)t^2} E(N^2; N > t\epsilon).$$

Thus, in view of Lemma 3,

$$\frac{\phi(t)}{t} = E_{ty} \left(\sum_{j=1}^N \frac{\phi(t - \tau_j)}{t - \tau_j} r_j(t) \right) + o(t^{-2}), \quad t \rightarrow \infty.$$

Multiplying this relation by t , we arrive at (19).

3.4. Proof of $\phi(t) \rightarrow 0$

Recall (20). If the non-decreasing function

$$M(t) := \max_{1 \leq j \leq t} m(j)$$

is bounded from above, then $\phi(t) = O\left(\frac{1}{\ln t}\right)$, proving that $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$. If $M(t) \rightarrow \infty$ as $t \rightarrow \infty$, then there is an integer-valued sequence $0 < t_1 < t_2 < \dots$, such that the sequence $M_n := M(t_n)$ is strictly increasing and converges to infinity. In this case,

$$m(t) \leq M_{n-1} < M_n, \quad 1 \leq t < t_n, \quad m(t_n) = M_n, \quad n \geq 1. \tag{32}$$

Since $|\phi(t)| \leq \frac{M_n}{\ln t_n}$ for $t_n \leq t < t_{n+1}$, to finish the proof of $\phi(t) \rightarrow 0$, it remains to verify that

$$M_n = o(\ln t_n), \quad n \rightarrow \infty. \tag{33}$$

Fix an arbitrary $y \in (0, 1)$. Putting $t = t_n$ in (21) and using (32), we find

$$M_n \leq M_n E_{tny} \left(\sum_{j=1}^N r_j(t_n) \frac{t_n \ln t_n}{(t_n - \tau_j) \ln(t_n - \tau_j)} \right) + (t_n^{-1} \ln t_n) o_n.$$

Here and elsewhere, o_n stands for a non-negative sequence such that $o_n \rightarrow 0$ as $n \rightarrow \infty$. In different formulas, the sign o_n represents different such sequences. Since

$$0 \leq \frac{t \ln t}{(t - u) \ln(t - u)} - 1 \leq \frac{u(1 + \ln t)}{(t - u) \ln(t - u)}, \quad 0 \leq u < t - 1,$$

and $r_j(t_n) \in [0, 1]$, it follows that

$$M_n - M_n E_{tny} \left(\sum_{j=1}^N r_j(t_n) \right) \leq M_n E_{tny} \left(\sum_{j=1}^N \frac{\tau_j(1 + \ln t_n)}{t_n(1 - y) \ln(t_n(1 - y))} \right) + (t_n^{-1} \ln t_n) o_n.$$

Recalling that $a = E(\sum_{j=1}^N \tau_j)$, observe that

$$E_{t_n y} \left(\sum_{j=1}^N \frac{\tau_j(1 + \ln t_n)}{t_n(1 - y) \ln(t_n(1 - y))} \right) \leq \frac{a(1 + \ln t_n)}{t_n(1 - y) \ln(t_n(1 - y))} = (a(1 - y)^{-1} + o_n) t_n^{-1}.$$

Combining the last two relations, we conclude

$$M_n E_{t_n y} \left(\sum_{j=1}^N (1 - r_j(t_n)) \right) \leq a(1 - y)^{-1} t_n^{-1} M_n + t_n^{-1} (M_n + \ln t_n) o_n. \tag{34}$$

Now it is time to unpack the term $r_j(t)$. By Lemma 4 with (30),

$$1 - r_j(t) = \sum_{i=1}^{j-1} \frac{h + \phi(t - \tau_i)}{t - \tau_i} + (N - j) \frac{h}{t} - R_j(t),$$

where, provided $\tau_j \leq ty$,

$$0 \leq R_j(t) \leq Nq_2 t^{-1} (1 - y)^{-1}, \quad R_j(t) \leq N^2 q_2^2 t^{-2} (1 - y)^{-2}, \quad t > t^*,$$

for a sufficiently large t^* . This allows us to rewrite (34) in the form

$$\begin{aligned} M_n E_{t_n y} \left(\sum_{j=1}^N \left(\sum_{i=1}^{j-1} \frac{h + \phi(t_n - \tau_i)}{t_n - \tau_i} + (N - j) \frac{h}{t_n} \right) \right) \\ \leq M_n E_{t_n y} \left(\sum_{j=1}^N R_j(t_n) \right) + a(1 - y)^{-1} t_n^{-1} M_n + t_n^{-1} (M_n + \ln t_n) o_n. \end{aligned}$$

To estimate the last expectation, observe that if $\tau_j \leq ty$, then for any $\epsilon > 0$,

$$R_j(t) \leq Nq_2 t^{-1} (1 - y)^{-1} 1_{\{N > t\epsilon\}} + N^2 q_2^2 t^{-2} (1 - y)^{-2} 1_{\{N \leq t\epsilon\}}, \quad t > t^*,$$

implying that for sufficiently large n ,

$$E_{t_n y} \left(\sum_{j=1}^N R_j(t_n) \right) \leq q_2 t_n^{-1} (1 - y)^{-1} E(N^2; N > t_n \epsilon) + q_2^2 \epsilon t_n^{-1} (1 - y)^{-2} E(N^2),$$

so that

$$\begin{aligned} M_n E_{t_n y} \left(\sum_{j=1}^N \left(\sum_{i=1}^{j-1} \frac{h + \phi(t_n - \tau_i)}{t_n - \tau_i} + (N - j) \frac{h}{t_n} \right) \right) \\ \leq a(1 - y)^{-1} t_n^{-1} M_n + t_n^{-1} (M_n + \ln t_n) o_n. \end{aligned}$$

Since

$$\sum_{j=1}^N \sum_{i=1}^{j-1} \left(\frac{h}{t_n - \tau_i} - \frac{h}{t_n} \right) \geq 0,$$

we obtain

$$\begin{aligned} M_n E_{t_n y} \left(\sum_{j=1}^N \left(\sum_{i=1}^{j-1} \frac{\phi(t_n - \tau_i)}{t_n - \tau_i} + (N - 1) \frac{h}{t_n} \right) \right) \\ \leq a(1 - y)^{-1} t_n^{-1} M_n + t_n^{-1} (M_n + \ln t_n) o_n. \end{aligned}$$

By (16) and (14), we have $\phi(t) \geq q_1 - h$ for $t \geq t_0$. Thus, for $\tau_j \leq L \leq t_n y$ and sufficiently large n ,

$$\frac{\phi(t_n - \tau_i)}{t_n - \tau_i} \geq \frac{q_1 - h}{t_n(1 - y)}.$$

This gives

$$\sum_{j=1}^N \left(\sum_{i=1}^{j-1} \frac{\phi(t_n - \tau_i)}{t_n - \tau_i} + (N - 1) \frac{h}{t_n} \right) \geq \left(h + \frac{q_1 - h}{2(1 - y)} \right) t_n^{-1} N(N - 1),$$

which, after multiplying by $t_n M_n$ and taking expectations, yields

$$\left(h + \frac{q_1 - h}{2(1 - y)} \right) M_n E_{t_n y}(N(N - 1)) \leq a(1 - y)^{-1} M_n + (M_n + \ln t_n) o_n.$$

Finally, since

$$E_{t_n y}(N(N - 1)) \rightarrow 2b, \quad n \rightarrow \infty,$$

we derive that for any $0 < \epsilon < y < 1$, there is a finite n_ϵ such that for all $n > n_\epsilon$,

$$M_n(2bh(1 - y) + bq_1 - bh - a - \epsilon) \leq \epsilon \ln t_n.$$

By (29), we have $bh \geq a$, and therefore

$$2bh(1 - y) + bq_1 - bh - a - \epsilon \geq bq_1 - 2bhy - y.$$

Thus, choosing $y = y_0$ such that $bq_1 - 2bhy_0 - y_0 = \frac{bq_1}{2}$, we see that

$$\limsup_{n \rightarrow \infty} \frac{M_n}{\ln t_n} \leq \frac{2\epsilon}{bq_1},$$

which implies (33) as $\epsilon \rightarrow 0$, concluding the proof of $\phi(t) \rightarrow 0$.

4. Proof of Theorem 1

We will use the following notational conventions for the k -dimensional probability generating function

$$E\left(z_1^{Z(t_1)} \dots z_k^{Z(t_k)}\right) = \sum_{i_1=0}^{\infty} \dots \sum_{i_k=0}^{\infty} P(Z(t_1) = i_1, \dots, Z(t_k) = i_k) z_1^{i_1} \dots z_k^{i_k},$$

with $0 < t_1 \leq \dots \leq t_k$ and $z_1, \dots, z_k \in [0, 1]$. We define

$$P_k(\vec{t}, \vec{z}) := P_k(t_1, \dots, t_n; z_1, \dots, z_k) := E\left(z_1^{Z(t_1)} \dots z_k^{Z(t_k)}\right)$$

and write, for $t \geq 0$,

$$P_k(t + \vec{t}, \vec{z}) := P_k(t + t_1, \dots, t + t_k; z_1, \dots, z_k).$$

Moreover, for $0 < y_1 < \dots < y_k$, we write

$$P_k(t\vec{y}, \vec{z}) := P_k(ty_1, \dots, ty_k; z_1, \dots, z_k),$$

and assuming $0 < y_1 < \dots < y_k < 1$,

$$P_k^*(t, \bar{y}, \bar{z}) := E\left(z_1^{Z(ty_1)} \dots z_k^{Z(ty_k)}; Z(t) = 0\right) = P_{k+1}(ty_1, \dots, ty_k, t; z_1, \dots, z_k, 0).$$

These conventions will be similarly applied to the functions

$$Q_k(\bar{t}, \bar{z}) := 1 - P_k(\bar{t}, \bar{z}), \quad Q_k^*(t, \bar{y}, \bar{z}) := 1 - P_k^*(t, \bar{y}, \bar{z}). \tag{35}$$

Our special interest is in the function

$$Q_k(t) := Q_k(t + \bar{t}, \bar{z}), \quad 0 = t_1 < \dots < t_k, \quad z_1, \dots, z_k \in [0, 1), \tag{36}$$

to be viewed as a counterpart of the function $Q(t)$ treated by Theorem 2. Recalling the compound parameters

$$h = \frac{a + \sqrt{a^2 + 4bd}}{2b}$$

and $c = 4bda^{-2}$, put

$$h_k := h \frac{1 + \sqrt{1 + c g_k}}{1 + \sqrt{1 + c}}, \quad g_k := g_k(\bar{y}, \bar{z}) := \sum_{i=1}^k z_1 \dots z_{i-1} (1 - z_i) y_i^{-2}. \tag{37}$$

The key step of the proof of Theorem 1 is to show that for any given $1 = y_1 < y_2 < \dots < y_k$,

$$tQ_k(t) \rightarrow h_k, \quad t_i := t(y_i - 1), \quad i = 1, \dots, k, \quad t \rightarrow \infty. \tag{38}$$

This is done following the steps of our proof of $tQ(t) \rightarrow h$ given in Section 3.

Unlike $Q(t)$, the function $Q_k(t)$ is not monotone over t . However, monotonicity of $Q(t)$ was used in the proof of Theorem 2 only for the proof of (14). The corresponding statement

$$0 < q_1 \leq tQ_k(t) \leq q_2 < \infty, \quad t \geq t_0,$$

follows from the bounds $(1 - z_1)Q(t) \leq Q_k(t) \leq Q(t)$, which hold by the monotonicity of the underlying generating functions over z_1, \dots, z_n . Indeed,

$$Q_k(t) \leq Q_k(t, t + t_2, \dots, t + t_k; 0, \dots, 0) = Q(t),$$

and on the other hand,

$$Q_k(t) = Q_k(t, t + t_2, \dots, t + t_k; z_1, \dots, z_k) = E\left(1 - z_1^{Z(t)} z_2^{Z(t+t_2)} \dots z_k^{Z(t+t_k)}\right) \geq E\left(1 - z_1^{Z(t)}\right),$$

where

$$E\left(1 - z_1^{Z(t)}\right) \geq E\left(1 - z_1^{Z(t)}; Z(t) \geq 1\right) \geq (1 - z_1)Q(t).$$

4.1. Proof of $tQ_k(t) \rightarrow h_k$

The branching property (8) of the GWO process gives

$$\prod_{i=1}^k z_i^{Z(t_i)} = \prod_{i=1}^k z_i^{1_{\{L > t_i\}}} \prod_{j=1}^N z_i^{Z_j(t_i - \tau_j)}.$$

Given $0 < t_1 < \dots < t_k < t_{k+1} = \infty$, we use

$$\prod_{i=1}^k z_i^{1_{\{L > t_i\}}} = 1_{\{L \leq t_1\}} + \sum_{i=1}^k z_1 \cdots z_i 1_{\{t_i < L \leq t_{i+1}\}}$$

to deduce the following counterpart of (9):

$$P_k(\bar{t}, \bar{z}) = E_{t_1} \left(\prod_{j=1}^N P_k(\bar{t} - \tau_j, \bar{z}) \right) + \sum_{i=1}^k z_1 \cdots z_i E \left(\prod_{j=1}^N P_k(\bar{t} - \tau_j, \bar{z}); t_i < L \leq t_{i+1} \right).$$

This implies

$$\begin{aligned} P_k(\bar{t}, \bar{z}) &= E_{t_1} \left(\prod_{j=1}^N P_k(\bar{t} - \tau_j, \bar{z}) \right) + \sum_{i=1}^k z_1 \cdots z_i P(t_i < L \leq t_{i+1}) \\ &\quad - \sum_{i=1}^k z_1 \cdots z_i E \left(1 - \prod_{j=1}^N P_k(\bar{t} - \tau_j, \bar{z}); t_i < L \leq t_{i+1} \right). \end{aligned} \tag{39}$$

Using this relation we establish the following counterpart of Lemma 1.

Lemma 5. Consider the function (36) and put $P_k(t) := 1 - Q_k(t) = P_k(t + \bar{t}, \bar{z})$. For $0 < u < t$, the relation

$$\begin{aligned} \Phi(h_k t^{-1}) &= P(L > t) - \sum_{i=1}^k z_1 \cdots z_i P(t + t_i < L \leq t + t_{i+1}) \\ &\quad + E_u \left(\sum_{j=1}^N Q_k(t - \tau_j) \right) - Q_k(t) + E_u(W_k(t)) + D_k(u, t) \end{aligned} \tag{40}$$

holds with $t_{k+1} = \infty$,

$$W_k(t) := (1 - h_k t^{-1})^N + N h_k t^{-1} - \sum_{j=1}^N Q_k(t - \tau_j) - \prod_{j=1}^N P_k(t - \tau_j), \tag{41}$$

and

$$\begin{aligned} D_k(u, t) &:= E \left(1 - \prod_{j=1}^N P_k(t - \tau_j); u < L \leq t \right) + E \left((1 - h_k t^{-1})^N - 1 + N h_k t^{-1}; L > u \right) \\ &\quad + \sum_{i=1}^k z_1 \cdots z_i E \left(1 - \prod_{j=1}^N P_k(t - \tau_j); t + t_i < L \leq t + t_{i+1} \right). \end{aligned} \tag{42}$$

Proof. According to (39),

$$\begin{aligned} P_k(t) &= E_u \left(\prod_{j=1}^N P_k(t - \tau_j) \right) + E \left(\prod_{j=1}^N P_k(t - \tau_j); u < L \leq t \right) \\ &\quad + \sum_{i=1}^k z_1 \cdots z_i P(t + t_i < L \leq t + t_{i+1}) \\ &\quad - \sum_{i=1}^k z_1 \cdots z_i E \left(1 - \prod_{j=1}^N P_k(t - \tau_j); t + t_i < L \leq t + t_{i+1} \right). \end{aligned}$$

By the definition of $\Phi(\cdot)$,

$$\begin{aligned}\Phi(h_k t^{-1}) + 1 &= E_u \left((1 - h_k t^{-1})^N + N h_k t^{-1} \right) + P(L > t) \\ &\quad + E \left((1 - h_k t^{-1})^N - 1 + N h_k t^{-1}; L > u \right) + P(u < L \leq t),\end{aligned}$$

and after subtracting the two last equations, we get

$$\begin{aligned}\Phi(h_k t^{-1}) + Q_k(t) &= E_u \left((1 - h_k t^{-1})^N + N h_k t^{-1} - \prod_{j=1}^N P_k(t - \tau_j) \right) + P(L > t) \\ &\quad - \sum_{i=1}^k z_1 \cdots z_i P(t + t_i < L \leq t + t_{i+1}) + D_k(u, t),\end{aligned}$$

with $D_k(u, t)$ satisfying (42). After a rearrangement, the relation (40) follows together with (41). \square

With Lemma 5 in hand, the convergence (38) is proven by applying almost exactly the same argument as used in the proof of $tQ(t) \rightarrow h$. An important new feature emerges because of the additional term in the asymptotic relation defining the limit h_k . Let $1 = y_1 < y_2 < \dots < y_k < y_{k+1} = \infty$. Since

$$\sum_{i=1}^k z_1 \cdots z_i P(ty_i < L \leq ty_{i+1}) \sim dt^{-2} \sum_{i=1}^k z_1 \cdots z_i (y_i^{-2} - y_{i+1}^{-2}),$$

we see that

$$P(L > t) - \sum_{i=1}^k z_1 \cdots z_i P(ty_i < L \leq ty_{i+1}) \sim dg_k t^{-2},$$

where g_k is defined by (37). Assuming $0 \leq z_1, \dots, z_k < 1$, we ensure that $g_k > 0$, and as a result, we arrive at a counterpart of the quadratic equation (29),

$$bh_k^2 = ah_k + dg_k,$$

which gives

$$h_k = \frac{a + \sqrt{a^2 + 4bdg_k}}{2b} = h \frac{1 + \sqrt{1 + cg_k}}{1 + \sqrt{1 + c}},$$

justifying our definition (37). We conclude that for $k \geq 1$,

$$\begin{aligned}\frac{Q_k(t\bar{y}, \bar{z})}{Q(t)} &\rightarrow \frac{1 + \sqrt{1 + c \sum_{i=1}^k z_1 \cdots z_{i-1} (1 - z_i) y_i^{-2}}}{1 + \sqrt{1 + c}}, \\ &1 = y_1 < \dots < y_k, \quad 0 \leq z_1, \dots, z_k < 1.\end{aligned}\tag{43}$$

4.2. Conditioned generating functions

To finish the proof of Theorem 1, consider the generating functions conditioned on the survival of the GWO process. Given (5) with $j \geq 1$, we have

$$\begin{aligned}Q(t) E \left(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)} | Z(t) > 0 \right) &= E \left(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)}; Z(t) > 0 \right) \\ &= P_k(t\bar{y}, \bar{z}) - E \left(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)}; Z(t) = 0 \right) \stackrel{(35)}{=} Q_j^*(t, \bar{y}, \bar{z}) - Q_k(t\bar{y}, \bar{z}),\end{aligned}$$

and therefore,

$$E\left(z_1^{Z(ty_1)} \dots z_k^{Z(ty_k)} | Z(t) > 0\right) = \frac{Q_j^*(t, \bar{y}, \bar{z})}{Q(t)} - \frac{Q_k(t\bar{y}, \bar{z})}{Q(t)}.$$

Similarly, if (5) holds with $j = 0$, then

$$E\left(z_1^{Z(ty_1)} \dots z_k^{Z(ty_k)} | Z(t) > 0\right) = 1 - \frac{Q_k(t\bar{y}, \bar{z})}{Q(t)}.$$

Letting $t' = ty_1$, we get

$$\frac{Q_k(t\bar{y}, \bar{z})}{Q(t)} = \frac{Q_k(t', t'y_2/y_1, \dots, t'y_k/y_1)}{Q(t')} \frac{Q(ty_1)}{Q(t)},$$

and applying the relation (43), we have

$$\frac{Q_k(t\bar{y}, \bar{z})}{Q(t)} \rightarrow \frac{1 + \sqrt{1 + \sum_{i=1}^k z_1 \dots z_{i-1} (1 - z_i) \Gamma_i}}{(1 + \sqrt{1 + c})y_1},$$

where $\Gamma_i = c(y_1/y_i)^2$. On the other hand, since

$$Q_j^*(t, \bar{y}, \bar{z}) = Q_{j+1}(ty_1, \dots, ty_j, t; z_1, \dots, z_j, 0), \quad j \geq 1,$$

we also get

$$\frac{Q_j^*(t, \bar{y}, \bar{z})}{Q(t)} \rightarrow \frac{1 + \sqrt{1 + \sum_{i=1}^j z_1 \dots z_{i-1} (1 - z_i) \Gamma_i + cz_1 \dots z_j y_1^2}}{(1 + \sqrt{1 + c})y_1}.$$

We conclude that as stated in Section 2,

$$E\left(z_1^{Z(ty_1)} \dots z_k^{Z(ty_k)} | Z(t) > 0\right) \rightarrow E\left(z_1^{\eta(y_1)} \dots z_k^{\eta(y_k)}\right).$$

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