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HOMOGENIZATION OF A NON–PERIODIC OSCILLATING BOUNDARY VIA PERIODIC UNFOLDING

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Abstract. This paper deals with the homogenization of an elliptic model problem in a two-dimensional domain with non-periodic oscillating boundary by the method of periodic unfolding. For the non-periodic oscillations, a modulated unfolding is used. The L^2 convergence of the solutions and their fluxes are shown, under natural hypotheses on the domain.

1. Introduction

This paper illustrates the homogenization of a second order elliptic boundary value problem posed on a domain with a non-periodically oscillating boundary. The homogeneous Neumann boundary condition is assumed at the oscillating part of the boundary.

The homogenization of boundary value problems with periodically oscillating boundary that is not asymptotically thin was first studied by Brizzi and Chalot [6]. Followed by this pioneering work, there has been many works till date on such periodic structure. A literature review on periodic oscillating domains can be found in [10]. Most of the articles on oscillating domains deal with periodic nature, where [2, 3, 8, 10] stand out.

The homogenization of non-periodic oscillating boundaries was first analyzed by Gaudiello, Guibé and Murat in [10], using the method of oscillating test functions. Locally periodic oscillations were studied in for instance [4, 12, 13, 2, 3]. Some cases of non-periodic oscillations using the unfolding technique were treated in the book [8].

An immediate apparent difficulty with the periodic unfolding method to overcome in the present non-periodic setting is the lack of Hausdorff convergence of the unfolded domains, which is one of the key hypotheses in the homogenization theorems such as Proposition 8.18 in [8]. Here two approaches seem to be natural. Either one attempts to prove Proposition 8.18 in [8] with classical periodic unfolding definition or one modulates the periodic unfolding with the change of variables in order to retain Hausdorff convergence in the spirit of [9]. In this paper, we follow the second approach because it appears to be technically less demanding.

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To motivate our work and relate it to [8], in Section 2.1 we give an example of a sequence of domains that goes into the framework of Proposition 8.18 in [8] except for the Hausdorff convergence hypothesis on the unfolded domains and the strong convergence of the corresponding characteristic functions.

In [10], the authors assume that the oscillating part is made up of pillars of uniform cross section while we consider non-uniform pillars. In particular, in [10], a possibly arbitrary number of pillars under fixed density constraint was considered, while we restrict to a very controlled number of pillars.

The rest of this paper is organized as follows. In Section 2, we describe the oscillating domain under consideration, and pose a model problem. In Section 3, we introduce the domain specific modulation to be used with periodic unfolding, which allows for passing of the problem in domain Ω_ε to a fixed unfolded domain Ω^u. In Section 4, we describe the limit problem, and in Section 5, we show that the model problem homogenizes to the limit problem, in the sense of weak convergence of the solutions u_ε, and their fluxes, of model problem to the solution of the limit problem. In Section 6, the convergence of the energies are established, resulting in some information about the strong L^2 convergence of the solutions u_ε and their fluxes in Ω_ε.

2. Domain description and problem statement

This section is devoted to the description of a category of non-periodic oscillating domains $\Omega_\varepsilon \subset \mathbb{R}^2$, given below by (2.4), and the statment of the model problem (2.5).

2.1. Domain description

Let us first define a lifted set $\Omega^u \subset \mathbb{R}^3$ which will be the main ingredient to define the oscillating upper part Ω_ε^+ of the rough domain Ω_ε. With the fixed lower part $\Omega^- = (0,1) \times (-1,0)$, we can define the rough domain under consideration as $\Omega_\varepsilon = \operatorname{Int}\overline{\Omega^- \cup \Omega_\varepsilon^+}$.

Let $\Omega^u \subset (0,1)^3$ be a bounded domain with finite boundary measure. Denote any $(x_1, x_2) \in \mathbb{R}^2$ by x. For each $x_1 \in (0,1)$, define the reference set

$$Z(x_1) = \{(y_1, x_2) \in \mathbb{R}^2 : \ (x_1, x_2, y_1) \in \Omega^u\}.$$

We choose Ω^u such that for each $x_1 \in (0,1)$, $Z(x_1)$ is a nonempty connected open set in $\mathbb{R}^2$ and the reference set

$$Y(x_1, x_2) = \{y_1 \in (0,1) : (y_1, x_2) \in \overline{Z(x_1)}\} \tag{2.1}$$

is an interval of positive measure. Further we assume that there exists $s > 0$ such that $|Y(x_1, 0)| \geqslant s$ for all $x_1 \in (0,1)$ where

$$Y(x_1, 0) = \{y_1 \in (0,1) : (y_1, 0) \in \overline{Z(x_1)}\}.$$

The reference set $Z(x_1)$ will be used to define the pillars in the oscillating domain and the last condition is to make sure that every pillar in the oscillating upper part Ω_ε^+ is connected to the fixed part Ω^-. Define the maximum function M as

$$M(x_1) = \sup\{x_2 : (y_1, x_2) \in Z(x_1)\}. \tag{2.2}$$

Further, we assume that M is a Lipschitz function. It will define the the top boundary of the limit domain.

Now, we will describe the oscillating upper part Ω_ε^+ of the domain. For simplicity, assume $\varepsilon = 1/m$ for $m \in \mathbb{N}$. For every tagging $x_k^\varepsilon \in [k\varepsilon, (k+1)\varepsilon)$ for $k = 0, 1, \ldots, m-1$, define the scaled and translated reference cell Z_k^ε as

$$Z_k^\varepsilon = \{(x_1, x_2) : x_1 \in x_k^\varepsilon + \varepsilon Y(x_k^\varepsilon, x_2), x_2 \in (0, M(x_k^\varepsilon))\}.$$

where the reference set $Y(x_1, x_2)$ and the maximum function $M(x_1)$ are as defined above. The tagging x_k^ε's are chosen such that Z_k^ε satisfies the compatibility condition

$$Z_k^\varepsilon \subset (k\varepsilon, (k+1)\varepsilon) \times (0, M(x_k^\varepsilon)). \tag{2.3}$$

The condition (2.3) ensures that the tagging x_k^ε is arbitrarily chosen in such a way that the scaled reference cell Z_k^ε is fully contained in $(k\varepsilon, (k+1)\varepsilon) \times (0, M(x_k^\varepsilon))$.

Now, we define the oscillating upper part Ω_ε^+ as

$$\Omega_\varepsilon^+ = \bigcup_{k=0}^{m-1} Z_k^\varepsilon.$$

The domain with oscillating boundary Ω_ε is given by

$$\Omega_\varepsilon = \mathrm{Int}\,\overline{\Omega^- \cup \Omega_\varepsilon^+}, \tag{2.4}$$

where the fixed part $\Omega^- = (0,1) \times (0,-1)$.

We denote the common boundary of Ω_ε^+ and Ω^- by γ_c^ε:

$$\gamma_c^\varepsilon = \{(x_1, x_2) \in \Omega_\varepsilon : x_2 = 0\}.$$

The full or limit domain which is the Hausdorff limit of Ω_ε is given by

$$\Omega = \{(x_1, x_2) : x_1 \in (0,1), -1 < x_2 < M(x_1)\},$$

while the upper part of the the limit domain Ω is defined by

$$\Omega^+ = \{(x_1, x_2) :\ x_1 \in (0,1),\ 0 < x_2 < M(x_1)\}.$$

The common boundary γ_c of Ω^+ and Ω^- is given by

$$\gamma_c = \{(x_1, 0) : x_1 \in (0,1)\}.$$

The following example provides a domain that falls in the category of Ω_ε^+ in this section.

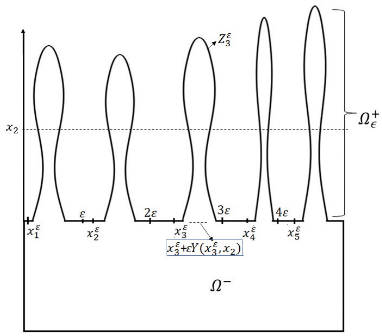

Figure 1: *Domain* Ω_ε.

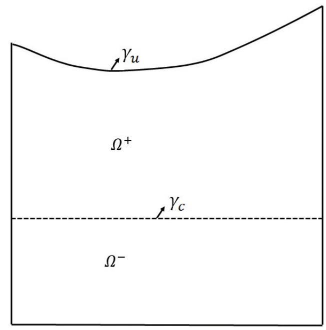

Figure 2: *Limit domain* Ω, $\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-}$.

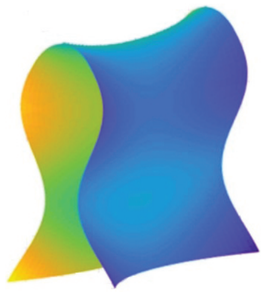

Figure 3: *Lateral surface of the domain* Ω^u.

EXAMPLE 2.1. In $\mathbb{R}^2$, for $\varepsilon = 1/N$, $N = 1,2,\ldots$, consider the following sequence of taggings x_k^ε of the partitions $\{[\varepsilon k, \varepsilon(k+1)] : k = 0,\ldots,1/\varepsilon\}$ of the interval $[0,1]$:

$$x_k^\varepsilon = \begin{cases} k\varepsilon + \varepsilon/3, & \text{if } \varepsilon = 1/N, N \text{ is odd;} \\ k\varepsilon + 2\varepsilon/3, & \text{if } \varepsilon = 1/N, N \text{ is even.} \end{cases}$$

Let the oscillating part of the domain be

$$\Omega_+^\varepsilon = \bigcup_{k=0}^{1/\varepsilon - 1} (x_k^\varepsilon + \varepsilon(-1/8, 1/8)) \times (0,1),$$

and $\Omega^- = (0,1) \times (-1,0)$ (see Figures 4 and 5). Then classically, the unfolded characteristic functions of the oscillating part of the domain are

$$\begin{aligned} T^\varepsilon \chi_{\Omega_+^\varepsilon}(x,y) &= \chi_{\Omega_+^\varepsilon}(\varepsilon[x_1/\varepsilon] + \varepsilon y, x_2) \\ &= \begin{cases} \chi_{(0,1)^2 \times (5/24, 11/24)}(x,y), & \text{if } \varepsilon = 1/N, N \text{ is odd;} \\ \chi_{(0,1)^2 \times (13/24, 19/24)}(x,y), & \text{if } \varepsilon = 1/N, N \text{ is even;} \end{cases} \end{aligned}$$

with $x \in (0,1)^2$, yielding the unfolded domains (see Figure 6)

$$\Omega_\varepsilon^u = \begin{cases} (0,1)^2 \times (5/24, 11/24), & \text{if } \varepsilon = 1/N, N \text{ is odd;} \\ (0,1)^2 \times (13/24, 19/24), & \text{if } \varepsilon = 1/N, N \text{ is even.} \end{cases}$$

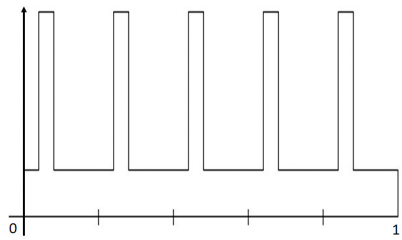

Figure 4: $\varepsilon = \frac{1}{5}$.

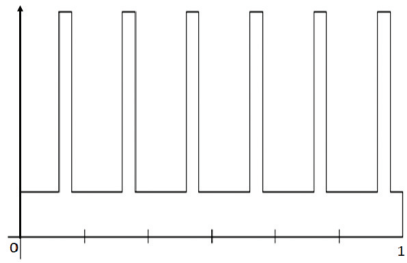

Figure 5: $\varepsilon = \frac{1}{6}$

The sequence of the unfolded domains Ω_ε^u does not converge in the sense of Hausdorff as for any $\varepsilon = 1/N$, the Hausdorff distance

$$\begin{aligned} & d_H(\Omega^u_{1/N}, \Omega^u_{1/(N+1)}) \\ & \quad = \max\Big\{ \sup_{x \in \Omega^u_{1/N}} \inf_{x' \in \Omega^u_{1/(N+1)}} |x - x'| \Big\}, \sup_{x \in \Omega^u_{1/(N+1)}} \inf_{x' \in \Omega^u_{1/N}} |x - x'| \Big\} \\ & \quad = 1/12. \end{aligned}$$

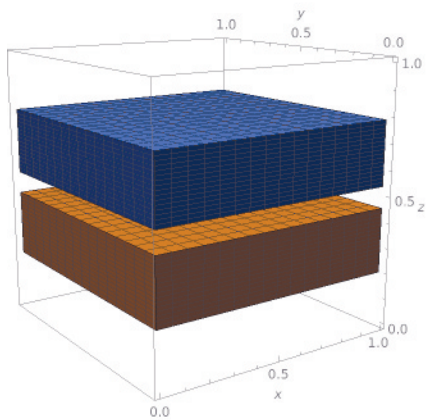

Figure 6: *Unfolded domain corresponding non-periodic pillar.*

2.2. Model problem

As a model to illustrate the application of the modulated unfolding operator, we consider the homogenization of following elliptic equation in divergence form on the domain Ω_ε given by (2.4), which contains the standard steps in homogenization:

$$\begin{cases} -\operatorname{div}(A\nabla u_\varepsilon) + u_\varepsilon = f \text{ in } \Omega_\varepsilon, \\ A\nabla u_\varepsilon \cdot \nu_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{2.5}$$

where $f \in L^2(\Omega)$, Ω is a domain containing all Ω_ε. Here, ν_ε denotes the unit outward normal vector to the boundary $\partial\Omega_\varepsilon$ of Ω_ε. The coefficient matrix $A = A(x)$ is assumed to be a 2×2 matrix with elements $a_{ij} = a_{ij}(x) : \Omega \to \mathbb{R}$ that are bounded measurable functions. We assume that A is uniformly elliptic and bounded in Ω, that is, there exists $\alpha, \beta \in \mathbb{R}^+$ such that,

$$(i)\ A(x)\xi\cdot\xi \geqslant \alpha|\xi|^2, \quad \xi \in \mathbb{R}^2, \text{ a.e. } x \in \Omega,$$
$$(ii)\ |A(x)\xi| \leqslant \beta|\xi|, \quad \xi \in \mathbb{R}^2, \text{ a.e. } x \in \Omega.$$

The weak formulation to the above problem is given by: Find $u_\varepsilon \in H^1(\Omega_\varepsilon)$ such that

$$\int_{\Omega_\varepsilon} A\nabla u_\varepsilon \cdot \nabla\phi\, dx + \int_{\Omega_\varepsilon} u_\varepsilon \phi\, dx = \int_{\Omega_\varepsilon} f\phi\, dx, \tag{2.6}$$

for all $\phi \in H^1(\Omega_\varepsilon)$. The Lax-Milgram lemma guarantees the existence and uniqueness of such $u_\varepsilon \in H^1(\Omega_\varepsilon)$.

By taking u_ε as test function in the weak formulation (2.6), we get a uniform estimate on $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$, that is, there exists a constant C independent of ε, such that

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C. \tag{2.7}$$

Our aim is to analyze the asymptotic behavior of u_ε as $\varepsilon \to 0$. The asymptotic analysis will be carried out using a modulated unfolding operator which will be the theme of the upcoming section.

3. Modulated unfolding operator

The periodic unfolding method was first introduced in [7]. A modified definition was used in [5] to do homogenization in pillar type oscillating domain. Later in [1], the authors further modified the unfolding operator for general periodic oscillating domains. For more literature on unfolding operators one can refer to the book [8] and references therein.

The usual periodic unfolding appears technically demanding to be applied here as the scaled pillars are kept in the ε-cells at arbitrary locations. In this paper, the oscillation is non periodic in the sense that for each ε, $m = \frac{1}{\varepsilon}$, the tagging

$x_k^\varepsilon \in (k\varepsilon,(k+1)\varepsilon)$ for $k=0,1,\dots,m-1$, have been chosen in a way that they are not necessarily be equidistant as long as they satisfy the compatibility condition (2.3).

For every $\varepsilon > 0$, we define the approximate unfolded domain corresponding to Ω_ε^+ as:

$$\Omega_\varepsilon^u = \bigcup_k [k\varepsilon,(k+1)\varepsilon) \times (0,M(x_k^\varepsilon)) \times Y(x_k^\varepsilon,x_2).$$

The approximate unfolded domain Ω_ε^u is close to the unfolded domain (lifted set) Ω^u in the sense that

$$\chi_{\Omega_\varepsilon^u} \to \chi_{\Omega^u} \text{ strongly in } L^p(\mathbb{R}^3) \text{ for } 1 \leqslant p < \infty.$$

DEFINITION 3.1. (Modulated periodic unfolding) The unfolding for a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$(T^\varepsilon\phi)(x_1,x_2,y_1) = \chi_{\overline{\Omega_\varepsilon^u}}(x,y)\phi(x_k^\varepsilon + \varepsilon y_1, x_2) \quad \text{if } x_1 \in [k\varepsilon,(k+1)\varepsilon),$$

where $k=0,1,\dots,m-1$.

The main properties of unfolding operators are given below.

LEMMA 3.2. *Let $u \in L^1(\Omega_\varepsilon^+)$. Then*

$$\int_{\Omega_\varepsilon^+} u\,dx = \int_{\Omega_\varepsilon^u} T^\varepsilon u\,dxdy_1.$$

Proof. For $\phi \in C_c^\infty(\Omega^+)$, we have, by the definition of unfolding

$$\begin{aligned}\int_{\Omega_\varepsilon^u} T^\varepsilon\phi(x_1,x_2,y_1)\,dxdy_1 &= \sum_{k=0}^m \int_{k\varepsilon}^{(k+1)\varepsilon}\int_0^{M(x_k^\varepsilon)}\int_{Y(x_k^\varepsilon,x_2)} \phi(x_k^\varepsilon+\varepsilon y_1,x_2)\,dxdy_1\\ &= \sum_{k=0}^m \varepsilon\int_0^{M(x_k^\varepsilon)}\int_{Y(x_k^\varepsilon,x_2)} \phi(x_k^\varepsilon+\varepsilon y_1,x_2)\,dx_2dy_1.\end{aligned}$$

The change of variable $x_1 = x_k^\varepsilon + \varepsilon y_1$ gives

$$\int_{\Omega_\varepsilon^u} T^\varepsilon\phi(x,y_1)\,dxdy_1 = \sum_{k=0}^m \int_0^{M(x_k^\varepsilon)}\int_{k\varepsilon+\varepsilon Y(x_k^\varepsilon,x_2)} \phi(x)\,dx = \int_{\Omega_\varepsilon^+}\phi\,dx.$$

Hence, the density of $C_c^\infty(\Omega^+)$ in $L^1(\Omega_\varepsilon^+)$ completes the proof. □

LEMMA 3.3. *For $v_\varepsilon \in L^p(\Omega_\varepsilon^+)$ bounded, $p>1$, we have*

$$\int_{\Omega_\varepsilon^+} v_\varepsilon\,dx = \int_{\Omega_\varepsilon^u} T^\varepsilon v_\varepsilon\,dxdy_1 = \int_{\Omega^u} T^\varepsilon v_\varepsilon\,dxdy_1 + o(1), \tag{3.1}$$

as $\varepsilon \to 0$.

Proof. Note that $T^\varepsilon \chi_{\Omega_\varepsilon^+} = \chi_{\Omega_\varepsilon^u}$. The discrepancy can be computed as follows

$$\begin{aligned}\int_{\Omega_\varepsilon^+} v_\varepsilon\, dx - \int_{\Omega^u} T^\varepsilon v_\varepsilon\, dxdy_1 &= \int_{\Omega^+\times(0,1)} (T^\varepsilon \chi_{\Omega_\varepsilon^+} T^\varepsilon v_\varepsilon - \chi_{\Omega^u} T^\varepsilon v_\varepsilon)\, dxdy_1 \\ &= \int_{\Omega^+\times(0,1)} (\chi_{\Omega_\varepsilon^u} - \chi_{\Omega^u}) T^\varepsilon v_\varepsilon\, dxdy_1.\end{aligned}$$

As the unfolded domain has finite boundary measure $\Omega_\varepsilon^u \backslash \Omega^u$ and $\Omega^u \backslash \Omega_\varepsilon^u$ are contained in some strip of measure $O(\varepsilon)$:

$$\left\{(x,y) \in (0,1)^3 : \mathrm{dist}((x,y), \partial\Omega^u) < C\varepsilon\right\},$$

where C can be chosen independent of ε. Hence by Hölder's inequality,

$$\lim_{\varepsilon\to 0} \left| \int_{\Omega_\varepsilon^+} v_\varepsilon\, dx - \int_{\Omega^u} T^\varepsilon v_\varepsilon\, dxdy_1 \right| = 0. \quad \square$$

4. Limit problem

In this section we introduce the limit problem with its associated function space.

4.1. Limit function space

We introduce the limit function space as it differs from the Sobolev space of the model problem (2.5). Let Ω be the limit domain: $\overline{\Omega}$ is the Hausdorff limit of $\overline{\Omega_\varepsilon}$. Define the density ω of Ω_ε in Ω as

$$\omega(x) = \begin{cases} |Y(x)|, & \text{if } x \in \overline{\Omega^+}; \\ 1, & \text{if } x \in \Omega^-, \end{cases} \tag{4.1}$$

where $|Y(x)|$ is the Lebesgue measure of the reference set $Y(x)$. Notice that if $x_2 = 0$, then $\omega(x) = |Y(x_1, 0)|$.

For any $\psi : \Omega \to \mathbb{R}$, we denote $\psi^+ = \psi\chi_{\Omega^+}$ and $\psi^- = \psi\chi_{\Omega^-}$.

Let $L^2(\Omega, \omega)$ be the weighted Lebesgue space $\{v : \int_\Omega v^2 \omega < \infty\}$ and

$$H(\Omega,\omega) = \left\{ v \in L^2(\Omega,\omega) : \frac{\partial v^-}{\partial x_1} \in L^2(\Omega^-),\ \frac{\partial v}{\partial x_2} \in L^2(\Omega,\omega) \right\}.$$

The space $H(\Omega,\omega)$ is a Hilbert space with respect to the scalar product

$$(u,v)_{H(\Omega,\omega)} = \int_\Omega uv\omega\, dx + \int_{\Omega^+} \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\omega\, dx + \int_{\Omega^-} \nabla u \cdot \nabla v\, dx.$$

In what follows, we will use the following notation $u = (u^+, u^-)$ for any function $u \in H(\Omega,\omega)$.

4.2. Limit problem

The homogenized problem is posed in Ω (see Figure 2), and reads as follows:

$$\begin{aligned}
-\frac{\partial}{\partial x_2}\left(\omega A_0^+\frac{\partial u^+}{\partial x_2}\right)+\omega u^+ &= \omega f && \text{in } \Omega^+,\\
-\operatorname{div}(A\nabla u^-)+u^- &= f && \text{in } \Omega^-,\\
\omega A_0^+\frac{\partial u^+}{\partial x_2}\nu_2 &= 0 && \text{on } \gamma_u,\\
u^+ &= u^- && \text{on } \gamma_c,\\
\omega A_0^+\frac{\partial u^+}{\partial x_2}-(a_{12}+a_{22})\frac{\partial u^-}{\partial x_2} &= 0 && \text{on } \gamma_c,\\
A\nabla u\cdot\nu &= 0 && \text{on } \partial\Omega^-\setminus\gamma_c,
\end{aligned} \tag{4.2}$$

where ω is the domain density defined in (4.1), and

$$A_0^+=\frac{\det(\mathrm{A})}{a_{11}}.$$

The weak formulation of the above system (4.2) is:

Find $u=(u^+,u^-)\in H(\Omega,\omega)$ such that

$$\begin{aligned}
&\int_{\Omega^+}\left(A_0^+\frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2}+u^+\phi\right)\omega\,dx+\int_{\Omega^-}(A\nabla u^-\cdot\nabla\phi+u^-\phi)\,dx\\
&\quad=\int_{\Omega^+}f\phi\omega\,dx+\int_{\Omega^-}f\phi\,dx,
\end{aligned} \tag{4.3}$$

for all $\phi\in H(\Omega,\omega)$.

As A and A_0^+ are uniformly elliptic and bounded (c.f. (5.13) below), by the Lax-Milgram lemma, there exists a unique solution $u\in H(\Omega,\omega)$ satisfying (4.3).

5. Homogenization

In this section, we establish the homogenization of our model elliptic problem (2.5) posed on the oscillating domain to the limit problem (4.2), and show the weak convergence of the solutions u_ε and the fluxes ∇u_ε.

First, let us verify an *a priori* estimate of the unfolded solutions $T^\varepsilon u_\varepsilon$. By using the *a priori* estimate on u_ε (2.7) and as $T^\varepsilon u_\varepsilon$ vanishes outside of Ω_ε^u, we get

$$\|T^\varepsilon u_\varepsilon\|^2_{L^2(\Omega^u)}+\|T^\varepsilon u_\varepsilon\|^2_{L^2(\Omega_\varepsilon^u\setminus\Omega^u)}=\|T^\varepsilon u_\varepsilon\|^2_{L^2(\Omega_\varepsilon^u)}=\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon^+)}\leqslant C$$

where ere $C>0$ is a constant independent of ε. Note that the inequality holds for $T^\varepsilon\nabla u_\varepsilon$ also. Thus, we have the following lemma.

LEMMA 5.1. *The sequence of solutions u_ε to the problem (2.5) satisfies the a priori estimates*

$$\|T^\varepsilon u_\varepsilon\|_{L^2(\Omega^u)} \leqslant C,$$
$$\|T^\varepsilon \nabla u_\varepsilon\|_{L^2(\Omega^u)} \leqslant C,$$

where $C>0$ is a constant independent of ε.

Now, we are in a position to state the main convergence result of this paper.

THEOREM 5.2. (Homogenization) *Let u_ε be the sequence of solutions to the problem (2.5), and let $u=(u^+,u^-)$ be the solutions to the problem (4.2). Then,*

i. $\widetilde{u_\varepsilon^+} \rightharpoonup \omega u^+$ *weakly in* $L^2(\Omega^+)$,

ii. $\widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_1}} \rightharpoonup -\omega \dfrac{a_{12}}{a_{11}} \dfrac{\partial u^+}{\partial x_2}$ *weakly in* $L^2(\Omega^+)$,

iii. $\widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_2}} \rightharpoonup \omega \dfrac{\partial u^+}{\partial x_2}$ *weakly in* $L^2(\Omega^+)$,

iv. $u_\varepsilon^- \rightharpoonup u^-$ *weakly in* $H^1(\Omega^-)$,

as $\varepsilon \to 0$. Here, $\sim$ denotes the zero extension.

Proof. First, let us look at the convergence on the upper part Ω^+. Apply unfolding operator in the weak formulation (2.6) to get

$$\int_{\Omega^u} (AT^\varepsilon \nabla u_\varepsilon \cdot T^\varepsilon \nabla\phi + T^\varepsilon u_\varepsilon T^\varepsilon \phi)\,dxdy_1 = \int_{\Omega^u} T^\varepsilon f\, T^\varepsilon \phi\, dxdy_1 + o(1), \tag{5.1}$$

for any test function $\phi \in C^\infty(\overline{\Omega})$, as $\varepsilon \to 0$.

By Lemma 5.1, we have the uniform estimate on $T^\varepsilon u_\varepsilon$. Thus, there exist u^+ and $P \in L^2(\Omega^u)$ such that, up to a subsequence, we have

$$T^\varepsilon u_\varepsilon^+ \rightharpoonup u^+ \qquad \text{weakly in} \quad L^2(\Omega^u),$$
$$T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_2} \rightharpoonup \frac{\partial u^+}{\partial x_2} \qquad \text{weakly in} \quad L^2(\Omega^u),$$
$$T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_1} \rightharpoonup P \qquad \text{weakly in} \quad L^2(\Omega^u),$$

as $\varepsilon \to 0$.

Step 1: Identification of P.

We will show that $P = -\dfrac{a_{12}}{a_{11}}\dfrac{\partial u^+}{\partial x_2}$ almost everywhere in Ω^u. Let $\phi \in C_c^\infty(\Omega^+)$. Let $\varphi \in C^\infty([0,2])$ and define $\psi(y) = \int_0^y \varphi(z)\,dz$. Then, $\psi \in C^\infty((0,2))$ and $\psi'(y) = \varphi(y)$. Consider the following oscillating test functions:

$$\psi^\varepsilon(x) = \varepsilon\phi(x)\psi\Big(\frac{x_1 - x_k^\varepsilon}{\varepsilon}\Big) \quad \text{when } x_1 \in [x_k^\varepsilon, x_{k+1}^\varepsilon).$$

Then $\psi^\varepsilon \in C^\infty(\overline{\Omega_\varepsilon})$, and (5.1) holds for such sequence of test functions. Moreover,

$$T^\varepsilon \psi^\varepsilon \to 0 \qquad \text{strongly in } L^2(\Omega^u), \tag{5.2}$$

$$T^\varepsilon \frac{\partial \psi^\varepsilon}{\partial x_1} \to \phi(x)\varphi(y_1) \qquad \text{strongly in } L^2(\Omega^u), \tag{5.3}$$

$$T^\varepsilon \frac{\partial \psi^\varepsilon}{\partial x_2} \to 0 \qquad \text{strongly in } L^2(\Omega^u), \tag{5.4}$$

as $\varepsilon \to 0$. Now, by taking ψ^ε as a test function in the weak from (5.1) and passing to the limit as $\varepsilon \to 0$, we get

$$\int_{\Omega^u} \Big(a_{11}P + a_{12}\frac{\partial u^+}{\partial x_2}\Big)\phi(x)\varphi(y_1)\,dx dy_1 = 0.$$

Since ϕ and ψ are arbitrary, the above equation implies that

$$P = -\frac{a_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2} \qquad \text{a.e. in } \Omega^u, \tag{5.5}$$

by the density of the tensor products.

Step 2: Limit equation in Ω^+ and Ω^-.

Let us write the weak formulation of (2.6) as

$$\int_{\Omega_\varepsilon^+} \big(A\nabla u_\varepsilon \cdot \nabla\phi + u_\varepsilon\phi\big)\,dx + \int_{\Omega^-} \big(A\nabla u_\varepsilon \cdot \nabla\phi + u_\varepsilon\phi\big)\,dx = \int_{\Omega_\varepsilon} f\phi\,dx, \tag{5.6}$$

for all $\phi \in C^\infty(\overline{\Omega})$. By using (5.1)–(5.5), the first term in the above equation becomes

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^+} \big(A\nabla u_\varepsilon \cdot \nabla\phi + u_\varepsilon\phi\big)\,dx = \int_{\Omega^+}\int_{Y(x)} \Big(\frac{\det(A)}{a_{11}}\frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + u^+\phi\Big)\,dx dy_1$$

for all $\phi \in C^\infty(\overline{\Omega})$. Note that $\dfrac{\partial}{\partial y_1}\big(T^\varepsilon u_\varepsilon\big) = \varepsilon T^\varepsilon \dfrac{\partial u_\varepsilon}{\partial x_1}$ and $T^\varepsilon \dfrac{\partial u_\varepsilon}{\partial x_1}$ is uniformly bounded by Lemma 5.1. Thus we have $\dfrac{\partial u^+}{\partial y_1} = 0$, in other words u^+ is independent of y_1, as the sections of $Y(x)$ are supposed to be connected. As u^+ is independent of y_1,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^+} \left(A\nabla u_\varepsilon\cdot\nabla\phi + u_\varepsilon\phi\right)dx \\
&\quad = \int_{\Omega^+}\left(\frac{\det(A)}{a_{11}}\frac{\partial u^+}{\partial x_2}\frac{\partial\phi}{\partial x_2} + u^+\phi\right)\omega\,dx \\
&\quad = \int_{\Omega^+}\left(A_0^+\frac{\partial u^+}{\partial x_2}\frac{\partial\phi}{\partial x_2} + u^+\phi\right)\omega\,dx.
\end{aligned} \tag{5.7}$$

We turn to the second term in (5.6).

By the *a priori* estimate (2.7), we have $\|u_\varepsilon\|_{H^1(\Omega^-)} \leqslant C$, where $C > 0$ is constant independent of ε. Then, by the weak compactness there exists $u^- \in H^1(\Omega^-)$ such that

$$u_\varepsilon \rightharpoonup u^- \qquad \text{weakly in } H^1(\Omega^-),$$

along some subsequence, as $\varepsilon \to 0$. Thus,

$$\lim_{\varepsilon\to 0}\int_{\Omega^-}\left(A\nabla u_\varepsilon\cdot\nabla\phi + u_\varepsilon\phi\right)dx = \int_{\Omega^-}\left(A\nabla u^-\cdot\nabla\phi + u^-\phi\right)dx, \tag{5.8}$$

for all $\phi \in C^\infty(\overline{\Omega})$. The right hand side of (5.6) becomes

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f\phi\,dx = \int_{\Omega^+} f\phi\omega\,dx + \int_{\Omega^-} f\phi\,dx. \tag{5.9}$$

The next step is to show $u^+ = u^-$ on the interface γ_c.

Step 3: Interface condition.

On the common interface γ_c^ε, $u_\varepsilon^+ = u_\varepsilon^-$, and because the interface is flat $\gamma_c^\varepsilon \subset \gamma_c$. It follows that

$$T^\varepsilon u_\varepsilon^+ = T^\varepsilon u_\varepsilon^- \qquad \text{on } \gamma_c^u, \tag{5.10}$$

where the unfolded interface is denoted by

$$\gamma_c^u = \{(x_1,x_2,y_1) \in \overline{\Omega^u} : x_2 = 0\}.$$

By the strong convergence of $T^\varepsilon\chi_{\gamma_c^\varepsilon}$ to $\chi_{\gamma_c^u}$ in $L^2(\gamma_c^u)$, and the weak sequential continuity of the trace, one obtains

$$u^+ = u^- \qquad \text{on } \gamma_c^u,$$

by passing to the limit in (5.10) as $\varepsilon \to 0$. Since u^- is independent of y_1, we have

$$u^+ = u^- \qquad \text{on } \gamma_c.$$

Step 4: Limit problem.

By combining (5.7), (5.8), and (5.9), we get as $\varepsilon \to 0$,

$$\begin{aligned}\int_{\Omega^+}\Big(A_0^+\frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2}+u^+\phi\Big)\omega\,dx+\int_{\Omega^-}(A\nabla u^-\cdot\nabla\phi+u^-\phi)\,dx \\ =\int_{\Omega^+}f\phi\omega\,dx+\int_{\Omega^-}f\phi\,dx,\end{aligned} \tag{5.11}$$

for all $\phi\in C^\infty(\overline{\Omega})$.

In general, one needs some condition on the weight ω for the smooth functions up to the boundary to be dense in the weighted limit space $H(\Omega,\omega)$. Here, we use the regularity of the unfolded domain to get density in weighted Sobolev space.

Let $\Omega^U=\text{Int}\Big(\overline{\Omega^u\cup(\Omega^-\times(0,1))}\Big)$ and

$$H(\Omega^U)=\Big\{\phi\in L^2(\Omega^U):\ \frac{\partial\phi}{\partial x_2},\frac{\partial\phi}{\partial x_1}\chi_{\Omega^-\times(0,1)}\in L^2(\Omega^U),\frac{\partial\phi}{\partial y_1}=0\Big\}.$$

This is a Hilbert space with the following inner product: For $\phi,\psi\in H(\Omega^U)$,

$$\langle\phi,\psi\rangle_{H(\Omega^U)}=\int_{\Omega^u}\frac{\partial\phi}{\partial x_2}\frac{\partial\psi}{\partial x_2}dxdy_1+\int_{\Omega^U}\big(\chi_{(\Omega^-\times(0,1))}\nabla\phi\cdot\nabla\psi+\phi\psi\big)\,dxdy_1.$$

Now, the above variational equality can be written in Ω^U as

$$\begin{aligned}\int_{\Omega^u}A_0^+\frac{\partial u^+}{\partial x_2}\frac{\partial\phi}{\partial x_2}dxdy_1+\int_{\Omega^U}\big(\chi_{(\Omega^-\times(0,1))}A\nabla u^-\cdot\nabla\phi+u\phi\big)\,dxdy_1 \\ =\int_{\Omega^U}f\phi\,dxdy_1\end{aligned} \tag{5.12}$$

for all $\phi\in H(\Omega^U)$ as $C^\infty(\overline{\Omega})$ is dense in $H(\Omega^U)$ because Ω^U is a bounded Lipschitz domain in $\mathbb{R}^3$, and the functions in $H(\Omega^U)$ are independent of y_1 by the connectedness of the sections.

To see the well-posedness of the problem (4.2), one can verify that the left hand side of (5.12) is a bounded and elliptic bilinear form on $H(\Omega^U)\times H(\Omega^U)$, due to the assumption on the matrix A and the scalar A_0^+ is strictly positive. In particular, by the assumptions on the matrix A,

$$\frac{\alpha^2}{\beta}\leqslant A_0^+\leqslant\frac{\beta^2}{\alpha}. \tag{5.13}$$

The right hand side in (5.12) is a bounded linear functional on $H(\Omega^U)$. By the Lax-Milgram lemma, there exists a unique solution $u\in H(\Omega^U)$ solving (5.12). Hence u satisfies the homogenized equation (4.2). By uniqueness of the solution $u=(u^+,u^-)$ the full ε sequence u_ε converges. □

6. Energy convergence

In this section, we show a strong L^2 estimate in Ω_ε for the solutions u_ε and their fluxes.

LEMMA 6.1. *Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the solution to (2.5) and $u \in H(\Omega,\omega)$ be the solution to (4.2). Then*

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(A\nabla u_\varepsilon\cdot\nabla u_\varepsilon + u_\varepsilon^2)\,dx$$
$$= \int_{\Omega^+}\left(\omega A_0^+\left(\frac{\partial u^+}{\partial x_2}\right)^2 + \omega(u^+)^2\right)dx + \int_{\Omega^-}(A\nabla u^-\cdot\nabla u^- + (u^-)^2)\,dx.$$

Proof. Using u_ε as a test function in (2.6), we have

$$\int_{\Omega_\varepsilon}(A\nabla u_\varepsilon\cdot\nabla u_\varepsilon + u_\varepsilon^2)\,dx = \int_{\Omega_\varepsilon} f u_\varepsilon\,dx. \tag{6.1}$$

Using the weak convergence of u_ε in the right hand side, we get

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(A\nabla u_\varepsilon\cdot\nabla u_\varepsilon + u_\varepsilon^2)\,dx = \int_{\Omega^+}\omega f u^+\,dx + \int_{\Omega^-} f u^-\,dx. \tag{6.2}$$

Since $u \in H(\Omega,\omega)$ is the solution of the homogenized system (4.2), right hand side of (6.2) matches with the right hand side of (4.3) and this completes the proof. □

THEOREM 6.2. *Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the solution to (2.5) and $u \in H(\Omega,\omega)$ be the solution to (4.2). Then*

i. $\left\|T^\varepsilon\nabla u_\varepsilon^+ - \left(-\frac{a_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2}, \frac{\partial u^+}{\partial x_2}\right)\right\|_{L^2(\Omega^u)} \to 0,$

ii. $\|T^\varepsilon u_\varepsilon^+ - u^+\|_{L^2(\Omega^u)} \to 0,$

iii. $\|u_\varepsilon^- - u^-\|_{H^1(\Omega^-)} \to 0,$

as $\varepsilon \to 0$.

Proof. The convergence in Lemma 6.1 is equivalent to the following convergence:

$$\lim_{\varepsilon\to 0}\int_{\Omega^u}(T^\varepsilon A\,T^\varepsilon\nabla u_\varepsilon^+\cdot T^\varepsilon\nabla u_\varepsilon^+ + (T^\varepsilon u_\varepsilon^+)^2)\,dxdy_1$$
$$+\lim_{\varepsilon\to 0}\int_{\Omega^-}(A\nabla u_\varepsilon^-\cdot\nabla u_\varepsilon^- + (u_\varepsilon^-)^2)\,dx$$
$$= \int_{\Omega^+}\left(\omega A_0^+\left(\frac{\partial u^+}{\partial x_2}\right)^2 + \omega(u^+)^2\right)dx + \int_{\Omega^-}(A\nabla u^-\cdot\nabla u^- + (u^-)^2)\,dx.$$

From the ellipticity assumption on A, we get

$$
\begin{aligned}
\alpha & \left\| T^{\varepsilon}\nabla u_{\varepsilon}^{+} - \left(-\frac{a_{12}}{a_{11}}\frac{\partial u^{+}}{\partial x_2}, \frac{\partial u^{+}}{\partial x_2} \right) \right\|_{L^2(\Omega^u)}^2 + \|T^{\varepsilon} u_{\varepsilon}^{+} - u^{+}\|_{L^2(\Omega^u)}^2 \\
& + \alpha \|\nabla u_{\varepsilon}^{-} - \nabla u^{-}\|_{L^2(\Omega^-)}^2 + \|u_{\varepsilon}^{-} - u^{-}\|_{L^2(\Omega^-)}^2 \\
& \leqslant \int_{\Omega^u} (T^{\varepsilon} A T^{\varepsilon} \nabla u_{\varepsilon}^{+} \cdot T^{\varepsilon}\nabla u_{\varepsilon}^{+} + |T^{\varepsilon} u_{\varepsilon}^{+}|^2)\, dx dy_1 + \int_{\Omega^-} (A\nabla u_{\varepsilon}^{-} \nabla \cdot u_{\varepsilon}^{-} + (u_{\varepsilon}^{-})^2)\, dx \\
& + \int_{\Omega^u} \left(A_0^{+} \left(\frac{\partial u^{+}}{\partial x_2} \right)^2 + (u^{+})^2 \right) dx dy_1 + \int_{\Omega^-} (A\nabla u^{-} \cdot \nabla u^{-} + (u^{-})^2)\, dx \\
& - 2 \int_{\Omega^u} \left(T^{\varepsilon} A T^{\varepsilon} \nabla u_{\varepsilon}^{+} \cdot \left(-\frac{a_{12}}{a_{11}}\frac{\partial u^{+}}{\partial x_2}, \frac{\partial u^{+}}{\partial x_2} \right) + T^{\varepsilon} u_{\varepsilon}^{+} u^{+} \right) dx dy_1 \\
& - 2 \int_{\Omega^-} \left(A\nabla u_{\varepsilon}^{-} \cdot \nabla u^{-} + u_{\varepsilon}^{-} u^{-} \right) dx.
\end{aligned}
$$

By letting $\varepsilon \to 0$, using Lemma 6.1, the right hand side of the above inequality vanishes. This completes the proof. □

REMARK 6.3. In the above Theorem, we have shown the converge in the limit unfolded domain Ω^u. Actually, we have the convergence in approximate unfolded domain Ω^u_{ε}, since, we have the following

$$
\begin{aligned}
& \int_{\Omega^+} \omega \left[A_0^{+} \left(\frac{\partial u^{+}}{\partial x_2} \right)^2 + (u^{+})^2 \right] dx + \int_{\Omega^-} (A\nabla u^{-} \cdot \nabla u^{-} + (u^{-})^2)\, dx \\
& \leqslant \liminf_{\varepsilon \to 0} \Bigg(\int_{\Omega^u} \omega \left(T^{\varepsilon} A T^{\varepsilon} \left(\frac{\partial u^{+}}{\partial x_2} \right)^2 + T^{\varepsilon}(u^{+})^2 \right) dx \\
& \qquad + \int_{\Omega^-} (A\nabla u_{\varepsilon}^{-} \cdot \nabla u_{\varepsilon}^{-} + (u_{\varepsilon}^{-})^2)\, dx \Bigg) \\
& \leqslant \liminf_{\varepsilon \to 0} \Bigg(\int_{\Omega^u_{\varepsilon}} \omega \left(T^{\varepsilon} A T^{\varepsilon} \left(\frac{\partial u^{+}}{\partial x_2} \right)^2 + T^{\varepsilon}(u^{+})^2 \right) dx \\
& \qquad + \int_{\Omega^-} (A\nabla u_{\varepsilon}^{-} \cdot \nabla u_{\varepsilon}^{-} + (u_{\varepsilon}^{-})^2)\, dx \Bigg) \\
& = \int_{\Omega^+} \omega f u^{+}\, dx + \int_{\Omega^-} f u^{-}\, dx \\
& = \int_{\Omega^+} \omega \left(A_0^{+} \left(\frac{\partial u^{+}}{\partial x_2} \right)^2 + (u^{+})^2 \right) dx + \int_{\Omega^-} (A\nabla u^{-} \cdot \nabla u^{-} + (u^{-})^2) dx.
\end{aligned}
$$

We have the following convergences:

i. $\left\| T^{\varepsilon}\nabla u_{\varepsilon}^{+} - \left(-\frac{a_{12}}{a_{11}}\frac{\partial u^{+}}{\partial x_2}, \frac{\partial u^{+}}{\partial x_2} \right) \right\|_{L^2(\Omega^u_{\varepsilon})} \to 0,$

ii. $\|T^{\varepsilon} u_{\varepsilon}^{+} - u^{+}\|_{L^2(\Omega^u_{\varepsilon})} \to 0,$

as $\varepsilon \to 0$.

We have the following corollary from Theorem 6.2.

COROLLARY 6.4. *Let u_ε and u be as in Lemma 6.1, then*

i. $\|u_\varepsilon^+ - u^+\|_{L^2(\Omega_\varepsilon^+,\omega)} \to 0,$

ii. $\left\|\nabla u_\varepsilon^+ - \left(-\frac{a_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2}, \frac{\partial u^+}{\partial x_2}\right)\right\|_{L^2(\Omega_\varepsilon^+,\omega)} \to 0,$

iii. $\|u_\varepsilon^- - u^-\|_{H^1(\Omega^-)} \to 0,$

as $\varepsilon \to 0$, *where* $L^2(\Omega_\varepsilon^+,\omega)$ *is the space of functions* v *such that* $v^2\omega \in L^1(\Omega_\varepsilon^+)$.

Proof. By Theorem 6.2,

$$\begin{aligned}
\|u_\varepsilon^+ - u^+\|^2_{L^2(\Omega_\varepsilon^+,\omega)} &= \int_{\Omega_\varepsilon^+} (u_\varepsilon^+ - u^+)^2 \omega \, dx \\
&= \int_{\Omega_\varepsilon^u} (T^\varepsilon u_\varepsilon^+ - T^\varepsilon u^+)^2 T^\varepsilon \omega \, dx dy_1 \\
&\leqslant \int_{\Omega_\varepsilon^u} (T^\varepsilon u_\varepsilon^+ - u^+)^2 \, dx dy_1 + \int_{\Omega_\varepsilon^u} (u^+ - T^\varepsilon u^+)^2 \, dx dy_1 \\
&\to 0,
\end{aligned}$$

as $\varepsilon \to 0$. Similarly, we have the convergences (ii) and (iii). □

REMARK 6.5. Suppose that Ω and Ω^U are open sets with continuous boundary, not necessarily Lipschitz. The solution $u \in H(\Omega,\omega)$ to the limit problem is then understood in the sense (4.3). Then Corollary 6.4 holds as long as (i) $C^\infty(\overline{\Omega})$ is dense in $H(\Omega,\omega)$, and (ii) $\partial\Omega$ and $\partial\Omega^U$ are of finite measure.

For example, suppose that there are $C_1, C_2, C_3 > 0$ such that for some $\gamma > 0$, in the local charts (x_{r1}, x_{r2}), M_r, (c.f. [11, Chapter 6]):

$$C_1\omega^\gamma \leqslant x_{r2} - M_r(x_{r1}) + \kappa_r(x_{r1}) \leqslant C_2\omega^\gamma, \tag{6.3}$$

and $0 \leqslant \kappa_r(x_{r1}) \leqslant C_3$. Then $C^\infty(\overline{\Omega})$ is dense in $H(\Omega,\omega^{\gamma\theta})$, $\theta \geqslant 0$. In particular, $C^\infty(\overline{\Omega})$ is dense in $H(\Omega,\omega)$.

Moreover, under condition (6.3), for $\theta > 1$, $H(\Omega,\omega^{\gamma\theta})$ is continuously embedded into $L^2(\Omega,\omega^{\gamma(\theta-2)})$ (c.f. [11, Chapter 6]). In particular, $H(\Omega,\omega)$ is continuously embedded into $L^2(\Omega)$ if the condition (6.3) holds for any $\gamma \in (0,1/2]$. In conclusion, $u^+ \in L^2(\Omega)$ if (6.3) holds for some $\gamma \in (0,1/2]$.

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