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An iterative algorithm for volume maximization of N-step backward reachable sets for constrained linear time-varying systems*

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***Abstract*— In this paper, we consider the computation of robust N-step backward reachable sets for state- and input constrained linear time-varying systems with additive uncertainty. We propose a method to compute a linear, time-varying control law that maximizes the volume of the robust N-step reachable set for the closed-loop system. The proposed method is an extension of recent developments and involves the recursive solution of N semi-definite programs (SDP). We demonstrate the performance of the proposed method on the lateral control problem for emergency maneuvers of autonomous vehicles and compare it to results obtained when backward reachability is applied to the same system and a naively designed controller.**

I. Introduction

For many types of controlled systems, it is important to derive bounds on the behaviour of the closed-loop system when subjected to bounded disturbances. For such purposes, reachability analysis of constrained linear systems is a commonly employed tool, see e.g. [1] and [2]. Within the automotive domain in particular, the development of autonomous vehicles has raised the need to bound the evolution of the vehicle to robustly ensure that it, e.g., stays in lane and avoids collisions with other road users. In this context, a number of contributions that leverage reachability analysis have been made. For instance, in [3] the set of all possible future occupancies of the autonomous vehicle and other traffic participants is calculated and safety guarantees are given if the occupancy of the autonomous vehicle does not intersect that of the other participants. In [4] and [5] reachability analysis is utilized for threat assessment that quantifies the risk of being involved in an accident at each time step.

In previous work [6], we proposed a safety methodology for automated vehicles that ensures a safety maneuver that leads the vehicle to a pre-defined safe state can be executed. The supervision of the system in this approach relies on N-step backward reachability analysis. In this and similar safety monitoring applications, it is desirable to use the N-step backward reachable set with the largest volume. Failure to do so introduces conservativeness in the safety monitor and causes unnecessary intervention.

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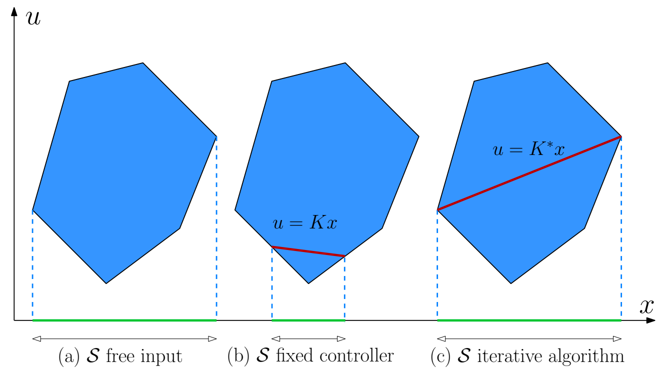

Fig. 1. Example of possible one-step backward reachable set ($\mathcal{S}$) computed with free input (a) using a pre-defined controller and the system closed-loop (b) and using the iterative algorithm (c)

The maximum backward reachable set for a system with state and input constraints can in principle be computed by repeatedly projecting the input constraint set on the state constraint set using Fourier-Motzkin elimination [7]. However, this procedure is known to have double-exponential algebraic complexity, making it intractable for real life application, like automated vehicles.

A way to employ reachability analysis for safety monitors while avoiding prohibitive computational issues, which was used in [6], is to first compute a linear control law $u = Kx$ for the system and compute the N-step backward reachable set for the closed-loop system rather than for its non-autonomous counterpart. Introducing a controller restricts the set of possible inputs and removes the expensive projection operation at the expense of the reachable set volume and thereby conservativeness. How much volume is lost with this approach is dependent on the design of the control gain K. It is thus desirable to find the control gain K^*, under which the backward reachable set for the closed-loop system is maximised. The idea is illustrated in Figure 1, where (a) shows the backward reachable set for the non-autonomous system, where all admissible inputs are considered, (b) shows the set obtained under the restriction of the input space to the output of a "bad" linear controller K, and (c) shows the maximum reachable set for the closed-loop system, obtained with a control gain K^*.

The contribution of this paper is an iterative algorithm that returns a control law such that the backward reachable set is of desirably large volume, approximating K^*. Similar algorithms and approaches have been developed for Robust Control Invariant (RCI) sets, see [8], [9] and [10]. These contributions find low-complexity RCI sets of desirably large volume by solving semidefinite programming problem. The theoretical aspects of the approach in this paper are based on the findings in [8] with the modification of computing N-step backward reachable sets. This leads to different derivations and LMI constraints that the optimization problem is subject to. The iterative algorithm solves a determinant maximization problem to compute backward reachable sets of increased volume at each step together with a state-feedback gain. The approach calculates a set of desirably large volume in comparison to a geometric approach that either is unable to compute a set in reasonable time or the set is under-approximated due to a controller choice.

The remainder of the paper is organized as follows: Section II formulates the problem that is solved in this paper. In Section III the LMI conditions for computing a backward reachable set with the specified control-law are derived, while Section IV presents the iterative algorithm for solving the optimization problem. An example of the computed sets with the proposed approach and a comparison with respect to a standard toolbox is given in Section V followed by final remarks in Section VI.

II. Problem statement

In this paper we consider a discrete time linear time-varying (LTV) system of the following form:

$$x(k+1) = A(k)x(k) + B(k)u(k) + E(k)w(k), \quad (1)$$

where x, u and w are the state, control input, and disturbance vectors respectively. We assume that the variation of the system matrices (A, B and E) over time is known and there is no uncertainty present in their formulation.

The system in eq. (1) is subject to state and input constraints and we assume that the disturbances are bounded and belong to a known set. In essence, we have:

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^n : Fx \leq \mathbf{1}\} \\ \mathcal{U} &= \{u \in \mathbb{R}^m : Gu \leq \mathbf{1}\} \\ \mathcal{W} &= \left\{w \in \mathbb{R}^d : |Dw| \leq \mathbf{1}\right\}, \end{aligned} \quad (2)$$

where n, m and d are the number of states, inputs, and disturbance the system has. The constraint inequalities are element-wise and $\mathbf{1}$ represents a vector of ones.

The main goal of this paper is to find a time-varying state feedback control law of the form:

$$u(k) = K(k)x(k), \quad (3)$$

where $K(k) \in \mathbb{R}^{m\times n}$ is the gain matrix to be found at each time-step, such that the one-step preimage set, denoted by $\mathcal{S}$, of a specified target set $\mathcal{H}$ is maximized. The closed-loop system dynamics now becomes:

$$x(k+1) = (A(k) + B(k)K(k)x(k) + E(k)w(k). \quad (4)$$

Assume that the preimage set and the target set are symmetric with respect to the origin therefore can be described as:

$$\begin{aligned} \mathcal{S} &= \{x \in \mathbb{R}^n : -\mathbf{1} \leq Px \leq \mathbf{1}\} \\ \mathcal{H} &= \left\{x^+ \in \mathbb{R}^n : -\mathbf{1} \leq Hx^+ \leq \mathbf{1}\right\}, \end{aligned} \quad (5)$$

where $P \in \mathbb{R}^{n\times n}$ is a square matrix that we need to find and $H \in \mathbb{R}^{n_x \times n}$ is the predefined target set matrix. For simplicity the time dependence notation (k) is dropped and $x(k+1)$ is substituted by x^+.

The state and input constraints need to be satisfied by the preimage set $\mathcal{S}$, meaning that $\mathcal{S} \subseteq \mathcal{X}$ and $K\mathcal{S} \subseteq \mathcal{U}$, which implies:

$$x \in \mathcal{S} \Rightarrow x \in \mathcal{X} \text{ and } u = Kx \in \mathcal{U}. \quad (6)$$

Taking the definition of the preimage set and the target set in eq. (5), we assume that the following relation holds for the set $\mathcal{S}$ to be a preimage set of the target set $\mathcal{H}$:

$$\{-\mathbf{1} \leq Px \leq \mathbf{1}\} \Rightarrow \left\{-\mathbf{1} \leq Hx^+ \leq \mathbf{1}\right\}, \forall w \in \mathcal{W}. \quad (7)$$

What eq. (7) describes is that if at the current time step the system is inside the set $\mathcal{S}$, for that set to be a one-step preimage set of the target set, the system must be in the set $\mathcal{H}$ at the next time step for all disturbances that belong to the disturbance set ($\mathcal{W}$), given the control law in eq. (3).

Taking these assumptions and formulations the problem in this paper can be summarized as:

***Problem** 1:* At each time-step, given the target set $\mathcal{H}$, find the matrices (P, K) for the system in eq. (1) subject to the constraints in eq. (2) such that:

1) The set $\mathcal{S}$ satisfies eq. (6) and eq. (7).
2) The matrix K, describing the time-varying control law, is such that the volume of the set $\mathcal{S}$ is maximized.

In the following, we will focus on forming the dependencies that fulfill the demands of our problem.

III. LMI conditions for computing a backward reachable set

In this section, we will derive LMI conditions for computing a one-step backward reachable set as specified in Problem 1.

A. Preimage set conditions

We first start by forming LMI conditions for the preimage set conditions by expanding eq. (7) to:

$$\{-\mathbf{1} \leq Px \leq \mathbf{1}\} \Rightarrow \{-\mathbf{1} \leq H(Ax + Bu + Ew) \leq \mathbf{1}\}, \quad \forall w \in \mathcal{W}. \quad (8)$$

Given the control law in eq. (3) that we can express the equation above as:

$$\{-\mathbf{1} \leq Px \leq \mathbf{1}\} \Rightarrow \{-\mathbf{1} \leq H((A + BK)x + Ew) \leq \mathbf{1}\}, \quad \forall w \in \mathcal{W}. \quad (9)$$

For the definition of a preimage set to hold, we need to ensure that for each $x \in \mathcal{S}$, each element of $H(:,i)x_i^+, i = 1,...,n$ must satisfy:

$$1 - (H(:,i)x_i^+)^2 = 1 - (e_i^T H x^+)^2 \geq 0, \quad i = 1,...,n, \tag{10}$$

where e_i is the i-th column of the identity matrix with size $n \times n$. We can get a sufficient condition for the inequality in eq. (10) by multiplying this inequality with a positive scalar variable (Φ_i) and by having this product to be greater or equal to an expression that is known to be non-negative for a disturbance $w \in \mathcal{W}$. This way we obtain a sufficient condition as:

$$\Phi_i \left(1 - \left(e_i^T H x^+\right)^2\right) \geq (\mathbf{1} - Px)^T \Gamma_i (\mathbf{1} + Px) + (\mathbf{1} - Dw)^T \Lambda_i (\mathbf{1} + Dw), \tag{11}$$

with $\Gamma_i = \text{diag}(\gamma_{i1},...,\gamma_{in}) \succeq 0$, $\Lambda_i = \text{diag}(\lambda_{i1},...,\lambda_{in}) \succeq 0$ and $\Phi_i \in \mathbb{R}^+$. Using standard manipulations we can express eq. (11) as:

$$\underbrace{\begin{bmatrix}\mathbf{1}\\ x\\ x^+\\ w\end{bmatrix}^T}_{\theta^T} \underbrace{\begin{bmatrix}\beta_1 & 0 & 0 & 0\\ 0 & P^T\Gamma_i P & 0 & 0\\ 0 & 0 & -\beta_2 & 0\\ 0 & 0 & 0 & D^T\Lambda_i D\end{bmatrix}}_{M} \underbrace{\begin{bmatrix}\mathbf{1}\\ x\\ x^+\\ w\end{bmatrix}}_{\theta} \succeq 0,$$
$$\forall \begin{bmatrix}0 & -(A+BK) & I & -E\end{bmatrix}\theta = 0, \tag{12}$$

where $\beta_1 = \Phi_i - \mathbf{1}^T\Gamma_i\mathbf{1} - \mathbf{1}^T\Lambda_i\mathbf{1}$ and $\beta_2 = \Phi_i H^T e_i e_i^T H$. As we can notice there are two non-linear terms present in eq. (12), the $P^T\Gamma_i P$ and β_2 terms. We start with the linearization of the β_2 term, by first applying the Finsler's lemma:

Lemma 1: (Finsler) [11]: Let $x \in \mathbb{R}^n, Q \in \mathbb{S}^n$ and $L \in \mathbb{R}^{m\times n}$ such that rank(L) $< n$. The following statements are equivalent:

i) $x^T Q x \succeq 0, \ \forall Lx = 0, \ x \neq 0$.
ii) $\left(L^\perp\right)^T Q L^\perp \succ 0$.
iii) $\exists \mu \in \mathbb{R} : Q - \mu L^T L \succ 0$.
iv) $\exists X \in \mathbb{R}^{n\times m} : Q + XL + L^T X^T \succ 0$.

The proof of *Lemma 1* is given in [11].

In particular we use the fact that i) and iv) are equivalent, where for our inequality in eq. (12) we have that $x = \theta$ and $Q = M$. Further for iv) we define:

$$L = \begin{bmatrix}0 & -(A+BK) & I & -E\end{bmatrix}, \quad X = \begin{bmatrix}0\\0\\S\\0\end{bmatrix}, \tag{13}$$

where S is a new optimization variable. We can notice that with L defined in this way the condition in i) of the *Lemma 1*, which is $\forall Lx = 0$, holds as this is the system's dynamics as stated in eq. (4). Using statement iv) from *Lemma 1* with the defined vectors and also applying the congruence transformation [12] on the third diagonal element of M matrix, gives a new expression for M:

$$\begin{bmatrix}\beta_1 & 0 & 0 & 0\\ 0 & P^T\Gamma_i P & * & 0\\ 0 & (A+BK) & S^{-1}+S^{-T}-S^{-1}\beta_2 S^{-T} & E\\ 0 & 0 & * & D^T\Lambda_i D\end{bmatrix} \succeq 0 \tag{14}$$

Finally, we apply the Schur complement w.r.t. the third diagonal element of M matrix and with that get the following two conditions:

$$z_i \geq S^{-1}\Phi_i H^T e_i e_i^T H S^{-T} \Rightarrow \begin{bmatrix}z_i & S^{-1}H^T e_i\\ * & \Phi_i^{-1}\end{bmatrix} \succeq 0 \tag{15}$$

$$\begin{bmatrix}\beta_1 & 0 & 0 & 0\\ 0 & P^T\Gamma_i P & * & 0\\ 0 & (A+BK) & S^{-1}+S^{-T}-z_i & E\\ 0 & 0 & * & D^T\Lambda_i D\end{bmatrix} \succeq 0. \tag{16}$$

The $P^T\Gamma_i P$ term is linearized by again applying the congruence transformation on the M matrix, this time on the second diagonal element of the matrix, which results in:

$$\begin{bmatrix}\beta_1 & 0 & 0 & 0\\ 0 & \Gamma_i & * & 0\\ 0 & (AW+BN) & S^{-1}+S^{-T}-z_i & E\\ 0 & 0 & * & D^T\Lambda_i D\end{bmatrix} \succeq 0, \tag{17}$$

where $W = P^{-1}$ and $N = KW$ are new optimization variables. With this, we have obtained LMI conditions for the calculation of the one-step preimage set.

B. System constraints

With W and N definied as above, we now need to formulate the input constraints to an LMI form. The input constraints can be stated as:

$$\begin{aligned} GKx &\leq \mathbf{1}, \ \forall x \in \mathcal{S}\\ GNW^{-1}x &\leq \mathbf{1}, \ \forall x \in \mathcal{S}, \end{aligned} \tag{18}$$

where $\mathcal{S} = \left\{x \in \mathbb{R}^n : -\mathbf{1} \leq W^{-1}x \leq \mathbf{1}\right\}$. By applying the S-procedure [13] we reformulate eq. (18) to:

$$e_j^T(\mathbf{1} - GNW^{-1}x) \geq (\mathbf{1} - W^{-1}x)^T \Theta_i (\mathbf{1} + W^{-1}x), \tag{19}$$

or in matrix form:

$$\begin{bmatrix}\mathbf{1}\\ x\end{bmatrix}^T \begin{bmatrix}\mathbf{1} - \mathbf{1}^T\Theta_i\mathbf{1} & \frac{1}{2}e_j^T GNW^{-1}\\ * & W^{-T}\Theta_i W^{-1}\end{bmatrix} \begin{bmatrix}\mathbf{1}\\ x\end{bmatrix} \succeq 0, \tag{20}$$

where e_j is the j-th column of the identity matrix with size $m \times m$ and $\Theta_i = diag(\theta_{i1},...,\theta_{in}) \succeq 0$. By applying the congruence transformation on the second diagonal element, we obtain the LMI condition for the input constraints as:

$$\begin{bmatrix}\mathbf{1} - \mathbf{1}^T\Theta_i\mathbf{1} & \frac{1}{2}e_j^T GN\\ * & \Theta_i\end{bmatrix} \succeq 0. \tag{21}$$

The state constraints can be handled by intersecting the preimage set with the state constraints set $\mathcal{X}$. Following the intersection, all redundant inequalities are removed and we obtain the minimal-representation of the set. An approach that describes how to find the minimal-representation of a polyhedron is presented in [14] and is used in this paper.

IV. Iterative algorithm

In this section, we will present an iterative algorithm that will maximize the volume of $\mathcal{S}$. As noted in [15] the volume of $\mathcal{S}$ is proportional to $|det(W)|$ and therefore we need to solve a determinant maximization problem subject to LMI constraints in order to obtain the set of enlarged volume. In [16] it is found that to solve such an optimization problem easily W must be symmetric. However, this will introduce conservatism and is thus undesirable.

In order to mitigate this we propose an iterative scheme such that the consecutive solution fulfills the following condition:

$$\left|\det\left(W^{r+1}\right)\right| \geq \left|\det\left(W^{r}\right)\right|. \tag{22}$$

We now introduce a new matrix variable Z required to satisfy:

$$W^T W \succeq Z \succ 0. \tag{23}$$

From the Minkowski determinant inequality it follows that:

$$\det(W^T W) = |\det(W)|^2 \geq \det(Z). \tag{24}$$

As eq. (23) is not an LMI, we can substitute it with:

$$W^T W^r + (W^r)^T W - (W^r)^T W^r \succeq Z \succ 0, \tag{25}$$

with W^r being the solution found from the previous iteration of the algorithm. In [8] it is proven that eq. (25) is a sufficient condition to eq. (23), so the goal of the algorithm is to maximize the determinant of Z.

The iterative algorithm needs an initial optimization in order to compute the W matrix for the first iteration of maximizing the determinant of Z. In this initial optimization, we maximize the determinant of $W + W^T$ in order to avoid conservatism due to a symmetric W.

A stopping criterion for the algorithm can be when the volume of the set $\mathcal{S}$ through W in the next iteration is not significantly increased with respect to the last iteration. This can be formulated as:

$$\left|\det\left(W^{r+1}\right)\right| - \left|\det\left(W^{r}\right)\right| \leq \epsilon \quad \text{or} \quad r > \eta, \tag{26}$$

where $\epsilon > 0$ is desirably small and η is the maximum amount of iterations we specify. The iterative algorithm is summarized in Algorithm 1.

Remark 1: At each iteration, the algorithm solves a generalized SDP. The algorithm would converge to a stationary point, however, this might not be the global optimum of the volume maximization problem under the constraints. The termination criterion must be selected such that numerical problems are avoided as well as an early termination.

In this paper, we are interested in obtaining an N-step backward reachable set. For that cause, we repeat Algorithm 1 N times where in each next step the target set for that step is the preimage set found in the N-1 step.

Algorithm 1 Calculate (P, K)

Input: $A, B, E, H, F, G, D, \epsilon, \eta$
Output: One-step backward reachable set $\mathcal{S}$ and control-law matrix K

```
for r = 1 : η
    if r = 1 then
        maximize  logdet(W + W^T)
        subject to eq. (15), (17), (21).
    else
        maximize  logdet(Z)
        subject to eq. (15), (17), (21) (25).
    end if
    if |det(W^{r+1})| − |det(W^r)| ≤ ε then
        break
    end if
end for
P = W^{-1}
S = {−1 ≤ Px ≤ 1} ∩ X
Obtain minimal-representation of S as per [14]
K = NW^{-1}
```

V. Example

This section gives an example of using Algorithm 1 to compute the N-step backward reachable set for a standard bicycle model for vehicle lateral dynamics, depicted in Figure 2. It also gives a comparison to the computed N-step backward reachable set using the Multi-Parametric Toolbox (MPT3) [17] with an LQR controller designed for the system. The continuous-time vehicle model that is used is similar to [18] and is described by the following equations:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & V_x & 0 \\ 0 & \frac{-171.2893}{V_x} & 0 & \frac{85.2523 - V_x^2}{V_x} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{42.1875}{V_x} & 0 & \frac{-199.6488}{V_x} \end{bmatrix}}_{A_c} \underbrace{\begin{bmatrix} e_y \\ \dot{y} \\ e_\psi \\ \dot{\psi} \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 65.8919 & 0 \\ 0 & 0 \\ 43.6411 & 0.2287 \end{bmatrix}}_{B_c} \underbrace{\begin{bmatrix} \delta \\ \mu_b \end{bmatrix}}_{u} + \underbrace{\begin{bmatrix} 0 \\ 0.0018 \\ 0 \\ -0.0022 \end{bmatrix}}_{E_c} \underbrace{V_w^2}_{w}, \tag{27}$$

where V_x is the longitudinal velocity $[m/s]$, e_y $[m]$ is the lateral error, $\dot{y}$ $[m/s]$ is the lateral velocity, e_ψ $[rad]$ is the orientation error, $\dot{\psi}$ $[rad/s]$ is the yaw rate, δ $[rad]$ is the steering angle, $\mu_b = 10^{-3} \cdot M_b$ where M_b $[Nm]$ is the braking yaw moment and V_w $[m/s]$ is the wind velocity. In this example, we assume that the vehicle performs a known maneuver, as described in [6], and thus we know the evolution of the speed over time. For this example, we take that the vehicle performs a constant braking maneuver starting from 60 $[km/h]$ with a constant deceleration of 3.3 $[m/s^2]$. The matrix A_c is dependent on the longitudinal velocity, V_x and thus varies over time with the change of the

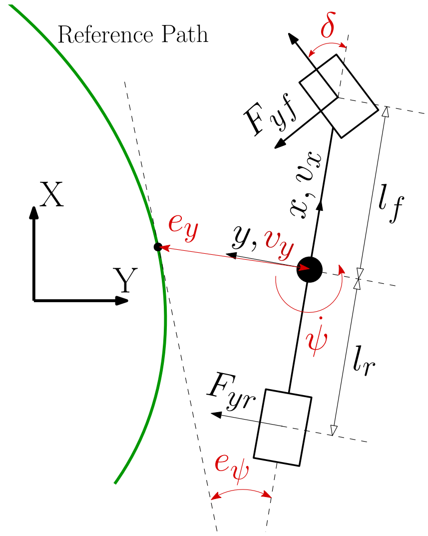

Fig. 2. Vehicle lateral dynamics

velocity. The state and input constraints used in this example are:

$$|e_y| \leq 0.4\ m, \quad |\dot{y}| \leq 3\ m/s \ \ |e_\psi| \leq \frac{10 \cdot \pi}{180} \text{ rad},$$
$$|\delta| \leq \frac{5 \cdot \pi}{180} \text{ rad}, \ \ |M_b| \leq 10^3 \text{ Nm} \ \ |V_w| \leq 10\ m/s. \quad (28)$$

The presented algorithm requires a discrete-time system and for that purpose using the zero-order hold discretization method with a sample time of 0.1 [s], we discretize the system in eq. (27) and obtain the A, B and E matrices. For this example, we have selected N = 30 for the amount of backward reachable sets to be computed. The target set for this example is equivalent to the state constraints set.

For the computations of the N-step backward reachable sets with MPT3, as explained before we first design an LQR controller for the system in eq. (27) with the weights $Q = R = I$ and then by using the discrete-time closed-loop system (obtained using the zero-order hold technique) and the same state, input and disturbance constraints as in eq. (28) we compute the backward reachable sets using the standard MPT3 commands.

Figure 3 (a) shows the projection in different dimensions of the given target set depicted with the bright green set, the one-step backward reachable set depicted with the blue set, and the N-step backward reachable set which is depicted with the red set both of which were computed using Algorithm 1. Part (b) of the figure depicts the sets at the same N iteration as (a) computed using the MPT3 algorithm for the computed LQR control law for the system. The sets contract for every N-step due to the disturbance acting on the system and because the control is bounded.

From Figure 3 the difference between the computed sets is evident for this problem and scenario. One can of course try to "blindly" tune the controller such that the sets would be larger than the ones obtained in this example with MPT3, however, there is no clear intuition of how to do that. With the proposed approach in this paper, we are able to compute less conservative sets and a time-varying control law that would lead us to the desired target set. The volume of the obtained N-step backward reachable set in this example (computed using the "Polyhedron.volume" command) for the iterative Algorithm 1 is 0.0153, whereas the volume of the N-step backward reachable set using MPT3 is 0.0009 which further illustrates the difference between these sets. Figure 4 depicts the increase of $|\det(W)|$ during the algorithm iterations for one of the N-backward reachable sets (in this particular case N = 27), which, as noted, corresponds to the increase of volume for the backward reachable set. We can notice the gradual increase until the tolerance for convergence is met.

Algorithm 1 was implemented in the YALMIP environment using a MOSEK solver on a 2.90GHZ Intel Xeon computer with 32GB of RAM. The total computational time for Algorithm 1 is 3935.31 seconds, while the total time of the computations using the MPT3 software is 12.82 seconds. We note that for our application as described in [6], the computation of the backward reachable sets is performed offline. For completeness, we note that attempting to compute the backwards reachable sets for the non-autonomous system with MPT3 on the same computer only reached 21-steps in 24 hours, which as noted in section I is due to the complexity of the Fourier-Motzkin elimination. Note that this computation would be even more demanding for a more complex system.

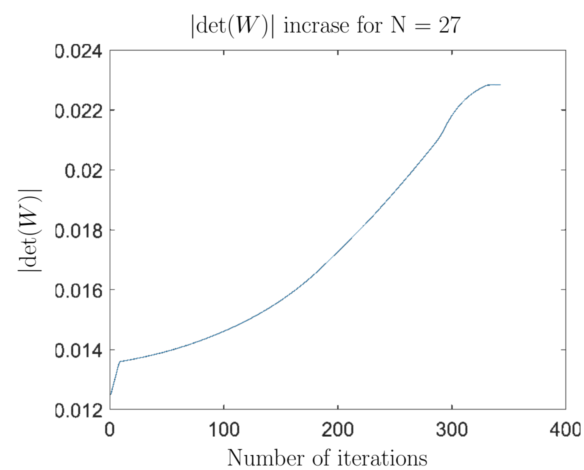

Fig. 4. Increase of $|\det(W)|$ during the algorithm iterations for one of the N-backward reachable sets (N = 27)

VI. Conclusions and Future Work

In this paper, we presented an iterative algorithm with LMI constraints for LTV systems for computing desirably large N-step backward reachable sets along with a time-varying control law. The approach is beneficial when backward reachable sets cannot be computed for the non-autonomous system. We demonstrated that the approach leads to less conservative sets for a realistic application when compared to those obtained with a naively tuned controller.

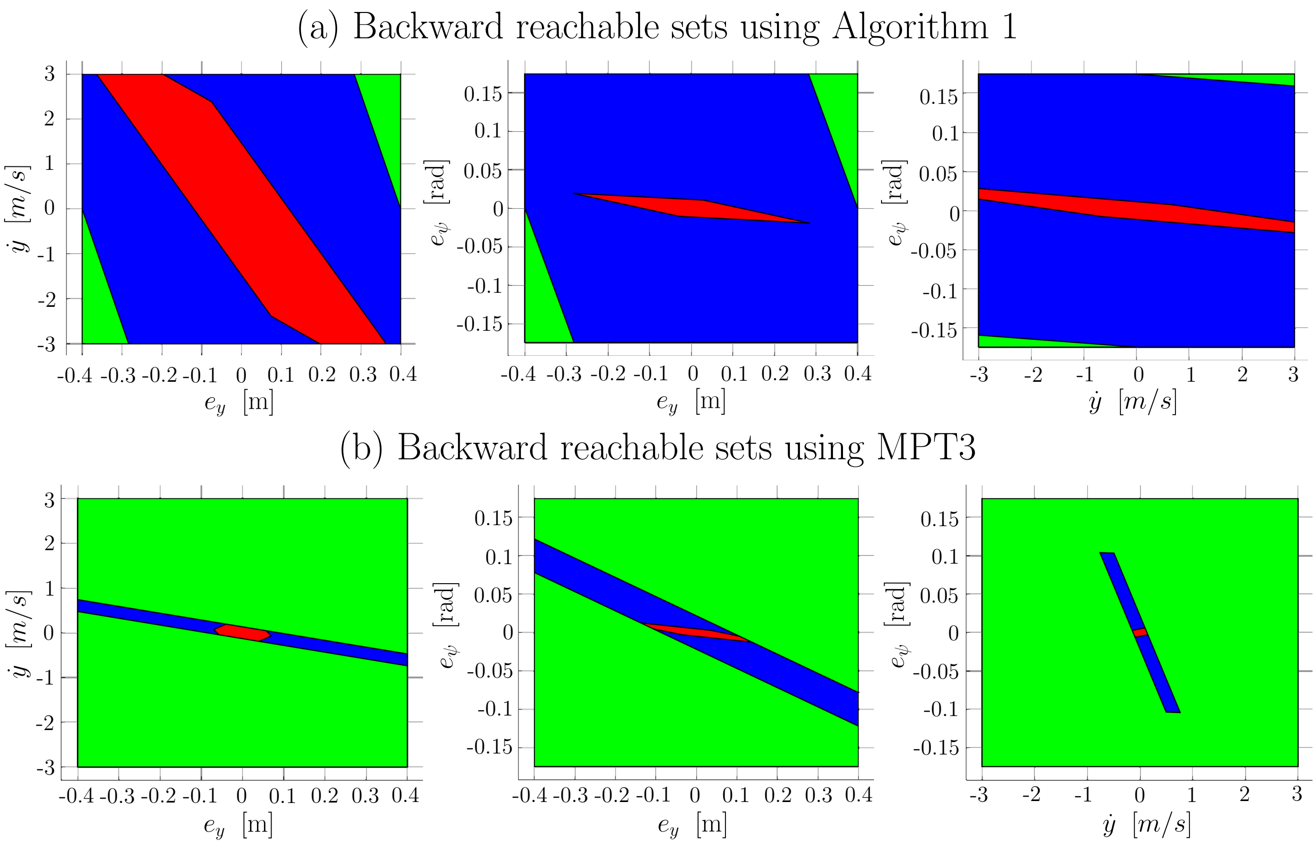

Fig. 3. Target set and backward reachable sets computed (a) using Algorithm 1 and (b) using MPT3.

Future extensions of the approach is to have a more complex control law that could further increase the volume of the backward reachable sets. Furthermore, it is interesting to investigate if it is possible to express the LMI conditions when the sets are not necessarily centered around the origin and increasing the dimension of the backward reachable set, this could further contribute to volume increase. Considering uncertain parameter varying systems is also part of future extensions to this work.

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