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# Crystallographic Groups, Strictly Tessellating Polytopes, and Analytic Eigenfunctions

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**Abstract.** The mathematics of crystalline structures connects analysis, geometry, algebra, and number theory. The planar crystallographic groups were classified in the late 19th century. One hundred years later, Bérard proved that the fundamental domains of all such groups satisfy a very special analytic property: the Dirichlet eigenfunctions for the Laplace eigenvalue equation are all trigonometric functions. In 2008, McCartin proved that in two dimensions, this special analytic property has both an equivalent algebraic formulation, as well as an equivalent geometric formulation. Here we generalize the results of Bérard and McCartin to all dimensions. We prove that the following are equivalent: the first Dirichlet eigenfunction for the Laplace eigenvalue equation on a polytope is real analytic, the polytope strictly tessellates space, and the polytope is the fundamental domain of a crystallographic Coxeter group. Moreover, we prove that under any of these equivalent conditions, all of the eigenfunctions are trigonometric functions. To conclude, we connect these topics to the Fuglede and Goldbach conjectures and give a purely geometric formulation of Goldbach's conjecture.

**1. INTRODUCTION.** In *The Grammar of Ornament*, published in 1856, Owen Jones wrote [20]:

Whenever any style of ornament commands universal admiration, it will always be found to be in accordance with the laws which regulate the distribution of forms in nature.

In the case of crystals, the laws that regulate their shape are dictated by the crystallographic groups.

**Crystallographic groups.** A crystal or crystalline solid is a solid material whose constituents, such as atoms, molecules, or ions, are arranged in a highly ordered microscopic structure; for a two-dimensional example, see Figure 1. The crystal is often described in terms of its *symmetries*, those isometries of the ambient space under which the crystal remains unchanged. The three basic types of isometries of  $\mathbb{R}^n$  are translations, rotations, and reflections. These form a group under composition. The patterns in Figure 2 have symmetry groups that are *plane crystallographic groups*. These are subgroups of the group of isometries of the plane that are topologically discrete and contain two linearly independent translations. Equivalently, a plane crystallographic group<sup>1</sup> is a co-compact subgroup of the group of isometries of the plane. A subgroup

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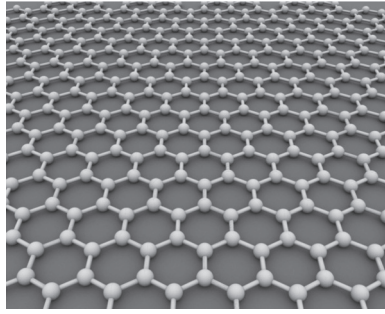
<sup>1</sup>These are also known as wallpaper groups.

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**Figure 1.** Graphene is an allotrope of carbon in the form of a two-dimensional, atomic-scale hexagonal lattice such that each point in the lattice corresponds to an atom. This image is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license at [commons.wikimedia.org/wiki/File:Graphen.jpg](https://commons.wikimedia.org/wiki/File:Graphen.jpg).

in this context is called co-compact if the quotient space  $\mathbb{R}^2 / \Gamma$  by the subgroup,  $\Gamma$ , is compact. The classification of these groups, up to equivalence, was achieved at the end of the 19th century by E. S. Fedorov [12–15] and A. Schoenflies [32–35]; for English references, see [15, 18, 36]. Two planar crystallographic groups are equivalent if they are isomorphic as abstract groups; equivalently if they are conjugate in the group of affine transformations of  $\mathbb{R}^2$ . In two dimensions, up to this notion of equivalence, there are seventeen crystallographic groups.

One can also consider crystals in three dimensions, and mathematically we may generalize all of these notions to  $\mathbb{R}^n$ . An  $n$ -dimensional crystallographic group is a discrete group of isometries of  $\mathbb{R}^n$  that is co-compact. Fedorov [12–15] and Schoenflies [32–35] proved that there are, up to equivalence, 219 crystallographic groups in  $\mathbb{R}^3$ . Two crystallographic groups in  $\mathbb{R}^n$  are equivalent if they are conjugate in the group of affine transformations of  $\mathbb{R}^n$ ; equivalently if they are isomorphic as abstract groups. In 1910, Bieberbach proved that, for any  $n$ , there are only finitely many  $n$ -dimensional crystallographic groups up to equivalence [3, 4], thereby solving Hilbert’s 18th problem; for an English reference, see [18, 36]. However, for general  $n$ , the precise number of crystallographic groups up to isometry in  $\mathbb{R}^n$  is unknown. In four dimensions, it was not known until the 1970s that there are 4783 crystallographic groups up to isometry [7]. Can one obtain upper and lower bounds for the number of crystallographic groups up to isometry in  $\mathbb{R}^n$  which depend on  $n$ ? If so, does the lower bound tend to infinity, or is there a uniform upper bound? For higher dimensions, the classification is still in progress; a nonexhaustive list of recent results includes [8, 28, 31].

**Strictly tessellating polytopes and our main result.** The constituents of a crystal create a perfectly regular pattern. Another way to create a perfectly regular pattern is by “strict tessellation.” This is a notion specific to polytopes.

**Definition 1.** The set of all *one-dimensional polytopes* is the set of all bounded open intervals

$$\wp_1 := \{(a, b) : -\infty < a < b < \infty\}.$$

A *domain* here is a connected, open set. Inductively, we define the set of polytopes  $\wp_n$  in  $\mathbb{R}^n$  for  $n \geq 2$  to be the set of bounded domains  $\Omega \subset \mathbb{R}^n$  such that

$$\partial\Omega = \bigcup_{j=1}^m \overline{P_j}, \quad P_j \cong Q_j \in \wp_{n-1}.$$



**Figure 2.** Two Egyptian patterns whose symmetry groups are planar crystallographic groups. These patterns were documented by Owen Jones in 1856 [20, Egyptian No. 7 (plate 10), images 8 and 13]. These images were obtained from Wikimedia Commons and are in the public domain in their country of origin and other countries and areas where the copyright term is the author’s life plus 100 years or fewer; this includes the United States commons.wikimedia.org/wiki/Commons:Copyright\_tags/Country-specific\_tags#United\_States\_of\_America.

Here, the boundary of  $\Omega$  consists of the closures of  $(n - 1)$ -dimensional polytopes,  $P_j$ . Each  $P_j$  is contained in an  $(n - 1)$ -dimensional hyperplane, which is a set of the form

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{M} \cdot \mathbf{x} = b\},$$

for some fixed  $\mathbf{M} \in \mathbb{R}^n$  and  $b \in \mathbb{R}$ . The meaning of  $P_j \cong Q_j$  is that the hyperplane above is isometrically identified with  $\mathbb{R}^{n-1}$ , and with this identification  $P_j$  is isometrically identified with  $Q_j$ . Note that our definition of polytope makes no assumption of convexity; polytopes as defined here can be nonconvex.

Next we introduce the notion of a strict tessellation. We are not aware of the term “strict tessellation” in the literature, but it might be known under a different name. An example of a strict tessellation of the plane is given in Figure 3; a tessellation of the plane which is not strict is given in Figure 4.

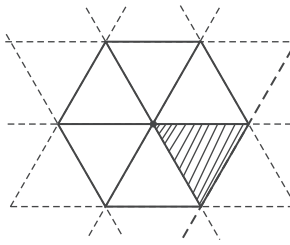
**Definition 2.** A polytope  $\Omega \in \wp_n$  strictly tessellates  $\mathbb{R}^n$  if

1.  $\mathbb{R}^n = \bigcup_{j \in \mathbb{Z}} \overline{\Omega_j}$ , such that each  $\Omega_j$  is isometric to  $\Omega$ , and  $\Omega_j \cap \Omega_k = \emptyset$  for any  $j \neq k$ .
2. Let  $m$  be the number of boundary faces of  $\Omega$ , and let  $\{\mathcal{H}_{j,i}\}_{i=1}^m$  be the corresponding  $m$  hyperplanes containing the  $m$  boundary faces of  $\Omega_j$ . Then  $\mathcal{H}_{j,i} \cap \Omega_k = \emptyset$  for all  $1 \leq i \leq m$ , and for all  $j$  and  $k \in \mathbb{Z}$  (including  $k = j$ ). Note that this immediately implies that the polytope is convex.
3. For each  $k \neq j$ , for some  $N \in \mathbb{N}$ ,  $\Omega_k = R_N \circ \dots \circ R_1(\Omega_j)$ . Here,  $R_1$  is reflection across one of the boundary faces of  $\Omega_j$ . For  $I \geq 2$ ,  $R_I$  is reflection across a boundary face of  $R_{I-1} \circ \dots \circ R_1(\Omega_j)$ .

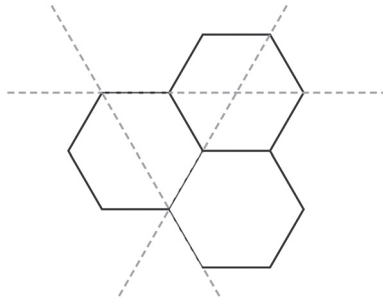
For any real numbers  $a < b$ ,

$$\mathbb{R} = \bigcup_{j \in \mathbb{Z}} \overline{\Omega_j}, \quad \text{where } \Omega_j := (j(b - a) + a, j(b - a) + b).$$

In this case, the boundary faces are points,  $\{j(b - a) + a, j(b - a) + b\}_{j \in \mathbb{Z}}$ , and therefore the hyperplanes that contain these faces are simply the points themselves. This



**Figure 3.** Equilateral triangles are shown here to strictly tessellate the plane.



**Figure 4.** Although it is well known that regular hexagons tessellate the plane by reflection, the tessellation is not strict, because the lines that contain the edges of the hexagon cut through the interior of the reflected copies.

shows that conditions (1) and (2) above are satisfied. Moreover, for any  $k \neq j$ , for example  $k = j + \ell$ , if  $\ell > 0$ , then  $\Omega_k$  is obtained by reflecting across the boundary faces  $(j + i)(b - a) + b$  for  $i = 0, \dots, \ell - 1$ . If  $\ell < 0$ , then  $\Omega_k$  is obtained by reflecting across the boundary faces  $(j + i)(b - a) + a$  for  $i = 0, \dots, \ell - 1$ . Consequently, every element of  $\wp_1$  strictly tessellates  $\mathbb{R}$ .

In 2008, McCartin proved a remarkable classification theorem [30], connecting geometry and analysis. Recall that the Laplacian on  $\mathbb{R}^n$  is the partial differential operator

$$\Delta := - \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}.$$

The Laplace eigenvalue problem for a domain  $\Omega \subset \mathbb{R}^n$  with the Dirichlet boundary condition is to find all functions  $u : \overline{\Omega} \rightarrow \mathbb{C}$  that are not identically zero and satisfy

$$\Delta u(\mathbf{x}) = \lambda u(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega, \quad \text{for some constant } \lambda, \quad \text{and } u|_{\partial\Omega} = 0.$$

This is a difficult problem, because in general it is impossible to compute the numbers  $\lambda$ . However, using the tools of functional analysis [9] one can prove that these eigenvalues are discrete and positive and therefore can be ordered, counting multiplicity, as

$$0 < \lambda_1 < \lambda_2 \leq \dots \uparrow \infty.$$

Since we define all domains here to be connected, the first eigenvalue is simple, and its corresponding eigenfunction is uniquely defined, up to multiplication by scalars. In this way we may speak of the first eigenfunction that has eigenvalue  $\lambda_1$ . In one dimension, by Definition 1, a polytope is a bounded open interval  $(a, b)$  for some

real numbers  $a < b$ . The Laplace eigenvalue equation with the Dirichlet boundary condition on such a polytope is to find all functions  $u$  defined on  $[a, b]$  such that there exists  $\lambda \in \mathbb{C}$  with

$$-u''(x) = \lambda u(x), \quad a < x < b, \quad u(a) = u(b) = 0.$$

This is a classical ordinary differential equation (see [6]), and all solutions to this equation are precisely (up to multiplication by constants) given by

$$u_k(x) = \sin\left(\frac{x-a}{b-a}k\pi\right), \quad \lambda_k = \frac{k^2\pi^2}{(b-a)^2}, \quad k \in \mathbb{N}.$$

These eigenfunctions are trigonometric functions. We can also define trigonometric functions on  $\mathbb{R}^n$ .

**Definition 3.** An eigenfunction  $u : \mathbb{R}^n \rightarrow \mathbb{C}$  for the Laplacian is *trigonometric* if it can be expressed as a finite sum of trigonometric functions

$$u(\mathbf{x}) = \sum_{j=1}^m a_j \sin(\mathbf{L}_j \cdot \mathbf{x}) + b_j \cos(\mathbf{M}_j \cdot \mathbf{x}).$$

Here,  $a_j, b_j, \in \mathbb{C}$  and  $\mathbf{L}_j, \mathbf{M}_j \in \mathbb{R}^n$  satisfy  $\|\mathbf{L}_j\|^2 = \|\mathbf{M}_j\|^2 = \lambda$  for all  $j = 1, \dots, m$ , where  $\lambda$  is the eigenvalue corresponding to  $u$ .

**Remark 1.** Since

$$\cos(t) = \sin(t + \pi/2), \quad \text{for all } t \in \mathbb{R},$$

it is equivalent to define a trigonometric eigenfunction to be a function of the form

$$u(\mathbf{x}) = \sum_{j=1}^m a_j \sin(\mathbf{L}_j \cdot \mathbf{x} + \phi_j).$$

Here,  $a_j \in \mathbb{C}$ ,  $\mathbf{L}_j \in \mathbb{R}^n$ ,  $\phi_j \in \{0, \frac{\pi}{2}\}$ , and  $\|\mathbf{L}_j\|$  are the same for all  $j = 1, \dots, m$ . We note that some authors refer to these functions as “quasi-periodic.”

In general, it is impossible to compute the eigenfunctions of an arbitrary polygonal domain. Nonetheless, McCartin proved the following classification theorem which shows the equivalence of the analytic property, having trigonometric eigenfunctions, with the geometric property, strictly tessellating.

**Theorem 1** (McCartin [30]). *Assume that  $\Omega$  is a polygonal domain in the plane (a two-dimensional polytope). Then the following are equivalent:*

1.  $\Omega$  has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue problem with the Dirichlet boundary condition.
2.  $\Omega$  strictly tessellates the plane.
3.  $\Omega$  is one of the following: a rectangle, an isosceles right triangle, an equilateral triangle, or a hemi-equilateral triangle, also known as 30-60-90 triangle because its interior angles have degree measures 30, 60, and 90.

We note that if any of the above three conditions are satisfied, it follows immediately that  $\Omega$  is convex.

**Remark 2.** The Laplace eigenfunctions for a rectangular domain with vertices at the points  $(0, 0)$ ,  $(a, 0)$ ,  $(0, b)$ , and  $(a, b)$  with the Dirichlet boundary condition can be computed using separation of variables, which reduces the problem to two one-dimensional problems. The resulting eigenfunctions are indexed by  $m, n \in \mathbb{N}$ . For Cartesian coordinates  $\mathbf{x} = (x, y) \in \mathbb{R}^2$ , the eigenfunctions are

$$u_{m,n}(x, y) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right).$$

Using trigonometric identities, we have

$$u_{m,n}(x, y) = \frac{1}{2} \left[ \cos\left(\left[\begin{array}{c} \frac{m\pi}{a} \\ -\frac{n\pi}{b} \end{array}\right] \cdot \mathbf{x}\right) - \cos\left(\left[\begin{array}{c} \frac{m\pi}{a} \\ \frac{n\pi}{b} \end{array}\right] \cdot \mathbf{x}\right) \right].$$

Consequently, these are trigonometric eigenfunctions.

Our main result is a generalization to all dimensions.

**Theorem 2.** *Assume that  $\Omega$  is a polytope in  $\mathbb{R}^n$ . Then the following are equivalent:*

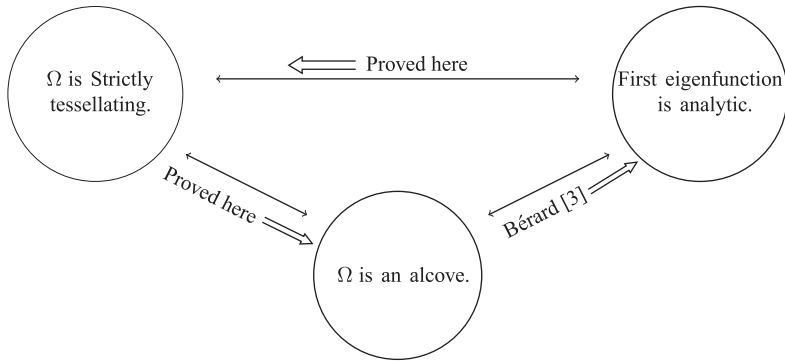
1. *The first eigenfunction for the Laplace eigenvalue equation with the Dirichlet boundary condition extends to a real analytic function on  $\mathbb{R}^n$ .*
2.  *$\Omega$  strictly tessellates  $\mathbb{R}^n$ .*
3.  *$\Omega$  is congruent to a fundamental domain of a crystallographic Coxeter group as defined in Bourbaki [5, VI.25, Proposition 9, p. 180], and is also known as an alcove [2, p. 179]; see also Section 3.*

The three equivalent statements in Theorem 2 are respectively analytic, geometric, and algebraic. These statements and how they were proved are depicted in Figure 5. Our work therefore reveals an intimate connection between analysis, geometry, and algebra. Moreover, combining our theorem with Bérard's proposition, see [2, Proposition 9, p. 181] or Proposition 2, we obtain the following rather remarkable result.

**Corollary 1.** *Assume that  $\Omega$  is a polytope in  $\mathbb{R}^n$ . If the first eigenfunction for the Laplace eigenvalue equation with the Dirichlet boundary condition extends to a real analytic function on  $\mathbb{R}^n$ , then it is a trigonometric eigenfunction. Moreover, in that case, all the eigenfunctions of  $\Omega$  are trigonometric.*

**Remark 3.** Every trigonometric eigenfunction satisfies the first condition of Theorem 2. However, there are many functions that satisfy this condition but are not trigonometric. Examples include the eigenfunctions for a disk in  $\mathbb{R}^2$  that are products of Bessel functions and trigonometric functions. There is no contradiction with Corollary 1 because a disk is not a polygonal domain.

**Organization.** In Section 2, we prove that if the first eigenfunction of a polytope satisfies the hypotheses of Theorem 2, then the polytope strictly tessellates  $\mathbb{R}^n$ . We prove this by generalizing classical results of Lamé [26]. In Section 3, we introduce the notions of root systems and alcoves and prove that all polytopes that strictly tessellate  $\mathbb{R}^n$  are alcoves. We then recall the result of Bérard [2]: all alcoves have a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition. These results together complete the proofs of Theorem 2 and Corollary 1. In Section 4, we discuss connections to the Fuglede and Goldbach conjectures. We make our own conjecture and conclude with a purely geometric conjecture which is equivalent to the strong Goldbach conjecture.



**Figure 5.** This diagram shows the three statements of [Theorem 2](#) and how they were proved.

**2. THE FIRST EIGENFUNCTION AND STRICT TESSELLATION.** There is no known method to explicitly compute the eigenfunctions and eigenvalues for an arbitrary polytope. However, using the tools of functional analysis, one can prove general facts about them. We summarize briefly here. Here a domain refers to an open, connected set. For the Dirichlet boundary condition for the Laplace eigenvalue equation on a bounded domain,  $\Omega \subset \mathbb{R}^n$ , the eigenvalues form a discrete positive set which accumulates only at infinity [9]. We can therefore order the eigenvalues as they increase and counting multiplicity by repeating an eigenvalue according to its multiplicity,

$$0 < \lambda_1 < \lambda_2 \cdots \uparrow \infty.$$

We may correspondingly order the eigenfunctions. Since we define all domains here to be connected, the first eigenvalue is simple, and its corresponding eigenfunction is uniquely defined, up to multiplication by scalars. In this way, we may speak of the “first” eigenfunction, which is the eigenfunction whose eigenvalue is equal to  $\lambda_1$ . The eigenfunctions form an orthogonal basis of the Hilbert space  $\mathcal{L}^2(\Omega)$ . We shall require the following well-known fact about the first eigenfunction. The proof of this theorem can be found in the classical PDE textbook of Evans [10, §6.5].

**Theorem 3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ . Then the first eigenfunction of the Laplace eigenvalue equation with the Dirichlet boundary condition does not vanish anywhere inside  $\Omega$ .*

The following result is originally due to Lamé [26] in two dimensions and restricted to trigonometric eigenfunctions. Here, we immediately obtain the following generalization to  $\mathbb{R}^n$  for all  $n$  as well as to real analytic functions by applying the identity theorem for real analytic functions; see [25].

**Lemma 1** (Vanishing planes). *Let  $u$  be a real analytic function on  $\mathbb{R}^n$ . Assume that  $u$  vanishes on an open, nonempty subset of a hyperplane*

$$\mathcal{P} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M} \cdot \mathbf{x} = b\}.$$

*Then  $u$  vanishes on all of  $\mathcal{P}$ .*

We will also generalize Lamé’s fundamental theorem, which was originally proved in two dimensions and for trigonometric functions, to  $n$  dimensions and real analytic eigenfunctions.

**Theorem 4** (Lamé's fundamental theorem). *Assume that  $u$  is a real analytic function on  $\mathbb{R}^n$  that satisfies the Laplace eigenvalue equation with the Dirichlet boundary condition on a polytope  $\Omega \in \wp_n$ . Then  $u$  is anti-symmetric with respect to all  $(n - 1)$ -dimensional hyperplanes on which  $u$  vanishes.*

*Proof.* Let  $\lambda$  be the eigenvalue corresponding to  $u$ , so that on  $\Omega$  we have

$$\Delta u(\mathbf{x}) = \lambda u(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega.$$

Then, since  $u$  is real analytic,  $\Delta u$  is also real analytic on  $\Omega$ . The function

$$\Delta u - \lambda u$$

is real analytic and vanishes on  $\Omega$  which is an open subset of  $\mathbb{R}^n$ . Consequently, by [Lemma 1](#) this function vanishes on all of  $\mathbb{R}^n$ , and therefore  $u$  satisfies the same Laplace eigenvalue equation on all of  $\mathbb{R}^n$ .

Now, let  $H$  be an  $(n - 1)$ -dimensional hyperplane on which  $u$  vanishes. Let  $\mathbf{v} \in \mathbb{R}^n$  be a normal vector to  $H$  of length one, such that  $\mathbf{v}$  points away from the interior of  $\Omega$ . Let

$$u(r, \mathbf{z}) := u(\mathbf{z} + r\mathbf{v}), \quad \text{for } \mathbf{z} \in H \text{ and } r \in \mathbb{R}.$$

The hyperplane  $H$  splits  $\mathbb{R}^n$  into the disjoint union

$$\mathbb{R}^n \setminus H = \mathcal{R}_+ \cup \mathcal{R}_-, \quad \mathbb{R}^n = \mathcal{R}_+ \cup H \cup \mathcal{R}_-,$$

such that  $\mathbf{v}$  points from  $\mathcal{R}_+$  to  $\mathcal{R}_-$ .

We now define the function

$$\tilde{u}(r, \mathbf{z}) := \begin{cases} u(r, \mathbf{z}), & (r, \mathbf{z}) \in \overline{\mathcal{R}_+}; \\ -u(-r, \mathbf{z}), & (r, \mathbf{z}) \in \mathcal{R}_-. \end{cases}$$

With this definition,  $\tilde{u}$  is anti-symmetric with respect to  $H$ . By the definition of  $u$ , there is an open, connected, nonempty subset  $O \subset \mathbb{R}^n$  that contains an open, connected, nonempty subset of  $\partial\Omega \subset H$ , and such that

$$(\Delta - \lambda)(u - \tilde{u}) = 0 \text{ on } O \setminus H, \quad u - \tilde{u} = 0 \text{ on } O \cap \overline{\mathcal{R}_+},$$

and the normal derivatives

$$\frac{\partial u}{\partial \mathbf{v}} = \frac{\partial \tilde{u}}{\partial \mathbf{v}} \text{ on } O \cap H.$$

Consequently, by standard uniqueness theory of partial differential equations [[9](#), [10](#)],  $u = \tilde{u}$  on  $O$ . It therefore follows that  $\tilde{u}$  is also real analytic on  $O$ . By the identity theorem for real analytic functions [[25](#), Chapter 2], we obtain that  $u = \tilde{u}$  on  $\mathbb{R}^n$ . We therefore obtain that  $u$ , like  $\tilde{u}$ , is anti-symmetric with respect to  $H$ . ■

We are now poised to prove the first implication in [Theorem 2](#).

**Proposition 1.** *Assume that  $\Omega$  is a polytope in  $\mathbb{R}^n$ , and the first eigenfunction satisfies the first condition of [Theorem 2](#). Then  $\Omega$  strictly tessellates  $\mathbb{R}^n$ .*

*Proof.* Let  $\Omega$  be a polytope in  $\mathbb{R}^n$  as in the statement of the proposition. If  $n = 1$ , then  $\Omega$  is a segment and may be written as  $(a, b)$  for some real numbers  $a < b$ . We have computed the eigenfunctions explicitly in this case. They are

$$u_k(x) = \sin\left(\frac{x - a}{b - a}k\pi\right).$$

The first eigenfunction in particular satisfies the hypotheses of [Theorem 2](#), and we have also shown that all one-dimensional polytopes strictly tessellate  $\mathbb{R}^1$ . Hence the proposition is proved in one dimension. So let us assume that  $n \geq 2$ . By [Lemma 1](#), for an affine hyperplane  $\mathcal{P}$  that contains a boundary face of  $\Omega$ , all eigenfunctions of  $\Omega$  vanish on  $\mathcal{P}$ . Since the first eigenfunction never vanishes in the interior of  $\Omega$  by [Theorem 3](#), it follows that all of the hyperplanes that contain the boundary faces of  $\Omega$  have empty intersection with the interior of  $\Omega$ . In simpler terms, this means that the polytope  $\Omega$  is convex.

Since the first eigenfunction,  $u_1$ , of  $\Omega$  satisfies  $(\Delta - \lambda_1)u_1 = 0$  on  $\Omega$  which is an open, connected, nonempty subset of  $\mathbb{R}^n$ , and  $u_1$  is real analytic, this equation is satisfied on all of  $\mathbb{R}^n$ . Consider a reflection of  $\Omega$  across one of its boundary faces. By [Theorem 4](#),  $u_1$  is odd with respect to this reflection and therefore satisfies the Dirichlet boundary condition as well as the Laplace eigenvalue equation on the reflected copy of  $\Omega$ . Consequently, by standard uniqueness theory [[9](#), [10](#)], the first eigenfunction on the reflected copy of  $\Omega$  is equal to a scalar multiple of  $u_1$ . Moreover, since the first eigenfunction does not vanish inside the domain, we obtain that  $u_1$  does not vanish in the reflected copy of  $\Omega$ . We repeat this argument to cover  $\mathbb{R}^n$  with copies of  $\Omega$  obtained by repeated reflections across boundary faces. Since  $u_1$  does not vanish inside any of the reflected copies of  $\Omega$ , by [Lemma 1](#) and [Definition 2](#) the tessellation must be strict. ■

**3. ROOT SYSTEMS, ALCOVES, AND STRICTLY TESSELLATING POLYTOPES.** In 1980, Pierre Bérard showed that a certain type of bounded domain in  $\mathbb{R}^n$ , known as an *alcove*, always has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition. To define alcoves, we must first define root systems. The concept of a root system was originally introduced by Wilhelm Killing in 1888 [[21](#), [22](#)]. His motivation was to classify all simple Lie algebras over the field of complex numbers. In this section, we will see how our analytic problem, the study of the Laplace eigenvalue equation, is connected to these abstract algebraic concepts from Lie theory and representation theory.

**Definition 4.** A *root system* in  $\mathbb{R}^n$  is a finite set  $R$  of vectors that satisfy:

1.  $0$  is not in  $R$ .
2. The vectors in  $R$  span  $\mathbb{R}^n$ .
3. For  $\mathbf{v} \in R$ , the only scalar multiples of  $\mathbf{v}$  that also belong to  $R$  are  $\pm\mathbf{v}$ .
4.  $R$  is closed with respect to reflection across any hyperplane whose normal is an element of  $R$ , that is,

$$\mathbf{v} - 2\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \in R, \quad \text{for all } \mathbf{u}, \mathbf{v} \in R;$$

5. If  $\mathbf{u}, \mathbf{v} \in R$ , then the projection of  $\mathbf{u}$  onto the line through  $\mathbf{v}$  is an integer or half-integer multiple of  $\mathbf{v}$ . The mathematical formulation of this is that

$$2\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \in \mathbb{Z}, \quad \text{for all } \mathbf{u}, \mathbf{v} \in R.$$

The elements of a root system are often referred to as *roots*. Four root systems in  $\mathbb{R}^2$  are shown in Figure 6.

**Remark 4.** There are different variations of Definition 4 of a root system depending on the context. Sometimes only conditions 1–4 are used to define a root system. When the additional assumption 5 is included, then the root system is said to be *crystallographic*. In other contexts, condition 3 is omitted, and one would call a root system that satisfies condition 3 *reduced*.

We will need the dual root system to define the eigenvalues of the polytope that will be naturally associated with the root system.

**Definition 5.** Let  $R$  be a root system. Then for  $\mathbf{v} \in R$  the *coroot*  $\mathbf{v}^\vee$  is defined to be

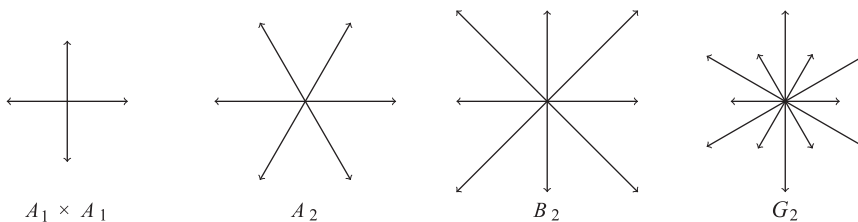
$$\mathbf{v}^\vee = \frac{2}{\|\mathbf{v}\|^2} \mathbf{v}.$$

The set of coroots  $R^\vee := \{\mathbf{v}^\vee\}_{\mathbf{v} \in R}$ . This is called the *dual root system*, and may also be called the *inverse root system*. It is a straightforward exercise requiring only the definitions to prove that the dual root system is itself a root system.

We associate a Weyl group to a root system. These Weyl groups are subgroups of the orthogonal group  $O(n)$ .

**Definition 6.** For any root system  $R \subset \mathbb{R}^n$  we associate a subgroup of the orthogonal group  $O(n)$  known as its *Weyl group*. This is the subgroup  $W < O(n)$  generated by the set of reflections by hyperplanes whose normal vectors are elements of  $R$ . For  $\mathbf{v} \in R$  reflection across the hyperplane with normal vector equal to  $\mathbf{v}$  is explicitly

$$\sigma_{\mathbf{v}} : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \sigma_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} - 2 \frac{(\mathbf{v} \cdot \mathbf{x})}{\|\mathbf{v}\|^2} \mathbf{v}.$$



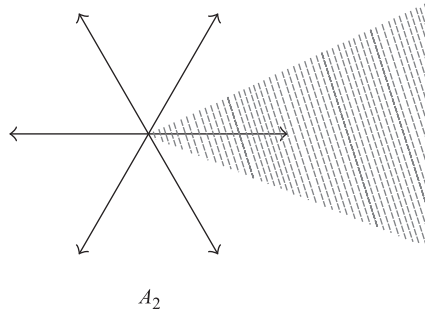
**Figure 6.** Here are four root systems in  $\mathbb{R}^2$ . Below each root system is the name of its Weyl group. The name of the Weyl group may also be used as the name of the root system.

By the definition of a root system, the associated Weyl group is finite. To explain what was proved in [2] by Bérard, we require the notion of Weyl chamber.

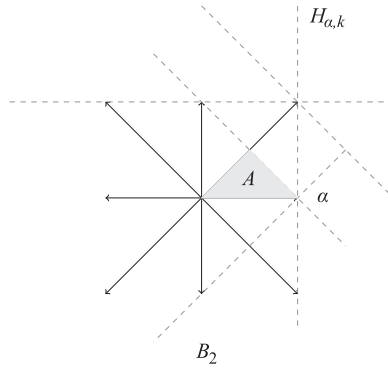
**Definition 7.** For a root system  $R \subset \mathbb{R}^n$  for each  $\mathbf{v} \in R$ , let  $H_{\mathbf{v}}$  denote the hyperplane that contains the origin and whose normal vector is  $\mathbf{v}$ . In particular,

$$H_{\mathbf{v}} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{v} = 0\}.$$

Let  $\mathcal{H} = \{H_{\mathbf{v}}\}_{\mathbf{v} \in R}$ . Then  $\mathbb{R}^n \setminus (\cup_{H \in \mathcal{H}} H)$  is disconnected, and each connected open component is known as a *Weyl chamber*. A Weyl chamber of the Weyl group  $A_2$  is shown in Figure 7.



**Figure 7.** Extending the shaded area to infinity shows a Weyl chamber of the Weyl group  $A_2$ .



**Figure 8.** This shows an alcove,  $A$ , corresponding to the root system with Weyl group  $B_2$ . For  $\alpha \in B_2$ , the hyperplanes  $H_{\alpha,k}$  for  $k \in \mathbb{Z}$  are the parallel hyperplanes which have normal vector equal to  $\alpha$ . Note that  $A$  is an isosceles right triangle.

**Definition 8.** Let  $R$  be a root system. Denote by  $H_{\mathbf{v}}$  the hyperplane in  $\mathbb{R}^n$  that contains the origin and whose normal vector is equal to  $\mathbf{v}$  for  $\mathbf{v} \in R$ . Let

$$H_{\mathbf{v},k} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{x} = k\},$$

for  $k \in \mathbb{Z}$ . Then  $H_{\mathbf{v},0} = H_{\mathbf{v}}$ . For  $k \neq 0$ , the hyperplane  $H_{\mathbf{v},k}$  is parallel to  $H_{\mathbf{v}}$ . We define an *alcove* to be a connected component of

$$\mathbb{R}^n \setminus \left\{ \bigcup_{\mathbf{v} \in R, k \in \mathbb{Z}} H_{\mathbf{v},k} \right\}.$$

We note that the definition of an alcove immediately implies that it is a polytope in  $\mathbb{R}^n$ . An example of an alcove is shown in [Figure 8](#).

**Proposition 2** ([2, Proposition 9, p. 181]). *Let  $\Omega \subset \mathbb{R}^n$  be an alcove. Then  $\Omega$  has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition.*

For readers who understand French and read [2], you may notice that the statement of [Proposition 2](#) is not the English translation of [2, Proposition 9, p. 181]. Bérard proved a stronger result; he specified the eigenvalues and corresponding eigenfunctions. To understand what Bérard proved, let  $R$  be a root system. Let  $C(R)$  denote a

Weyl chamber, and let  $D(R)$  denote an alcove that is contained in the Weyl chamber  $C(R)$ . Consider the dual root system  $R^\vee$ . The vertices of the closures of the alcoves associated to  $R^\vee$  create a lattice. Let us denote this lattice by  $\Gamma$ . The dual lattice is

$$\Gamma^* := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \boldsymbol{\gamma} \in \mathbb{Z}, \quad \forall \boldsymbol{\gamma} \in \Gamma\}.$$

Bérard referred to the points contained in this dual lattice as “the group of weights of  $R$ ” (“le groupe des poids de  $R$ ”) [2]. He proved that the eigenvalues for the alcove  $D(R)$  are given by

$$\{4\pi^2 \|\mathbf{q}\|^2 : \mathbf{q} \in \Gamma^* \cap C(R)\}.$$

The multiplicity of the eigenvalue  $\lambda = 4\pi^2 \|\mathbf{q}\|^2$  is equal to the number of vectors  $\mathbf{q} \in \Gamma^* \cap C(R)$  that satisfy  $\lambda = 4\pi^2 \|\mathbf{q}\|^2$ . The eigenfunctions are certain linear combinations of  $e^{2\pi i \mathbf{x} \cdot w(\mathbf{q})}$ , where  $w(\mathbf{q})$  is in the affine Weyl group of  $R$ . The affine Weyl group of  $R$  is the semi-direct product of the Weyl group and the lattice  $\Gamma$ . Combining our Proposition 1 with Bérard’s Proposition 2, we obtain the following corollary which states that every alcove is a strictly tessellating polytope.

**Corollary 2.** *Let  $\Omega \subset \mathbb{R}^n$  be an alcove. Then  $\Omega$  is a polytope that strictly tessellates  $\mathbb{R}^n$ .*

In the following proposition, we prove the converse: every strictly tessellating polytope is an alcove of a root system.

**Proposition 3.** *Let  $\Omega \subset \mathbb{R}^n$  be a polytope that strictly tessellates  $\mathbb{R}^n$ . Then  $\Omega$  is an alcove.*

*Proof.* We will build a root system,  $R$ , using the fact that  $\Omega$  strictly tessellates space. The tessellation defines hyperplanes in  $\mathbb{R}^n$  that contain the boundary faces of the copies of  $\Omega$  in the tessellation. Assume that  $\Omega$  has  $m$  boundary faces. By the definition of strict tessellation, there is a discrete set of vectors

$$\{v_{j,k}\}_{j \in \mathbb{Z}, 1 \leq k \leq m},$$

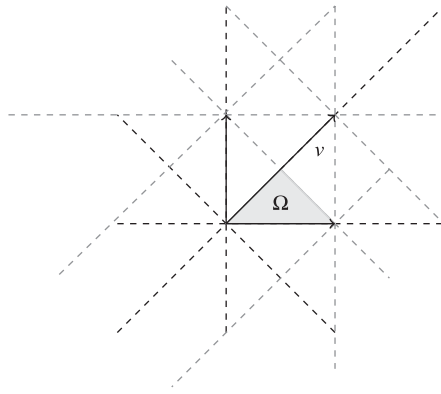
where  $v_{j,k}$  is a unit normal vector to the hyperplane containing the  $k$ th boundary face of  $\Omega_j$ . We first define  $\mathcal{R}$  to be the set that contains each distinct  $v_{j,k}$  together with its opposite  $-v_{j,k}$ . Since  $\Omega$  is a bounded, connected, open set with boundary consisting of flat faces, the set of vectors  $\mathcal{R}$  defined in this way spans  $\mathbb{R}^n$ . To see this, we observe that if this were not the case, then  $\Omega$  would be contained in a  $k$ -dimensional hyperplane in  $\mathbb{R}^n$  and thus would not be an open set in  $\mathbb{R}^n$ . By definition, we note that  $0 \notin \mathcal{R}$ . By Definition 2 the set of vectors  $\mathcal{R}$  is finite.

Since  $\mathcal{R}$  is a finite set, and there are countably many hyperplanes defined by the tessellation, this means that for each  $v \in \mathcal{R}$ , there are countably infinitely many hyperplanes whose normal direction is  $\pm v$ . Fix some  $v \in \mathcal{R}$ , and by possibly moving the entire picture, assume that there is a hyperplane  $H_{v,0}$  with normal direction  $\pm v$  that contains the origin. Let the closest parallel hyperplane to  $H_{v,0}$  in the direction of  $v$  be  $H_{v,1}$ . We repeat this process for each  $v \in \mathcal{R}$  and then define

$$R := \left\{ \mathbf{v} := \frac{v}{\|v\| \operatorname{dist}(0, H_{v,1})} \right\}_{v \in \mathcal{R}}.$$

We therefore have

$$H_{v,1} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{v} = 1.\}$$



**Figure 9.** Given a polytope  $\Omega$ , we construct the hyperplanes  $H_{v,0}$ , here in the thicker black dotted lines, and the normal vectors  $\mathbf{v}$ . The set  $\{H_{v,k}\}$  includes the thinner gray dotted lines.

Since the tessellation is unchanged by reflection in the direction of  $\pm v$ , the distance between adjacent parallel hyperplanes with normal vector  $\pm v$  is equal to  $\text{dist}(0, H_{v,1})$ . Consequently, we may enumerate the parallel hyperplanes as

$$H_{v,j} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{v} = j\}, \quad j \in \mathbb{Z}.$$

A schematic image is given in Figure 9. For ease of notation, let us define  $H_{v,k} := H_{v,k}$ .

Let  $\mathbf{w} \in R$ . By possibly translating the entire picture, assume that there is a hyperplane in the tessellation with normal direction  $\pm \mathbf{w}$  and that contains the origin, such that the origin is a vertex of a copy of  $\Omega$  in the tessellation. Thus  $H_{\mathbf{w},0}$  is a hyperplane in the tessellation. Consider the reflection with normal direction  $\mathbf{v}$ , denoted by  $\sigma_{\mathbf{v}}$ , that is,

$$\sigma_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} - 2 \frac{\mathbf{x} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}.$$

Then  $\sigma_{\mathbf{v}}(0) = 0$ . Consequently,  $\sigma_{\mathbf{v}}(H_{\mathbf{w},0})$  is another hyperplane in the strict tessellation which also contains the origin: thus it is  $H_{\mathbf{u},0}$  for some  $\mathbf{u} \in R$ . Similarly, we also have  $\sigma_{\mathbf{v}}(H_{\mathbf{w},1}) = H_{\mathbf{u},j}$  for some  $j \in \mathbb{Z}$ . Since  $\sigma_{\mathbf{v}}$  preserves the scalar product, for  $\mathbf{x} \in H_{\mathbf{w},1}$ , by definition we have

$$\mathbf{x} \cdot \mathbf{w} = 1 \implies \sigma_{\mathbf{v}}(\mathbf{x}) \cdot \sigma_{\mathbf{v}}(\mathbf{w}) = 1.$$

Since  $\sigma_{\mathbf{v}}$  sends  $\mathbf{x}$  to a point in  $H_{\mathbf{u},j}$  we also have

$$\sigma_{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{u} = j.$$

Since  $\sigma_{\mathbf{v}}(H_{\mathbf{w},0}) = H_{\mathbf{u},0}$ , we must have that  $\sigma_{\mathbf{v}}(\mathbf{w}) = \alpha \mathbf{u}$  for some  $\alpha \in \mathbb{R}$ . Therefore, combining with the above, we obtain

$$1 = \mathbf{x} \cdot \mathbf{w} = \sigma_{\mathbf{v}}(\mathbf{x}) \cdot \sigma_{\mathbf{v}}(\mathbf{w}) = \alpha \sigma_{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{u} = \alpha j \implies \alpha = \frac{1}{j}.$$

So we have proved that

$$\sigma_{\mathbf{v}}(\mathbf{w}) = \frac{1}{j} \mathbf{u}.$$

The vector  $\mathbf{y}_w := \text{dist}(0, H_{w,1}) \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{\mathbf{w}}{\|\mathbf{w}\|^2}$  is orthogonal to the hyperplanes  $H_{w,0}$  and  $H_{w,1}$  and connects the origin to the nearest point in  $H_{w,1}$ . When this vector is reflected by  $\sigma_v$ , it will again start from the origin and have its endpoint lying on one of the parallel hyperplanes, by virtue of the strict tessellation. Let us define the vector  $\mathbf{y}_v$  in the analogous way. We compute explicitly that

$$\sigma_v(\mathbf{y}_w) = \mathbf{y}_w - 2\mathbf{y}_w \cdot \mathbf{v} \frac{\mathbf{v}}{\|\mathbf{v}\|^2} = \mathbf{y}_w - 2(\mathbf{y}_w \cdot \mathbf{v})\mathbf{y}_v.$$

On the other hand, since  $\sigma_v(\mathbf{w}) = \frac{1}{j}\mathbf{u}$ , we compute that

$$\sigma_v(\mathbf{y}_w) = \sigma_v\left(\frac{\mathbf{w}}{\|\mathbf{w}\|^2}\right) = \frac{1}{\|\mathbf{w}\|^2} \sigma_v(\mathbf{w}) = \frac{1}{\|\mathbf{w}\|^2} \frac{1}{j} \mathbf{u}.$$

Now, since  $\|\mathbf{u}\|^2 = j^2\|\mathbf{w}\|^2$ , we have  $\sigma_v(\mathbf{y}_w) = j \left(\frac{\mathbf{u}}{\|\mathbf{w}\|^2 j^2}\right) = j\mathbf{y}_u$ . Combining these calculations, we obtain

$$\sigma_v(\mathbf{y}_w) = \mathbf{y}_w - 2(\mathbf{y}_w \cdot \mathbf{v})\mathbf{y}_v = j\mathbf{y}_u \implies 2(\mathbf{y}_w \cdot \mathbf{v})\mathbf{y}_v = \mathbf{y}_w - j\mathbf{y}_u.$$

The vector  $\mathbf{y}_w$  goes from the origin to  $H_{w,1}$ , while the vector  $-j\mathbf{y}_u$  goes from the origin to  $H_{u,-j}$ . By vector addition and the strict tessellation, the sum  $\mathbf{y}_w - j\mathbf{y}_u$  must go from the origin and end precisely at one of the parallel hyperplanes. Consequently, the vector

$$2(\mathbf{y}_w \cdot \mathbf{v})\mathbf{y}_v$$

must be an integer multiple of  $\mathbf{y}_v$  because it goes from the origin in the direction of  $\mathbf{y}_v$  and lands at one of the parallel hyperplanes  $H_{v,k}$  for some  $k \in \mathbb{Z}$ . Therefore,

$$2(\mathbf{y}_w \cdot \mathbf{v}) = k \in \mathbb{Z}.$$

By the definitions of  $\mathbf{y}_w$  and  $\mathbf{v}$ ,

$$2(\mathbf{y}_w \cdot \mathbf{v}) = 2 \frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{w}\|^2} = k \in \mathbb{Z}.$$

In a similar way, reversing the roles of  $\mathbf{w}$  and  $\mathbf{v}$ , we also obtain

$$2 \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2} \in \mathbb{Z}.$$

Since  $\mathbf{w}, \mathbf{v} \in R$  were arbitrary, this shows the final condition needed for  $R$  to be a root system in [Definition 4](#) is satisfied. We conclude that  $R$  is a root system and that  $\Omega$  is one of its alcoves. ■

The proofs of [Theorem 2](#) and [Corollary 1](#) will now follow from [Propositions 1](#) and [3](#) and [Bérard's Proposition 2](#).

*Proof of Theorem 2* By [Proposition 1](#), if  $\Omega$  is a polytope, and its first eigenfunction is real analytic on  $\mathbb{R}^n$ , then  $\Omega$  strictly tessellates  $\mathbb{R}^n$ . By [Proposition 3](#), if  $\Omega$  is a polytope that strictly tessellates  $\mathbb{R}^n$ , then  $\Omega$  is an alcove. By [Bérard's Proposition 2](#), if  $\Omega$  is an alcove, then all its eigenfunctions are trigonometric. We have therefore proved that the statements in [Theorem 2](#) satisfy:  $1 \implies 2 \implies 3 \implies 1$ . ■

*Proof of Corollary 1* If the first eigenfunction of a polytope in  $\mathbb{R}^n$  satisfies the hypotheses of [Theorem 2](#), then the polytope is an alcove. By Bérard's [Proposition 2](#), all of the eigenfunctions of the polytope are trigonometric. ■

**4. CONCLUDING REMARKS AND CONJECTURES.** We have now answered the analysis question: When does a polytope in  $\mathbb{R}^n$  have a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation? In geometric terms, the necessary and sufficient condition for a polytope to have a complete set of trigonometric eigenfunctions is that the polytope strictly tessellates  $\mathbb{R}^n$ . In algebraic terms, in the language of Bourbaki, the equivalent necessary and sufficient condition is that the polytope is congruent to a fundamental domain of a crystallographic Coxeter group [[2](#), p. 179], [[5](#), VI.25, Proposition 9, p. 180]. Returning to the analysis problem, it is interesting to note that it is enough to know that the first eigenfunction is real analytic and satisfies the Laplace eigenvalue equation on  $\mathbb{R}^n$  to conclude that it is a trigonometric function and moreover, *all* the eigenfunctions are trigonometric. This is a remarkable fact. Moreover, the equivalence of analytic, geometric, and algebraic statements shows that these different areas of mathematics are intimately connected. The Fuglede conjecture similarly brings together different areas of mathematics in the study of a single question.

**The Fuglede conjecture.** To state the Fuglede conjecture, we introduce a few concepts.

**Definition 9.** A domain  $\Omega \subset \mathbb{R}^d$  is said to be a *spectral set* if there exists  $\Lambda \subset \mathbb{R}^n$  such that the functions

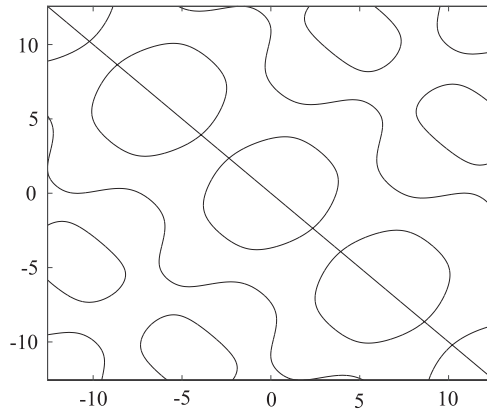
$$\{e^{2\pi i\lambda \cdot \mathbf{x}}\}_{\lambda \in \Lambda}$$

are an orthogonal basis for  $\mathcal{L}^2(\Omega)$ . The set  $\Lambda$  is then said to be a *spectrum of  $\Omega$* , and  $(\Omega, \Lambda)$  is called a *spectral pair*.

To relate these notions to our work here, we observe that if a domain  $\Omega$  were to have all its eigenfunctions for the Laplace eigenvalue equation of the form  $e^{2\pi i\lambda \cdot \mathbf{x}}$ , then these functions would comprise an orthogonal basis for  $\mathcal{L}^2(\Omega)$ . Consequently, knowing that the eigenfunctions are precisely of this form implies that the domain is a spectral set. However, the converse is not true, in the sense that if  $\Omega$  is a spectral set, then its eigenfunctions are not necessarily individual complex exponential functions. If  $\Omega$  is a spectral set, then the eigenfunctions must be linear combinations of the  $e^{2\pi i\lambda \cdot \mathbf{x}}$ , since these are a basis for  $\mathcal{L}^2(\Omega)$ . However, the linear combinations could have countably infinitely many terms, so it is not clear what precise form the eigenfunctions will take.

**Conjecture 1** (Fuglede [[16](#)]). *Every domain of  $\mathbb{R}^n$  that has positive Lebesgue measure is a spectral set if and only if it tiles  $\mathbb{R}^n$  by translation.*

Fuglede proved in 1974 that the conjecture holds if one assumes that the domain is the fundamental domain of a lattice [[16](#)]. Only several years later, in 2003, was further progress made by Iosevich, Katz, and Tao [[19](#)] who proved that the Fuglede conjecture is true if one restricts to convex planar domains. In the following year, Tao proved that the Fuglede conjecture is false in dimension 5 and higher [[37](#)]. In 2006, the works of Farkas, Kolounzakis, Matolcsi, and Mora [[11](#), [23](#), [24](#), [29](#)] proved that the conjecture is also false for dimensions 3 and 4. In 2017, Greenfeld and Lev proved that Fuglede's conjecture is true if one restricts attention to domains that are convex polytopes, but



**Figure 10.** This figure shows the null set of the function  $u(x, y) = \sin(x) + \sin(y) + \sin((x+y)/\sqrt{2})$  in a square-shaped region of  $\mathbb{R}^2$ . The null set includes the line  $y = -x$  as well as the other curves in the region. Consequently, by uniqueness, this function is the first eigenfunction of the connected, open domains that are bounded by these curves, since it vanishes on the boundary but not on the interior and satisfies the Laplace eigenvalue equation. Hence the first eigenfunction satisfies the first condition of [Theorem 2](#), but we do not obtain any further conclusions because the domain is not a polytope.

only in  $\mathbb{R}^3$  [17]. In 2019, Lev and Matolcsi proved that Fuglede’s conjecture is true if one restricts attention to convex domains, in any dimension [27]. Interestingly, the Fuglede conjecture is still an open problem for arbitrary domains in dimensions one and two. Here we make the following conjecture which is related to yet independent from Fuglede’s.

**Conjecture 2.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$ . Then  $\Omega$  has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition if and only if  $\Omega$  is a polytope that strictly tessellates  $\mathbb{R}^n$ . Equivalently,  $\Omega$  has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition if and only if  $\Omega$  is an alcove.*

The difficulty in treating arbitrary domains is that we do not have a replacement for Lamé’s results which are central to our proof. Moreover, it is possible to construct linear combinations of trigonometric functions that vanish on curved regions; an example is given in [Figure 10](#). Consequently, we cannot immediately conclude that domains that have trigonometric eigenfunctions have flat boundary faces, and hence they are polytopes. A domain with a curved boundary could have a few trigonometric eigenfunctions. What is reasonable to expect, however, is that it does not have a complete set of trigonometric eigenfunctions.

**The crystallographic restriction theorem and a geometric approach to the Goldbach conjecture.** The vertices of the strict tessellation given by a polytope that is an alcove are in fact the set of points in a full-rank lattice. We note that two different polytopes may give rise to the same lattice; for example, an isosceles right triangle and the square obtained by two copies of that triangle will produce the same lattice. For any discrete group of isometries of  $\mathbb{R}^n$ , an element  $g$  in such a group has finite order if there is an integer  $k > 0$  such that  $g$  composed with itself  $k$  times is the identity. The minimal such  $k$  is the order of  $g$ . To state the crystallographic restriction theorem, we define a function which is like an extension of the Euler totient function. For an odd

prime  $p$  and  $r \geq 1$ ,

$$\psi(p^r) := \phi(p^r), \quad \phi(p^r) = p^r - p^{r-1}.$$

Here  $\phi$  denotes the Euler totient function. The Euler totient function of a positive integer  $n$  counts the positive integers that are relatively prime to, and at most  $n$ . So, for example, for an odd prime  $p$ , the positive integers that are not relatively prime to  $p^r$  are  $p, 2p, 3p, \dots, p^{r-1}p = p^r$ . There are  $p^{r-1}$  of these. All other positive integers are relatively prime to  $p^r$ , hence  $\phi(p^r) = p^r - p^{r-1}$ . The function  $\psi$  is further defined as follows:

$$\psi(1) = \psi(2) = 0, \quad \psi(2^r) := \phi(2^r) \text{ for } r > 1,$$

and

$$\text{for } m = \prod_i p_i^{r_i}, \quad \psi(m) := \sum_i \psi(p_i^{r_i}).$$

**Theorem 5** (Crystallographic restriction I). *For any discrete group  $G$  of isometries of  $\mathbb{R}^n$ , for  $n \geq 2$  the set of orders of the elements  $G$  that have finite order is equal to*

$$\text{Ord}_n = \{m \in \mathbb{N} : \psi(m) \leq n\}.$$

The crystallographic restriction theorem is connected to the mathematics of crystals when we reformulate the theorem in the context of lattices. A full-rank lattice is a set of points in  $\mathbb{R}^n$  of the form

$$\Gamma = \{\mathbf{p} \in \mathbb{R}^n : \mathbf{p} = L\mathbf{x}, \quad L \in \text{GL}(n, \mathbb{R}), \quad \mathbf{x} \in \mathbb{Z}^n\}. \quad (1)$$

Here  $\text{GL}(n, \mathbb{R})$  is the set of  $n \times n$  invertible matrices with real entries, and  $\mathbb{Z}^n$  are the elements of  $\mathbb{R}^n$  whose entries are integers. We say that the matrix  $L$  *generates* the lattice  $\Gamma$ . The generating matrix  $L$  is not unique, because for any  $M \in \text{GL}(n, \mathbb{Z})$  the set of points in (1) is equal to

$$\{\mathbf{p} \in \mathbb{R}^n : \mathbf{p} = LM\mathbf{x}, \quad \mathbf{x} \in \mathbb{Z}^n\}.$$

Here  $\text{GL}(n, \mathbb{Z})$  is the group of invertible  $n \times n$  matrices whose entries are integers. Note that to be a group, this requires the determinant of all elements of  $\text{GL}(n, \mathbb{Z})$  to be equal to  $\pm 1$ . Two matrices  $L_1, L_2 \in \text{GL}(n, \mathbb{R})$  generate the same lattice if and only if there is an  $M \in \text{GL}(n, \mathbb{Z})$  such that  $L_1 = L_2M$ . For a matrix  $M \in \text{GL}(n, \mathbb{Z})$ , we identify it with the isometry of  $\mathbb{R}^n$  that maps  $\mathbf{x} \in \mathbb{R}^n$  to  $M\mathbf{x}$ . The matrices in  $\text{GL}(n, \mathbb{Z})$  can therefore be identified with the group of symmetries of the crystal whose atoms lie on the points of the lattice. Hence, the order of  $M$  is equal to the smallest positive integer  $k$  such that  $M^k$  is the identity matrix. It turns out that the set of orders of the elements of any discrete group  $G$  of isometries of  $\mathbb{R}^n$  that have finite order is equal to the set of orders of the elements of  $\text{GL}(n, \mathbb{Z})$ . Consequently, the crystallographic restriction may be reformulated as follows.

**Theorem 6** (Crystallographic restriction II). *For any  $n \geq 2$ , the set of orders of the elements of  $\text{GL}(n, \mathbb{Z})$  is equal to*

$$\text{Ord}_n = \{m \in \mathbb{N} : \psi(m) \leq n\}.$$

In [1], Bamberg, Cairns, and Kilminster proved that one may reformulate the strong Goldbach conjecture in terms of the orders of elements of  $\text{GL}(n, \mathbb{Z})$ .

**Conjecture 3** (Strong Goldbach). *Every even natural number greater than six can be written as the sum of two distinct odd primes.*

**Theorem 7** (Theorem 3 of [1]). *The following statements are equivalent:*

1. *The strong Goldbach conjecture is true;*
2. *For each even  $n \geq 6$  there is a matrix  $M \in \text{GL}(n, \mathbb{Z})$  that has order  $pq$  for distinct primes  $p$  and  $q$ , and there is no matrix in  $\text{GL}(k, \mathbb{Z})$  of order  $pq$  for any  $k < n$ .*

The Goldbach conjecture is an extremely difficult problem. Difficult, long-standing open problems have sometimes been solved by translating the problem into a different field of mathematics. The proof of Fermat's last theorem, also a statement in number theory, was achieved using newly-developed techniques in algebraic geometry [38, 39]. To approach the Goldbach conjecture geometrically, we ask

**Question 1.** Is there a geometric reason for the existence of a symmetry for full-rank lattices in  $\mathbb{R}^n$ , with  $n \geq 6$  an even number, such that this symmetry is of order  $pq$  for two odd primes  $p \neq q$  such that  $p + q = n + 2$ ?

The condition that there is no matrix in  $\text{GL}(k, \mathbb{Z})$  of order  $pq$  for any  $k < n$  is equivalent to requiring  $p + q = n + 2$ . This follows from Theorem 6, which states that the orders of the elements of  $\text{GL}(k, \mathbb{Z})$  are equal to the set of nonnegative integers  $m$  with  $\psi(m) \leq k$ . In order to guarantee that

$$\psi(pq) = p + q - 2 > k \text{ for all } k < n, \text{ but } \psi(pq) \leq n,$$

we must have  $\psi(pq) = p + q - 2 = n$ .

Consequently, the symmetry of order  $pq$  would correspond to a matrix  $M \in \text{GL}(n, \mathbb{Z})$  that does not admit a diagonal decomposition into two matrices of smaller dimensions. Geometrically, this matrix would not arise as a product of symmetries of  $\mathbb{R}^k$  and  $\mathbb{R}^{n-k}$  for any  $k = 1, \dots, n - 1$ . It would be a new symmetry occurring first in  $\mathbb{R}^n$ . Since [1] already realized the connection between the Goldbach conjecture and the crystallographic restriction theorem, this geometric approach would seem unlikely to lead to any new developments. Nonetheless, it is interesting that a famous number-theoretic conjecture can be equivalently phrased as a simple question about the orders of symmetries of full-rank lattices in  $\mathbb{R}^n$ .

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