THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Some Cases of Kudla's Modularity Conjecture for Unitary Shimura Varieties

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Abstract

A common theme of the thesis is the interplay of symmetry and rigidity, which is a general phenomenon in mathematics. Symmetry is a notion related to the degree to which an object remains unchanged under transformations, and rigidity is a notion that in terms of physics can be thought of as a lack of freedom, which leads to stronger properties of an object than we normally expect. An object of higher symmetry often also exhibits a higher extent of rigidity, and vice versa.

In the introduction of the thesis, we provide some background on modular forms, number theory, and geometry in a way that does not require familiarity with these subjects. The contributions of this thesis are presented in three articles.

In Article I, we establish the existence of rational geometric designs for rational polytopes via the circle method and convex geometry, and discuss the existence of rational spherical designs which relates to Lehmer's conjecture on the Ramanujan tau function.

In Article II, we break the barrier of expressing weight-2 modular forms of higher level whose central L-values vanish by products of at most two Eisenstein series. This work shows the power of Rankin–Selberg method and also contributes to the computation of elliptic modular forms.

In Preprint III, we prove unconditionally some cases of Kudla's conjecture on the modularity of generating functions of special cycles on unitary Shimura varieties, for norm-Euclidean imaginary quadratic fields. Our method is based on a result of Liu and work of Bruinier–Raum, who confirmed the orthogonal Kudla conjecture over $\mathbb{Q}$.

Key words and phrases: Kudla's modularity conjecture, unitary Shimura varieties, special cycles, Eisenstein series, theta series, Jacobi forms, generating functions, spherical designs, rational points, central L-values, Rankin–Selberg method, the circle method.

List of appended publications

The following three publications are appended to the thesis:

I Zhen Cui, Jiacheng Xia, Ziqing Xiang,
Rational designs.
Advances in Mathematics, Volume 352 (2019), 541–571.

II Martin Raum, Jiacheng Xia,
All modular forms of weight 2 can be expressed by Eisenstein series.
Research in Number Theory 6 (2020), Issue 3, Article No. 32.

III Jiacheng Xia,
Some cases of Kudla's modularity conjecture for unitary Shimura varieties.
Preprint, submitted.

Contributions

I I proposed the original idea of using results from analytic number theory to show the existence of rational designs on the unit interval conjectured by Eiichi Bannai. Based on Ziqing Xiang's idea on finding an algebraic path, we generalized the results to open connected spaces, with the help from Zhen Cui on a key reference about the Hilbert–Kamke problem. I was responsible mainly for the number-theoretic part of the proofs, and shared equally with Ziqing the responsibility of writing up the framework and working out the details.

II The article is based on the work of my Licentiate thesis, which was carried out under the guidance of my supervisor Martin Raum, who also wrote the framework of the article and helped with the literature about the non-vanishing properties of certain L-values.

III Single-authored manuscript.

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I thank Yifeng Liu for reading my manuscript and suggesting the concept of norm-Euclidean number fields. I also thank Chao Li for useful comments on my manuscript. I thank Jan-Hendrik Bruinier for his encouragement and useful discussion. And I thank Yota Maeda for informing me his work on the unitary Kudla conjecture over general CM-fields.

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Jiacheng Xia
Gothenburg, April 16, 2021

Contents

Chapter 1

Introduction

This dissertation is devoted to the study of modular forms, number theory, and algebraic geometry in three interrelated problems of "symmetry-rigidity" flavor, where theta series, a unifying object in modular forms, number theory, geometry, representation theory, differential equations, and many other mathematical delights as well as string theory, play prominent roles. In this introductory chapter we provide some background, define key notions, and state the main results of my thesis, first in their qualitative nature, and gradually towards their quantitative forms.

This introductory chapter is organized as follows. In Section 1.1, we start with an overview highlighting a central theme of all three papers and motivate the notion of modular forms; we then proceed to define them in a quantitative way in Section 1.2; however, the existence of them is yet to be studied, and we construct Eisenstein series for this aim, and present our work on products of Eisenstein series in Section 1.3; theta series were among the first examples beyond Eisenstein series, and they have an interesting connection to spherical designs, which we illustrate in Section 1.4; the theme of symmetry and rigidity is prominent when it comes to the modularity of special cycles on unitary Shimura varieties, and we first define the relevant geometric objects in Section 1.5; finally we state the unitary Kudla conjecture and announce our main result in Section 1.6. Section 1.7 features a more formal statement of the main results in each appended paper.

1.1 Symmetry and rigidity

The interplay between symmetry and rigidity is a characteristic feature in many areas of mathematics. We discuss some well-known examples illustrating this interplay.

Polynomials in one variable are among functions of the simplest kind. The first and foremost fact about polynomials is the fundamental theorem of algebra, which asserts the existence of a zero point of any polynomial with complex coefficients. One consequence of this fact is the polynomial interpolation in numerical analysis, which features both symmetry and rigidity. From the perspective of rigidity, the polynomial interpolation claims that for any positive integer n, given $n+1$ distinct points $x_0, x_1, \ldots, x_n$ with corresponding values $y_0, y_1, \ldots, y_n$, there exists a unique polynomial in one variable of degree at most n that interpolates (x_i, y_i) for $i = 0, 1, \ldots, n$. From the symmetry point of view, the way we construct such a polynomial is via a linear combination of the Lagrange basis functions which are symmetric when we permute the variables x_i and y_i, respectively.

The next simplest kind of functions are holomorphic functions in one variable, and one of the profound facts about them is the Riemann mapping theorem, which asserts that any non-empty simply connected open proper subset of the complex plane is biholomorphically isomorphic to the open unit disk.

First of all, this theorem can be viewed as a very strong rigidity statement, that the seemingly weak topological assumption about the domain automatically implies its holomorphic structure, which belongs to the strongest properties in function theory. Bearing in mind the fact that the boundaries of simply connected open subsets of the complex plane can be nowhere-differentiable, not even a Jordan curve, and highly intricate, we see how amazing (another way of saying rigid) the Riemann mapping theorem is: it claims those very intricate domains can always be mapped bijectively in an angle-preserving way to the highly regular unit disk!

Secondly, from the symmetry perspective, the Riemann mapping theorem says that symmetries on the complex plane behave very uniformly: that is, the automorphism groups of these simply connected domains are all isomorphic via conjugations by biholomorphic maps between these domains.

Moreover in this context, the symmetry and rigidity are deeply intertwined. In fact, the abundance of biholomorphic maps that appear in the Riemann mapping theorem naturally arises from the lack of rigidity of the so-called Beltrami equation in the 2-dimensional case, which is a type of differential equation historically constructed as a tool in local differential geometry. By contrast, for higher dimension this equation exhibits a high extent of rigidity, and consequently we have a theorem of Liouville which states that any smooth conformal mapping on a domain of $\mathbb{R}^n$ for $n > 2$ is a Möbius transformation. In other words, conformal mappings of higher dimension are highly restrictive, as they can be expressed as a composition of translations, similarities, orthogonal transformations and inversions, and thus most higher-dimensional analogues of Riemann mapping theorem are far from true.

Last but not least, it is worth mentioning that Poincaré first proved that the Riemann mapping is essentially unique. His idea is nowadays known as a consequence of the Schwarz lemma, which follows from another wonder of rigidity: the maximum modulus principle for holomorphic functions, or more generally the maximum principle for harmonic functions.

This interplay goes on when arithmetic becomes part of the problem. For the next example, let us consider holomorphic functions in one variable that are periodic. From the symmetry point of view, they can be defined by means of group theory and functional equations. For a positive real number T, consider the functional equation $f(x+T) = f(x)$ for all $x \in \mathbb{C}$. A solution f to this equation can be viewed as an invariant in the space of all holomorphic functions un-

der the group action of $\mathbb{Z}$ via translation. From the rigidity perspective, if f has an absolutely convergent Fourier series, then it is uniquely determined by the Fourier coefficients. In order to keep track of such a nice periodic holomorphic function, all we need to know is a sequence of countably many numbers!

What if we add more symmetries from arithmetic to the functional equation? In other words, let Γ be a discrete group of rank higher than 1 acting on a certain class of holomorphic functions (in one or several variables). The invariant functions under the action of Γ that at the same time behave regularly will be of most interest to us. To begin with, let us note that they are also periodic functions, and hence Fourier expansions are important tools to study them. Furthermore, the rigidity side is so strong that these functions span a finite dimensional vector space! It is therefore not surprising that these functions play important roles for computational purposes. They are objects known as modular forms, which are introduced in the next few sections.

1.2 Elliptic modular forms and Fourier expansions

In this thesis we discuss modular forms of integral weights for our aim. And in this section we introduce elliptic modular forms for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Let $\mathfrak{H}$ denote the upper half plane, that is, the set of all complex numbers with positive imaginary part. The special linear group $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathfrak{H}$ via Möbius transformations: for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\tau \in \mathfrak{H}$, we define

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

It is then clear to see, that for each $k \in \mathbb{Z}$ we have a group action of the special linear group $\mathrm{SL}_2(\mathbb{R})$ on the space of holomorphic functions $f : \mathfrak{H} \longrightarrow \mathbb{C}$, defined as follows. For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and holomorphic function $f : \mathfrak{H} \longrightarrow \mathbb{C}$, we define the weight-$k$ slash action via the formula

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau).$$

Let N be positive integer, and let

$$\Gamma(N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N},\, b \equiv c \equiv 0 \pmod{N}\right\}$$

denote the principal congruence subgroup of level N. A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if there is some positive integer N such that

$$\Gamma(N) \subseteq \Gamma.$$

We are now ready to define elliptic modular forms for a congruence subgroup Γ and weight k.

Definition 1.2.1. A modular form of level Γ and weight k is a holomorphic function $f : \mathfrak{H} \longrightarrow \mathbb{C}$ satisfying

1. (modular transformation property) invariance under the weight-k slash action of Γ, that is, $f|_k\gamma = f$ for all $\gamma \in \Gamma$, and

2. (holomorphy at all cusps) that for every $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$, the corresponding cusp expansion $(f|_k\gamma')(\tau)$ is holomorphic at infinity, that is, bounded as $\mathrm{Im}(\tau) \to \infty$.

In addition, a modular form f is called a cusp form of level Γ and weight k if it satisfies the cuspidal condition: for any $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$, the corresponding cusp expansion vanishes at infinity, that is, $(f|_k\gamma')(\tau) \to 0$ as $\mathrm{Im}(\tau) \to \infty$.

Let $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) denote the vector space spanned by modular forms (resp. cusp forms) of level Γ and weight k. Note that from the definition of congruence subgroups, for each congruence subgroup Γ, there is a minimal positive integer $w \in \mathbb{Z}_{\geq 1}$, called the fan width of the congruence subgroup, such that $\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. Consequently, every modular form f of level Γ and weight k is a holomorphic function on the upper half plane $\mathfrak{H}$ satisfying the periodicity $f(\tau + w) = f(\tau)$ for all $\tau \in \mathfrak{H}$. By the holomorphy of f at infinity, we can expand $f(\tau)$ as a power series in $\exp(\frac{2\pi i\tau}{w})$. In other words, f has a Fourier expansion at infinity

$$f(\tau) = \sum_{n=0}^{\infty} c_n \exp\left(\frac{2\pi i n\tau}{w}\right)$$

that features fractional exponents, where the denominator w depends on the congruence subgroup Γ.

For instance, the fan width of $\Gamma = \Gamma(N)$ is N, so that denominator we expect is N. On the other hand, for the congruence subgroup

$$\Gamma = \Gamma_1(N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N},\, c \equiv 0 \pmod{N}\right\},$$

its fan width is $w = 1$, so every modular form $f \in M_k(\Gamma_1(N))$ has a Fourier expansion at infinity with integral exponents.

We have seen that modular forms are a special kind of periodic functions, so it is not surprising that Fourier expansions play an important role in the study of modular forms. Moreover, Fourier expansions of modular forms are very accessible to computation. In fact, for a fixed congruence group Γ and integer k,

there is a uniform constant $C \in \mathbb{Z}$ we can compute, such that every modular form $f \in M_k(\Gamma)$ is determined by its first C Fourier coefficients. We look at the computational aspect of modular forms in Article II.

In addition, what makes modular forms really interesting and connects them to many other subjects is that Fourier coefficients themselves are often objects of great interest in number theory, geometry, and many other areas of mathematics. For instance, the well-known Fermat's Last Theorem claims that there is no non-zero integral solutions to the equation

$$x^n + y^n = z^n,$$

when n is an integer greater than 2. Part of the reason that this theorem became so well known is the simple shape of this equation in sheer contrast to the 358 years of great effort by mathematicians. The final resolution of this problem by Andrew Wiles, is based on the approach of relating a central object in mathematics, called elliptic curves, to Fourier coefficients of modular forms.

In this thesis, we connect modular forms to geometric and combinatorial objects of number-theoretic interests. Article I addresses questions motivated by a specific relation of Fourier coefficients of modular forms with "spherical designs", a mathematical structure featuring a high degree of symmetry and combinatorial flavor. Preprint III studies the modularity of "special cycles" on unitary Shimura varieties, a family of algebro-geometric objects that arise from group theory (representation theory) and can be defined in a purely number-theoretic way. All of them are strongly influenced by the interplay of symmetry of rigidity in mathematics.

1.3 Eisenstein series and their products

Like most subjects in mathematics, the very first question on modular forms we want to ask is: are there any examples of nonzero modular forms?

A natural idea to construct functions that are invariant under the group action of a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is to sum over the group action in such a way that yields an absolute convergent series. It turns out we can do this if we take care of certain quotient of the group Γ. This construction defines an Eisenstein series, which are among the simplest examples of modular forms.

Another natural question that follows the construction of Eisenstein series is: are there any modular forms out there that are more "sophisticated" than Eisenstein series? A short answer is: there are many, and historically theta series were among the first few examples (even though they are also connected to Eisenstein series). In this dissertation, we use theta series to establish results of

arithmetic interest, which are introduced in the next few sections. On the other hand, no matter how complicated modular forms may be, we show that they can always be expressed via Eisenstein series. More precisely, we prove a rigidity result expressing every modular form by a linear combination of products of at most two Eisenstein series. In this section, we define Eisenstein series that we need in this thesis, and explain the meaning of our main results in Article II.

1.3.1 Eisenstein series of level $\mathrm{SL}_2(\mathbb{Z})$

For simplicity, let us first define Eisenstein series for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We start from the constant function 1, and consider the infinite sum over the group action of Γ on this function $\sum_{\gamma\in\Gamma} 1|_k\gamma$. However, such an infinite sum does not converge. Fortunately, this is only caused by the action of the infinite subgroup $\Gamma_\infty = \{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\}$. After taking quotient by this subgroup, we can show that the series

$$E_k(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} 1|_k\gamma(\tau) = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k}$$

converges absolutely when k is greater than 2. Futhermore, note that this series E_k vanishes unless k is an even integer. There is a way to express this series E_k in terms of a sum over a lattice, which is useful for computing its Fourier coefficients. This normalization is usually denoted by

$$G_k(\tau) := \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ (c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^k}, \tag{1.3.1}$$

and is related to the group-theoretic series $E_k(\tau)$ via the formula

$$G_k(\tau) = \zeta(k)E_k(\tau),$$

where $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ is the value at k of the Riemman zeta function.

Next, we want to extend holomorphic Eisenstein series $G_k(\tau)$ to an Eisenstein series of weight 2. Note that there is no non-zero holomorphic modular forms of level $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and weight 2, so we expect a non-holomorphic Eisenstein series in this case. We construct such an extension in two steps.

First, although the sum (1.3.1) does not converge absolutely for $k = 2$, its Fourier expansion at infinity

$$G_k(\tau) = \frac{(2\pi i)^k}{(k-1)!}\left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)\exp(2\pi i n\tau)\right)$$

nevertheless converges so rapidly that it also defines a holomorphic function $G_2(\tau)$ for $k=2$. Here B_k is the k-th Bernoulli number defined via its generating function $\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x-1}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ denotes the $(k-1)$st divisor sum.

Then, we note that $G_2(\tau)$ transforms according to the law

$$(G_2|_2\gamma)(\tau) = G_2(\tau) - \frac{\pi i c}{c\tau + d}$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$. If we introduce a non-holomorphic "correction" term and define $g_2(\tau) := G_2(\tau) - \frac{\pi}{2\mathrm{Im}(\tau)}$, then we see that $g_2(\tau)$ transforms like a modular form of weight 2.

This approach was systematically studied by Hecke, who introduced a method known as "Hecke's trick" to construct Eisenstein series of lower weight in a general setting. Let $s > 0$ be a positive real number, and consider an infinite series

$$G_{2,s}(\tau) := \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^2|c\tau+d|^{2s}}. \tag{1.3.2}$$

This series converges absolutely and transforms via

$$(G_{2,s}|_2\gamma)(\tau) = |c\tau + d|^{2s} G_{2,s}(\tau)$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$. Moreover, we can show that the limit

$$\lim_{s\to 0} G_{2,s}(\tau)$$

exists, and is equal to the non-holomorphic Eisenstein series $g_2(\tau)$ constructed above.

1.3.2 Products of Eisenstein series

This joint work with Martin Raum, aims to strengthen and apply our previous work on vector-valued modular forms, and compute certain effective bounds for the computation of modular forms. A novel method in our work is that we use deeper results on non-vanishing of special L-values to refine previous results.

Pioneering work

Let us first recall the aim of this work. Eisenstein series provide us the simplest examples of non-constant modular forms. The very basic idea of expressing "more sophisticated" modular forms, for instance theta functions, by a linear

combination of products of at most two Eisenstein series, is both conceptually insightful and computationally fruitful. Historically this idea came from the work of Kohnen–Zagier [KZ84] on periods of modular forms, where they obtained the complete result in this respect for modular forms of level $\mathrm{SL}_2(\mathbb{Z})$. Their insight is to relate the expressions for cuspidal Hecke eigenforms with the special values of the associated L-functions, which we further explore in our work for elliptic modular forms of higher levels.

For higher levels however, the situation is more complicated. Borisov and Gunnells [BG01a; BG01b; BG03] first studied modular forms associated with toric varieties, and they wanted to understand what kind of modular forms can occur in this setting. For weight greater than 2, they showed that modulo Eisenstein series, every toric modular form can be generated by products of two toric Eisenstein series. However, it turns out that for the weight-2 modular forms, those arising from toric varieties coincide exactly with the cusp eigenforms with non-vanishing central L-values.

It is natural for us to turn around their question in the context of Kohnen–Zagier, and ask what kind of products of Eisenstein series coincide with all modular forms, even those with vanishing central L-values? In other work generalizing Kohnen–Zagier's results to higher levels, in particular the works of Kohnen–Martin [KM08] and Dickson–Neururer [DN18], the settings again restrict in analogy to the work of Borisov–Gunnells, to the cusp forms of weight 2 with non-vanishing central L-values.

However, this restriction excludes an important class of modular forms in light of Birch and Swinnerton-Dyer Conjecture (which by the conjecture should correspond to elliptic curves over $\mathbb{Q}$ of rank at least 1), and the rank-distribution prediction of (twists of) elliptic curves based on the work of Goldfeld and Katz–Sarnak. If these predictions are true, this restriction may well exclude about half of the most interesting cases. In our work, we break this barrier and for the first time we use a method of Rankin-Selberg type to show that indeed all modular forms, regardless the vanishing of their central L-values, can be expressed by products of at most two Eisenstein series, if we consider slightly more general Eisenstein series.

Eisenstein series of level $\Gamma_1(N)$ and $\Gamma(N)$

To state our results, we first define Eisenstein series of higher levels in this context. Both the large weight and small weight cases are treated in a similar way to the ones of level $\mathrm{SL}_2(\mathbb{Z})$ which we have discussed in details. There are several ways to do so, and we choose the classical approach for self-containment.

The most general approach to define the space of Eisenstein series for a con-

gruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is via the Petersson inner product. Recall that the cusp forms $S_k(\Gamma)$ of level Γ and weight k is a subspace of the modular forms $M_k(\Gamma)$, and we define the space of Eisenstein series $\mathscr{E}_k(\Gamma)$ to be the orthogonal complement of $S_k(\Gamma)$ in $M_k(\Gamma)$. We refer the reader to [Miy89] for more details.

The other approaches essentially use Hecke's trick. From the standpoint of vector-valued modular forms, we can define the Eisenstein series $\mathscr{E}_k(\Gamma)$ to be the space spanned by the components of vector-valued Eisenstein series. See [Wes17] for more details.

Now we present a classical approach by constructing an explicit set of generators for $\mathscr{E}_k(N) := \mathscr{E}_k(\Gamma(N))$. We can show that this space is also the space of all cusp expansions of Eisenstein series of level $\Gamma_1(N)$ and weight k. The space of Eisenstein series of level $\Gamma_1(N)$ is denoted by $\mathscr{E}_k(N)_\infty := \mathscr{E}_k(\Gamma_1(N)) = \mathscr{E}_k(N) \cap M_k(\Gamma_1(N))$, as it corresponds to Fourier expansions at infinity.

For large weights $k > 2$, Eisenstein series can be defined via a double sum similar to the sum (1.3.1) for those of level $\mathrm{SL}_2(\mathbb{Z})$. For a pair of integers (c', d') such that $\gcd(c', d', N) = 1$, we define an Eisenstein series

$$G_{k,N,c',d'}(\tau) := \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ c\equiv c' \,(\mathrm{mod}\, N) \\ d\equiv d' \,(\mathrm{mod}\, N)}} \frac{1}{(c\tau + d)^k},$$

which is of level $\Gamma(N)$ and weight k. We then define $\mathscr{E}_k(\Gamma(N))$ to be the space spanned by all these Eisenstein series $G_{k,N,c',d'}$.

For weight $k = 2$, we use Hecke's trick to define non-holomorphic Eisenstein series $G_{2,N,c',d'}(\tau, s)$ for $s \in \mathbb{R}_{>0}$ similar to (1.3.2), and we can show that $G_{2,N,c',d'}(\tau, s)$ has a limit $g_{2,N,c',d'}(\tau)$ as $s \to 0$. Although each $g_{2,N,c',d'}$ is non-holomorphic, the pairwise differences of these functions are holomorphic and satisfy the modular transformation property for $\Gamma = \Gamma(N)$. We define $\mathscr{E}_2(\Gamma(N))$ to be the space spanned by all these differences.

Finally for weight $k = 1$ and each (c', d'), similar to the function G_2 defined for level $\mathrm{SL}_2(\mathbb{Z})$, we can either use the Fourier expansions or rearrange the double sum to define a holomorphic function $G_{1,N,c',d'}(\tau)$. Then we correct these functions by certain constant functions to obtain holomorphic Eisenstein series

$$g_{1,N,c',d'}(\tau) := G_{1,N,c',d'}(\tau) + \frac{2\pi i}{N}\left(\frac{c'}{N} - \frac{1}{2}\right),$$

and define $\mathscr{E}_1(\Gamma(N))$ to be their linear span.

Our contribution

Given the importance of weight-2 modular forms of higher levels in number theory and geometry, we manage to cover all cases in our results.

We prove that for every fixed positive integer N, there is a positive integer N_0 that is effective in principle, such that every weight-2 modular form of level $\Gamma(N)$ can be written as a sum of an Eisenstein series in $\mathscr{E}_2(N)$ and a linear combination of products of two Eisenstein series in $\mathscr{E}_1(N_0)$.

Another novel feature of our work is that we fix the weights of Eisenstein series which express all modular forms of a given weight, as opposed to earlier results which require contribution from all possible weights. As a further refinement, we show that every modular form of weight at least 3 can be expressed by a linear combination of products of exactly two Eisenstein series of certain specified levels.

Moreover, our work is accessible for computation of modular forms. As this thesis is completed, Tobias Magnusson and Martin Raum have implemented a large part of our work, and related research articles are in preparation.

Methods and further investigations

We work with the framework of vector-valued modular forms and view classical modular forms as their components. This point of view helps us conceptualize, organize, and simplify our work by means of representation theory.

We employ the Rankin–Selberg method to connect products of Eisenstein series with special values of L-functions associated with twists of new forms, and use results on non-vanishing of these values and the Shimura–Waldspurger correspondence to reduce the problem to non-vanishing of Fourier coefficients of certain half-integral weight modular forms. Note that the main tools that we use are also available for more general types of automorphic forms. In particular, we expect similar methods can be carried out for half-integral modular forms and modular forms of higher degree.

Historically the Rankin–Selberg method dates back to Riemann, who constructed his Riemann zeta function as the Mellin transform of Jacobi's theta series. It was the asymptotics and automorphy of theta series that allowed Riemann to obtain the well-known analytic continuation and functional equation of the zeta function, respectively. We introduce theta series and their central role in the next few sections.

As a final remark about products of Eisenstein series, we may turn around the question again, and ask what types of more general geometric objects are such products associated with? They are more general than toric varieties, and

we expect them to be twisted toric varieties.

1.4 Theta series and spherical designs

The history of theta series is almost as old as that of arithmetic. Euler studied infinite products of arithmetic interest, and proved the formula

$$\prod_{n=1}^{\infty}(1-q^n)^{-1}=\sum_{n=0}^{\infty}p(n)q^n,$$

which relates the infinite product to the number $p(n)$ of partitions of n. In the quest for this type of identities, he discovered the celebrated pentagonal number theorem, which relates the infinite product representation and the series representation of the Euler function $\phi(q)$ by

$$\phi(q)=\prod_{n=1}^{\infty}(1-q^n)=\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{3n^2-n}{2}},$$

where $\frac{3n^2-n}{2}$ is called the n-th pentagonal number because it counts the dots in a certain diagram associated to a regular pentagon whose sides consist of n dots.

The Euler function $\phi(q)$ was later generalized to Jacobi's theta functions. In fact, another similar well-known example $\sum_{n\in\mathbb{Z}}q^{n^2}$ had already appeared in the work of Fourier, but it was not until the time of Jacobi and Abel, who systematically studied these functions by algebraic means, especially after Jacobi's theory of elliptic functions was born, that theta series finally became part of modern number theory. The Jacobi theta functions also motivate the notion of Jacobi forms of higher degrees considered in this dissertation, and serve as a bridge in Article I and a powerful tool in Preprint III.

In modern times, Weil (1964) reformulated classical theta series from the perspective of representation theory, which we adopt in Preprint III. The Weil representation has a far-reaching generalization, namely the theta correspondence (Howe correspondence), which relates representations of a reductive dual pair, in both local and global versions. A special case of the theta correspondence, known as the Shimura–Waldspurger correspondence, is used as a powerful tool in Article II.

Our quest in this dissertation starts from the Poisson summation formula. Let f be a function defined and continuous on the reals, such that the series $\sum_{n\in\mathbb{Z}}\hat{f}(n)$ of the Fourier coefficients $\hat{f}(n)$ converges absolutely, and on every compact set, the series $\sum_{n\in\mathbb{Z}}\|f_n\|_\infty$ converges for $f_n(x)=f(x+n)$ defined on the compact set, then the Poisson summation formula asserts that for every $x\in\mathbb{R}$,

we have

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\exp(2\pi i n x).$$

This seemingly simple fact is nonetheless ubiquitous in modular forms and number theory. For example in this dissertation, it gives rise to the modularity of the theta functions considered in Article I and Preprint III; it is also a key ingredient to prove the modularity of the generating functions of special cycles considered in Preprint III. We motivate and present the main results in this respect. A more formal summary of our results is presented in Section 1.7.

In this section, we introduce the notion of spherical designs, and their connection to a well-known conjecture on the non-vanishing of Fourier coefficients of certain theta series. This motivates us to ask a more general question regarding the existence of rational geometric designs.

1.4.1 Spherical designs and Lehmer's conjecture

Just like modular forms, spherical designs are a very rare object in mathematics, and can be best illustrated from the interplay of symmetry and rigidity. Spherical designs are closely related to spherical codes, and both are finite subsets of the unit sphere in a Euclidean space. Roughly speaking, the idea of spherical designs is more based on the symmetry standpoint, which is to find such a finite subset that "interpolates" (in a dual way) integrals of polynomial functions up to a certain degree on the unit sphere; and the idea of spherical codes is more related to the rigidity point of view, which aims to find finitely many points whose minimum pairwise distance is as large as possible. More generally, the theories of designs and codes are unified under the framework of association schemes, and the latter is a generalization of groups and the character theory of group representations. We focus on spherical designs in this section.

Definition 1.4.1. Let t be a natural number. A finite subset $X \subseteq S^{n-1} \subseteq \mathbb{R}^n$ is called a spherical t-design, also known as spherical design of strength t, if for all polynomials f of degree at most t on the unit sphere S^{n-1}, we have

$$\frac{1}{\mu(S^{n-1})}\int_{S^{n-1}} f\,\mathrm{d}\mu = \frac{1}{|X|}\sum_{x\in X} f(x),$$

where μ is the ordinary (Lebesgue) measure on the unit sphere.

Example 1.4.2. The following well-known examples are summarised in [BB09].

1. On $S^1 \subseteq \mathbb{R}^2$, the regular N-gon gives rise to a spherical $(N-1)$-design.

2. On $S^2 \subseteq \mathbb{R}^3$, the regular tetrahedron (resp. cube, regular octahedron, regular dodecahedron, regular icosahedron) gives rise to a spherical 2- (resp. 3-, 3-, 5-, 5-) design.

3. On $S^3 \subseteq \mathbb{R}^4$, regular polytopes give rise to spherical designs of strength $t \leq 11$. For every large n, regular polytopes give rise to spherical designs of strength $t \leq 3$ on $S^{n-1} \subseteq \mathbb{R}^n$.

4. The root system of type A_n, D_4, D_n, E_6, E_7, and E_8 give rise to spherical designs of strength 3, 5, 3, 5, 5, and 7, respectively.

5. The set of minimal vectors of the Leech lattice gives rise to a spherical 11-design on $S^{23} \subseteq \mathbb{R}^{24}$.

We observe from these examples that spherical designs of smaller cardinality, higher strength, and in higher dimensions are rarer to find. Similar to the Eisenstein series, there are two natural ways to construct spherical designs: via the orbit of a point under the group action of a suitable finite subgroup of the real orthogonal group $O(n)$ on $S^{n-1} \subseteq \mathbb{R}^n$, and via a shell of a lattice in $\mathbb{R}^n$, that is, a set of points in a lattice with a fixed distance from the origin. We are interested in taking shells of lattices, as it yields spherical designs of number-theoretic nature. Let us look at two examples listed above more closely, which are discovered by Venkov [Ven84]. A striking feature is that they consist solely of rational points, and hence we call them rational spherical designs.

Example 1.4.3. For an integral lattice Λ and a natural number m, let $\Lambda_m := \{x \in \Lambda : \|x\|^2 = m\}$ be the shell of lattice points of norm m.

1. Let $\Lambda \subseteq \frac{1}{2}\mathbb{Z}^8$ be the E_8-lattice. For every $m \in \mathbb{Z}_{\geq 1}$, $\frac{1}{2m}\Lambda_{4m^2}$ is a rational spherical 7-design on S^7 (over $\mathbb{Q}$).

2. Let $\Lambda \subseteq \frac{1}{\sqrt{8}}\mathbb{Z}^{24}$ be the Leech lattice. For every $m \in \mathbb{Z}_{\geq 2}$, $\frac{1}{\sqrt{2}m}\Lambda_{2m^2}$ is a rational spherical 11-design on S^{23} (over $\mathbb{Q}$).

On the other hand, shells of lattices are only known to yield spherical designs of strength t at most 11. In fact, based on the work of Venkov [Ven84], de la Harpe–Pache–Venkov [HPV06; Pac05] related spherical designs (of arithmetic nature) to the well-known Lehmer conjecture, which asserts that the Ramanujan τ-function never vanishes, where $\tau(n)$ is the n-th Fourier coefficient of the discriminant modular form $\Delta(\tau)$. The latter is related to the Euler function in (1.4) via the formula

$$\Delta(\tau) = \sum_{n \geq 1} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

for $q = \exp(2\pi i\tau)$ and τ is in the upper half plane of the complex numbers.

Proposition 1.4.4 ([Pac05]). *Let $\Lambda \subseteq \frac{1}{2}\mathbb{Z}^8$ be the E_8-lattice. For any positive integer m, the following statements are equivalent:*

(i) $\tau(m) = 0$.

(ii) $\frac{1}{\sqrt{2m}}\Lambda_{2m}$ *is a spherical* 8*-design.*

More generally, for any integral lattice $\Lambda \subseteq \mathbb{R}^n$ and every polynomial function P in n variables, we consider the theta series $\Theta_{\Lambda,P}$ associated to Λ weighted by P with the m-th Fourier coefficient $c_{m,P}$ for each $m \geq 0$. Then we can relate spherical designs similarly to variations of Lehmer's conjecture.

Proposition 1.4.5 ([Pac05]). *For every positive integer m such that the shell of lattice Λ_m is nonempty, the following statements are equivalent:*

(i) $c_{m,P} = 0$ *for every harmonic polynomial P of even degree up to t.*

(ii) Λ_m *yields a spherical t-design.*

Assuming that Lehmer's conjecture and all of its variations for integral lattices (for instance the Leech lattice) are true, "pure" shells of lattices cannot yield spherical 12-designs in any dimension. A natural question of arithmetic nature is then, can we mix shells of different integral lattices to find spherical designs of higher strength $t \geq 12$?

1.4.2 Rational geometric designs and our contribution

It turns out that the question raised here can be asked in a more general manner. Instead of asking about the existence of rational spherical designs, we may define geometric designs in analogy to spherical designs and many other combinatorial designs, and ask about the existence of rational geometric designs.

For a good space Z in our context, there is usually a natural measure μ_Z and a concept of polynomial functions on Z. A t-design X on Z is a finite subset such that for every polynomial f on Z of degree at most t, the equation

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|Z|}\int_Z f \,\mathrm{d}\mu_Z$$

holds for the total measure $|Z|$ of the space Z. There is also a similar notion about combinatorial designs when the space Z is discrete, which we omit here.

The notion of geometric design is also known as the cubature formula [GS81; Möl79] and averaging sets [SZ84], and was historically first introduced by Delsarte, Goethals and Seidel in [DGS77]. There is a large literature on geometric designs, among which the most relevant to this dissertation are about interval designs [RB91; Wag91], Euclidean designs [DS89; NS88], unitary designs [RS09], and designs on homogeneous spaces, and most of them are closely related to each other via certain mappings. We refer to the survey by Eiichi Bannai and Etsuko Bannai [BB09] for further information.

For a good space Z in the context of geometric designs, we say that Z is geometrically designed over a field $F = F(Z)$, if F is the smallest field such that for every t there is a t-design on Z consisting of F-rational points. So the aforementioned question can be formulated more strongly as: for a positive integer d, is the unit sphere S^d geometrically designed over a number field? And what kind of geometric spaces Z are geometrically designed over a number field?

This project joint with Zhen Cui and Ziqing Xiang, aims to attack such kind of questions for the first time. Motivated by a question of Eiichi Bannai on the unit interval, we showed in Article I that every rationally defined "ambient" space is geometrically designed over $\mathbb{Q}$. In particular, this is true for every open connected subspace satisfying a certain rational condition and every rationally defined convex polytope in the Euclidean spaces.

Consequently, we proved that for any positive integer d, the field $F(S^d)$ over which S^d is geometrically designed, satisfies

$$\mathbb{Q} \subseteq F(S^d) \subseteq \mathbb{Q}(\{\sqrt{p} : p \text{ is a prime number}\}).$$

Note that from this result, the Galois theory, and the Kronecker–Weber theorem, if $F(S^d)$ is a number field, then there is a minimal cyclotomic field $\mathbb{Q}(\zeta_n)$ such that $F(S^d) \subseteq \mathbb{Q}(\zeta_n)$. Furthermore, from standard constructions in design theory, we know that such hypothetical n must be essentially independent of d; this provides evidence that S^d might be geometrically designed over $\mathbb{Q}$ for any d.

1.5 Special cycles on unitary Shimura varieties

Recall that modular forms can also be viewed as functions on the "moduli space" of lattices in the complex plane, hence the word "modular". The idea of a moduli space is to identify a geometric space whose points represent algebro-geometric objects of a fixed type. Historically, moduli spaces were viewed as spaces of parameters in a classification problem. For instance, lines on a plane passing through a fixed point can be classified by the angles between a fixed line and them.

If we want to classify these lattices up to homotheties, we can choose an oriented basis (ω_1, ω_2) for each lattice Λ, such that the ratio $\frac{\omega_2}{\omega_1}$ is in the upper half plane $\mathfrak{H}$. However, there are different ways to choose such a basis, which are transformed under the group action of $\mathrm{SL}_2(\mathbb{Z})$. It turns out that the quotient space $Y(1) := \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ is the desired moduli space. An amazing fact is that although we start from the upper half plane $\mathfrak{H}$ which is of complex-analytic nature, the moduli space $Y(1)$ is an algebraic curve, and is moreover of arithmetic nature. We may in general construct such a quotient space by a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, and obtain a modular curve $Y(\Gamma) := \Gamma\backslash\mathfrak{H}$.

Elliptic curves became a central object in modern mathematics, particularly in geometry and number theory. Every elliptic curve defined over $\mathbb{C}$ can be represented by a quotient $\mathbb{C}/\Lambda$ for a lattice $\Lambda \subseteq \mathbb{C}$. Therefore, the modular curve $Y(1)$ can also be viewed as the moduli space of the isomorphism classes of complex elliptic curves. We can show that the endomorphisms of an elliptic curve $E = \mathbb{C}/\Lambda$, that is, morphisms from E to itself both as a group and as an algebraic curve, correspond to complex numbers λ such that $\lambda\Lambda \subseteq \Lambda$. For a generic elliptic curve, the only possible values of λ satisfying this condition are the integers $\mathbb{Z}$. We say an elliptic curve admits complex multiplication if there are other values of λ than integers satisfying this condition, and the corresponding points on the modular curve $Y(1)$ are called CM-points.

The theory of complex multiplication was historically developed as a central topic in number theory. Heegner applied CM-points on modular curves to study the class number problem for imaginary quadratic fields. In their celebrated work, Gross and Zagier used CM-points to construct a rational point of infinite order on any modular elliptic curve of analytic rank 1. Hilbert first proposed to extend this theory to higher dimension, and the Hilbert modular varieties was a first step in his program.

The modern theory of complex multiplication was established in the study of abelian varieties with complex multiplication, which are analogues of elliptic curves in higher dimension, by Shimura, Taniyama and Weil. Shimura varieties are generalizations of modular curves to higher dimension, which can also be defined over a number field, and they were reformulated by Deligne as moduli spaces of Hodge structures. In this thesis we work with certain unitary Shimura varieties. In particular, the group $\mathrm{SL}_2(\mathbb{R})$ (and its discrete subgroup $\mathrm{SL}_2(\mathbb{Z})$) that we see for the modular curve $Y(1)$ is replaced by a unitary group of signature $(n-1, 1)$ (and a discrete subgroup). And the upper half space $\mathfrak{H}$ is replaced by a hermitian symmetric space defined below.

Since Grothendieck, the study of subvarieties plays a prominent role for understanding geometric problems. This modern point of view emphasizes the idea that if we know enough information about subvarieties on some algebraic

variety, then we should be able to understand this variety as well. Algebraic cycles provide a way to linearize subvarieties, and can be naively viewed as linear combinations of subvarieties of the same dimension modulo certain linear relations. In light of the modern point of view in geometry, it is highly desirable to construct as many algebraic cycles as possible on Shimura varieties. Special cycles form an attractive family of algebraic cycles on unitary Shimura varieties, and they are indexed by positive definite Hermitian matrices, which are analogues of CM-points on modular curves in higher dimension. Just like Shimura varieties, they also arise from representation theory of groups, and are linear combinations of Shimura varieties themselves. They are geometric objects of highly symmetric nature, and we expect that they also exhibit a high degree of rigidity in the next section.

In the rest of this section, we introduce these special cycles quantitatively based on lattices and group theory from a classical approach. To further simplify the situation, we focus on open connected Shimura varieties. Let $E/\mathbb{Q}$ be an imaginary quadratic field and fix a complex embedding $E \hookrightarrow \mathbb{C}$. Let $n \geq 2$ be an integer. Let (V, Q) be an E-Hermitian space of signature $(n-1, 1)$ and write $(\cdot, \cdot)$ for the Hermitian form corresponding to the quadratic form Q, and $H = \mathrm{U}(V)$ be the unitary group of V, which we view as a classical (matrix) group. Let $V(\mathbb{C}) := V \otimes_E \mathbb{C}$ be the complexification of V and $P(V(\mathbb{C})) := (V(\mathbb{C}) \setminus \{0\})/\mathbb{C}^*$ the corresponding projective space. Let $\mathbb{D}$ be the Hermitian symmetric space attached to the group H, which in a projective model consists of all negative $\mathbb{C}$-lines in $V(\mathbb{C})$. In other words, this space can be realized as

$$\mathbb{D} = \big\{[z] \in P(V(\mathbb{C})) : (z, z) < 0\big\},$$

which is isomorphic to the open unit ball in $\mathbb{C}^{n-1}$.

Let $L \subseteq V$ be an even lattice, that is, the norm of every lattice point is an even integer. In particular, L is an integral lattice. Let $L^\vee$ be the dual lattice, given by

$$L^\vee = \big\{x \in V : (x, y) \in \mathbb{Z} \text{ for all } y \in L\big\}.$$

As L is an integral lattice, we have $L \subseteq L^\vee$, and the quotient $L^\vee/L$ is a finite abelian group, known as the discriminant group. Let $\Gamma \subseteq \mathrm{U}(L)$ be a subgroup of finite index which acts trivially on the discriminant group $L^\vee/L$. Note that this is an analogue of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Similar to the construction of modular curves we consider the quotient

$$X_\Gamma = \Gamma \backslash \mathbb{D},$$

which has a structure as a quasi-projective algebraic variety of dimension $(n-1)$, by the work of Bailey-Borel, Shimura, Deligne and others. This is a unitary Shimura variety that we work with.

We then proceed to define special cycles. Let g be an integer such that $1 \le g \le n-1$. For a g-tuple $\lambda = (\lambda_i)_i \in V(E)^g$, we define $V_\lambda := \{v \in V : \forall i (v \perp \lambda_i)\}$, $H_\lambda := \mathrm{U}(V_\lambda)$, and $\mathbb{D}_\lambda := \{w \in \mathbb{D} : \forall i (w \perp \lambda_i)\}$. If the $g \times g$ Hermitian matrix $Q(\lambda) := \frac{1}{2}\big((\lambda_i, \lambda_j)\big)_{i,j}$ is positive semidefinite and $\mathrm{rk}(Q(\lambda)) = \mathrm{rk}\{\lambda_i : 1 \le i \le g\}$, then this extra data allows us to define an analytic subvariety Z_λ (not necessarily algebraic) in essentially the same way as the unitary Shimura variety X_Γ. For each Hermitian positive semidefinite matrix $T \in \mathrm{Herm}_g(E)_{\ge 0}$, and each "shift" $\mu \in L^\vee / L$, we define a "translated cycle"

$$Z(T, \mu) := \sum_{\substack{\lambda \in \mu + L \\ Q(\lambda) = T \\ (\mathrm{mod}\, \Gamma)}} Z_\lambda.$$

Since Γ acts trivially on $L^\vee / L$, by Chow's lemma this cycle descends to an algebraic cycle on X_Γ.

Finally, for weight functions $\varphi : L^\vee / L \longrightarrow \mathbb{C}$, we form a family of special cycles indexed by (T, φ) via

$$Z(T, \varphi) := \sum_{\mu \in L^\vee / L} \varphi(\mu) Z(T, \mu),$$

which are Γ-invariant and descend to an algebraic cycle on the unitary Shimura variety X_Γ.

1.6 Kudla's modularity conjecture

Our aim in this work is to show some cases of the unitary Kudla conjecture unconditionally.

1.6.1 Modularity of special cycles

In the last section, we see that special cycles are a family of weighted sums of sub-Shimura varieties indexed by positive semidefinite Hermitian matrices, so it is desirable to study their generating functions. Special cycles arise from group theory, and carry a high degree of symmetry. For this reason, we expect them to exhibit a certain level of rigidity. Recall from the beginning of this thesis that modularity is a very strong type of rigidity, and mathematical objects having modularity are automatically more accessible for computation. Given that relations of algebraic cycles constitute a central theme in geometry while they are at the same time hard to compute, it is desirable to establish modularity of a suitable generating series of special cycles. This idea was formulated by Kudla and others in a series of work in the 1990's.

Modularity of generating series of objects in a more classical context that are similar to the idea of special cycles dates back to Jacobi, who showed that the generating series of the number of solutions to the equation $Q(x) = t$ defines an elliptic modular form. Here Q is a positive definite quadratic form, the unknown x lies in a lattice L, and t runs through a family of non-negative integers. In the twentieth century, Siegel and Hecke extended Jacobi's work to higher rank and they also studied the case of indefinite quadratic forms Q.

Kudla's modularity conjecture is about the modularity of the generating functions of special cycles on Shimura varieties of orthogonal and unitary types, valued in the Chow groups and their variations. This conjecture lies at the heart of Kudla's Program, which is a development of the earlier work of Hirzebruch–Zagier [HZ76], Gross–Zagier [GZ86], Kudla–Millson [KM86; KM87; KM90], and Gross–Keating [GK93] among others, formulated by Kudla in [Kud97].

1.6.2 The unitary Kudla conjecture

Let us state the unitary Kudla conjecture based on the notation from the last section. Let $\mathscr{L}_{\mathbb{D}}$ be the tautological line bundle over $\mathbb{D}$, that is, the restriction to $\mathbb{D}$ of the line bundle $\mathscr{O}(-1)$ over $\mathbb{P}(V(\mathbb{C}))$. Since the action of $H = \mathrm{U}(V)$ on $\mathbb{D}$ lifts naturally to a certain action on $\mathscr{L}_{\mathbb{D}}$, the line bundle $\mathscr{L}_{\mathbb{D}}$ descends to a line bundle $\mathscr{L}_\Gamma$ over the Shimura variety X_Γ. We write $\mathscr{L}_\Gamma^\vee$ for the class of the dual line bundle in the first Chow group $\mathrm{CH}^1(X_\Gamma)$. It is clear that $Z(T,\varphi)$ is a cycle of codimension $r(T)$ (the rank of T) from the definition of special cycles, and we write $Z(T,\varphi)$ also for its class in the Chow group $\mathrm{CH}^{r(T)}(X_\Gamma)$. Finally, the generating series for special cycles on X_Γ is defined to be the formal sum

$$\psi^{\mathrm{CH}}_{g,\varphi}(\tau) := \sum_{T \in \mathrm{Herm}_g(E)_{\geq 0}} Z(T,\varphi) \cdot (\mathscr{L}_\Gamma^\vee)^{g-r(T)} \exp\big(2\pi i \operatorname{tr}(T\tau)\big),$$

where the dot "$\cdot$" denotes the intersection product in the Chow ring $\mathrm{CH}^\bullet(X_\Gamma)$, and τ lies in the Hermitian upper half space $\mathbb{H}_g$ (in analogy to the upper half plane $\mathfrak{H}$ defined at the beginning of the thesis). Note that the intersection product with the dual line bundle $\mathscr{L}_\Gamma^\vee$ is nothing but taking a hyperplane section. The unitary Kudla conjecture states that, for any linear functional ℓ on the Chow group $\mathrm{CH}^g(X_\Gamma)$, the termwise application of the functional to the generating series, denoted by $\ell \circ \psi^{\mathrm{CH}}_{g,\varphi}(\tau)$, is a Hermitian modular form over $E/\mathbb{Q}$ of genus g and weight n, that is, absolutely convergent and invariant under a certain weight-n slash action of a congruence subgroup of $\mathrm{U}(g,g)$ over the integers in E.

1.6.3 Previous work and arithmetic applications

First of all, an immediate consequence of this conjecture is that the $\mathbb{C}$-span of special cycles $Z(T,\varphi)$ in the complexification of the Chow group $\mathrm{CH}^g(X_K)_{\mathbb{C}}$ is finite dimensional. A further appealing feature is that relations between Fourier coefficients of certain Hermitian modular forms give rise to the corresponding relations between special cycles $Z(T,\varphi)$ in the complexification $\mathrm{CH}^g(X_K)_{\mathbb{C}}$, which are otherwise inaccessible in the literature. See [BW15; Wes15] for this computational aspect in the orthogonal case.

The modularity of generating series for geometric cycles has been studied since the seminal work of Hirzebruch–Zagier [HZ76]. In a long collaboration, Kudla–Millson [KM86; KM87; KM90] established the modularity of generating series of special cycles valued in cohomology. The analogous generating series valued in Chow groups can also be defined, and it is natural to ask if they are already modular at this level. In the case of Shimura varieties of orthogonal type $(n-2,2)$, this question was raised in [Kud97].

Two years later, Borcherds [Bor99] proved the modularity of generating series for Heegner (special) divisors valued in the first Chow group inspired by the work of Gross–Kohnen–Zagier [GKZ87] on the images of Heegner points in the Jacobian of a modular curve, and also based on his celebrated work [Bor98] on the construction of a family of meromorphic modular functions via regularized theta lift. Building upon Borcherds' work, W. Zhang [Zha09] proved that assuming absolute convergence, the generating series for special cycles valued in the Chow groups are Siegel modular forms. Finally, Bruinier–Raum [BW15] completed the proof of Kudla's modularity conjecture over $\mathbb{Q}$ by showing that such a convergence assumption holds unconditionally.

In a recent colloboration of Bruinier–Howard–Kudla–Rapoport–Yang, modularity of generating series for arithmetic special divisors over certain imaginary quadratic fields was established [Bru+20a], and more arithmetic applications were found [Bru+20b], including relations between derivatives of L-functions and arithmetic intersection pairings à la Gross–Zagier, as well as a special case of Colmez's Conjecture on the Faltings heights of abelian varieties with complex multiplication, in line with the work of Andreatta–Goren–Howard–Pera [And+18]. In an announced work by Bruinier–Howard, they use an inductive argument to compute the arithmetic volumes of unitary Shimura varieties of higher dimension, based on the work of Howard [How20] for unitary Shimura curves.

Most recently under the assumption of the unitary Kudla conjecture for general CM-fields, Chao Li and Yifeng Liu [LL21] proved the arithmetic inner product formula, which can be viewed as a generalization of the Gross–Zagier formula to higher dimensional motives. Their approach is via arithmetic theta lift-

ing. While the exact cases needed by the work of Li–Liu are not covered in this thesis as they require a more general CM-fields than the imaginary quadratic fields, this illustrates the arithmetic indications of the presented thesis work.

1.6.4 The main result

In Preprint III, we combine a result of Yifeng Liu [Liu11] on the formal modularity of such generating series in the cases of arbitrary CM-fields and the method of Bruinier–Raum to show the following main result.

Theorem 1.6.1 (Preprint III). *The unitary Kudla conjecture for open Shimura varieties is true in the cases of norm-Euclidean imaginary quadratic fields, which are the ones generated by a square root of* $-1, -2, -3, -7, -11$*, repectively.*

Remark 1.6.2. After uploading a first version of this preprint on arXiv, the author learned that Yuxiang Wang partially proved similar results in his thesis work [Wan20]. Note that Wang claims to prove the general case of imaginary quadratic fields. However, in the proof of Lemma 4.6 on page 39 of his thesis, the choice of r is not justified, and in fact cannot be made in the general case. One of the first counterexamples arises from the case of $E = \mathbb{Q}(\sqrt{-5})$, where in Wang's notations for an arbitrarily fixed $a \in \frac{1}{m}(\mathscr{O}^\#)^g/\mathscr{O}^g$, one cannot even choose $r = (r_1, \dots, r_g)$ such that $\frac{1}{m} r \equiv a \pmod{\mathscr{O}^g}$ and that $|r_g|^2 < m^2$ (but Wang claims $|r_g|^2 \leq \frac{m^2}{D}$ for $D = 20$).

1.7 Summaries of the appended publications

1.7.1 Article I

The existence of geometric designs on path-connected spaces was first proved by Seymour–Zaslavsky [SZ84]. Inspired by their work, we defined a suitable notion of path-connected spaces for designs over $\mathbb{Q}$, which we call algebraically path-connected spaces, and proved the following analogous result for rational geometric designs. Let t and d be natural numbers in the following discussion.

Theorem 1.7.1 (Article I). *For every algebraically path-connected space defined in the paper, there exists* t*-designs on it of size* n *for arbitrary* t *and every sufficiently large integer* n*.*

A natural consequence of this result is the existence of rational t-designs on rational convex polytopes.

Theorem 1.7.2 (Article I). *Let $Z \subseteq \mathbb{R}^d$ be a d-dimensional convex polytope whose vertices are in $\mathbb{Q}^d$, and μ_Z the Lebesgue measure. Then, there exists a natural number n_0 such that for every natural number $n \geq n_0$, there exist rational t-designs on (Z, μ_Z) of size n. In particular, there exist rational t-designs on the unit interval $[0,1]$.*

On the unit circle S^1, the notion of regular $(t+1)$-gon provides a spherical t-design over the field $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$, where $\mathbb{Q}^{\mathrm{ab}}$ is the abelian closure of $\mathbb{Q}$. Combining this aspect with our main result, we showed the following result about spherical designs.

Theorem 1.7.3 (Article I). *There exists a natural number n_0 such that for every even natural number $n \geq n_0$, there exist spherical t-designs on S^d of size n where for each point in the design, all of its coordinates are rational numbers except possibly for the first coordinate. In particular, there exist spherical t-designs of size n over the field $\mathbb{Q}(\{\sqrt{q} : q \text{ prime}\})$.*

1.7.2 Article II

Let k and N be positive integers. Let $\mathscr{E}_k(N)$ denote the space spanned by Fourier expansions at every cusp of the congruence subgroup $\Gamma_1(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ of all Eisenstein series of weight k and level N. Let $M_k(\Gamma(N))$ denote the space of weight-k modular forms for the principal congruence subgroup $\Gamma(N)$. We proved the following main result.

Theorem 1.7.4 (Article II). *Let k, l, and N be positive integers. Then there is a positive integer N_0 such that*

$$M_{k+l}(\Gamma(N)) \subseteq \mathscr{E}_{k+l}(N) + \mathscr{E}_k(N_0) \cdot \mathscr{E}_l(N_0).$$

Moreover, if $k+l \geq 3$, then a suitable N_0 is explicitly specified in the paper, and there is a positive integer N_1, specified explicitly in the paper such that

$$M_{k+l}(\Gamma(N)) \subseteq \mathscr{E}_k(\mathrm{lcm}(N_0, NN_1)) \cdot \mathscr{E}_l(\mathrm{lcm}(N_0, N_1)).$$

Remark 1.7.5. This main result is a consequence of two theorems in the paper expressing cusp forms and Eisenstein series by linear combinations of products of exactly two Eisenstein series, respectively. Our results in the paper also cover the more general case of arbitrary arithmetic type, by general theory about the connection between vector-valued elliptic modular forms and classical elliptic modular forms. Moreover, we obtain precise statements about which subspace is equal to $M_k(\chi)$ for a Dirichlet character χ modulo N.

1.7.3 Preprint III

Following the approach of Bruinier–Raum [BW15] in the orthogonal case over the rationals $\mathbb{Q}$, I proved the following result in some unitary cases over imaginary quadratic fields.

Theorem 1.7.6 (Preprint III). *The unitary Kudla conjecture for open Shimura varieties is true in the cases of norm-Euclidean imaginary quadratic fields.*

Remark 1.7.7. Note that if the Hermitian space V is anisotropic, so that X_K is compact, my result automatically covers the whole conjecture in this case.

The proof of Theorem 1.7.6 combines Theorem 3.5 of [Liu11] about formal modularity of the generating series and my result stated below that such formal series satisfying the modularity automatically converge absolutely. To state this rigidity result, we first introduce the notion of symmetric formal Fourier–Jacobi series, which was shown in [Liu11; YZZ09; Zha09] to encode among other things special cycles on Shimura varieties of orthogonal and unitary types. Fourier–Jacobi expansions of automorphic forms, first defined in [Pia66], are among prominent tools to study automorphic forms. For example, they played important roles in the proof of Saito–Kurokawa conjecture [And79; Maa79a; Maa79b; Maa79c; Zag81].

Let $E/\mathbb{Q}$ be an imaginary quadratic field. For integers g, k, l such that $0 \le l \le g$, every Hermitian modular form f of degree g, weight k has a Fourier–Jacobi expansion of cogenus l. More precisely, if we write the variable $\tau \in \mathbb{H}_g$ in the Hermitian upper half space $\mathbb{H}_g$ as

$$\tau = \begin{pmatrix} \tau_1 & w \\ z & \tau_2 \end{pmatrix},$$

for $\tau_1 \in \mathbb{H}_{g-l}$, $\tau_2 \in \mathbb{H}_l$, $w \in \mathrm{Mat}_{g-l,l}(\mathbb{C})$, and $z \in \mathrm{Mat}_{l,g-l}(\mathbb{C})$, then every Hermitian modular form f of degree g and arithmetic type ρ has a Fourier–Jacobi expansion of the form

$$f(\tau) = \sum_{m \in \mathrm{Herm}_l(E)_{\ge 0}} \phi_m(\tau_1, w, z) e(m\tau_2),$$

where the sum runs over all the $l \times l$ positive semidefinite Hermitian matrices m with entries in E, and $e(x) := \exp\left(2\pi i \cdot \mathrm{tr}(x)\right)$ for a square matrix x, and the coefficients ϕ_m are Hermitian Jacobi forms of degree $(g-l)$, weight k, type ρ and index m, which satisfy a certain symmetry condition for their ordinary Fourier coefficients from the modularity of f.

By general definition, every Hermitian modular form is a formal Fourier–Jacobi series satisfying a certain symmetry condition and absolute convergence. This point of view motivates us to define symmetric formal Fourier–Jacobi series, which are formal Fourier–Jacobi series that satisfy the symmetry condition, without assuming absolute convergence. By the definition, if such a series converges absolutely, then it is a Hermitian modular form. One natural question is: does every symmetric formal Fourier–Jacobi series automatically converge absolutely?

In the case of Siegel modular forms, Ibukiyama–Poor–Yuen also raised this question for Siegel paramodular forms in [IPY13]. The first breakthrough in this direction belongs to J. Bruinier [Bru15] and M. Raum [Wes15], who proved independently the case of genus 2 and arbitrary type for Siegel modular forms over $\mathbb{Q}$. Their joint work [BW15] then resolved the case of higher genus and arbitrary type over $\mathbb{Q}$.

For Hermitian modular forms, we show the following rigidity result, which is a key step to prove Theorem 1.7.6, and will probably be of independent interest for the theory of Jacobi forms and formal geometry.

Theorem 1.7.8 (Preprint III). *Every symmetric formal Fourier–Jacobi series of arbitrary arithmetic type for the unitary group* $\mathrm{U}(g,g)(\mathbb{Z})$ *in the cases of norm-Euclidean imaginary quadratic fields, converges absolutely to a Hermitian modular form.*

Bibliography

[And+18] F. Andreatta, E. Z. Goren, B. Howard, and K. Madapusi Pera. "Faltings heights of abelian varieties with complex multiplication". *Ann. of Math. (2)* 187.2 (2018). ISSN: 0003-486X. DOI: 10.4007/annals.2018.187.2.3.

[And79] A. N. Andrianov. "Modular descent and the Saito-Kurokawa conjecture". *Invent. Math.* 53.3 (1979).

[BB09] E. Bannai and E. Bannai. "A survey on spherical designs and algebraic combinatorics on spheres". *European Journal of Combinatorics* 30.6 (2009). DOI: 10.1016/j.ejc.2008.11.007.

[BG01a] L. A. Borisov and P. E. Gunnells. "Toric modular forms and nonvanishing of L-functions". *J. Reine Angew. Math.* 539 (2001).

[BG01b] L. A. Borisov and P. E. Gunnells. "Toric varieties and modular forms". *Invent. Math.* 144.2 (2001).

[BG03] L. A. Borisov and P. E. Gunnells. "Toric modular forms of higher weight". *J. Reine Angew. Math.* 560 (2003).

[Bor98] R. E. Borcherds. "Automorphic forms with singularities on Grassmannians". *Invent. Math.* 132.3 (1998).

[Bor99] R. E. Borcherds. "The Gross-Kohnen-Zagier theorem in higher dimensions". *Duke Math. J.* 97.2 (1999).

[Bru+20a] J. Bruinier, B. Howard, S. S. Kudla, M. Rapoport, and T. Yang. "Modularity of generating series of divisors on unitary Shimura varieties" (2020). arXiv: 1702.07812.

[Bru+20b] J. Bruinier, B. Howard, S. S. Kudla, M. Rapoport, and T. Yang. "Modularity of generating series of divisors on unitary Shimura varieties II: arithmetic applications" (2020). arXiv: 1710.00628.

[Bru15] J. H. Bruinier. "Vector valued formal Fourier-Jacobi series". *Proc. Amer. Math. Soc.* 143.2 (2015).

[BW15] J. H. Bruinier and M. Westerholt-Raum. “Kudla’s modularity conjecture and formal Fourier-Jacobi series”. *Forum Math. Pi* 3 (2015).

[DGS77] P. Delsarte, J.-M. Goethals, and J. J. Seidel. “Spherical codes and designs”. *Geometriae Dedicata* 6.3 (1977). DOI: 10.1007/BF03187604.

[DN18] M. Dickson and M. Neururer. “Products of Eisenstein series and Fourier expansions of modular forms at cusps”. *J. Number Theory* 188 (2018). ISSN: 0022-314X. DOI: 10.1016/j.jnt.2017.12.013.

[DS89] P. Delsarte and J. J. Seidel. “Fisher type inequalities for Euclidean t-designs”. *Linear Algebra and its Applications* 114.Supplement C (1989). DOI: 10.1016/0024-3795(89)90462-X.

[GK93] B. H. Gross and K. Keating. “On the intersection of modular correspondences”. *Invent. Math.* 112.2 (1993). ISSN: 0020-9910. DOI: 10.1007/BF01232433.

[GKZ87] B. Gross, W. Kohnen, and D. B. Zagier. “Heegner points and derivatives of L-series. II”. *Math. Ann.* 278.1-4 (1987).

[GS81] J.-M. Goethals and J. J. Seidel. “Cubature Formulae, Polytopes, and Spherical Designs”. *The Geometric Vein: The Coxeter Festschrift*. 1981. DOI: 10.1007/978-1-4612-5648-9_13.

[GZ86] B. H. Gross and D. B. Zagier. “Heegner points and derivatives of L-series”. *Invent. Math.* 84.2 (1986).

[How20] B. Howard. “Arithmetic volumes of unitary Shimura curves” (2020). arXiv: 2010.07362.

[HPV06] P. de la Harpe, C. Pache, and B. Venkov. “Construction of spherical cubature formulas using lattices”. *Algebra i Analiz* 18.1 (2006). ISSN: 0234-0852. DOI: 10.1090/S1061-0022-07-00946-6.

[HZ76] F. Hirzebruch and D. B. Zagier. “Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus”. *Invent. Math.* 36 (1976).

[IPY13] T. Ibukiyama, C. Poor, and D. S. Yuen. “Jacobi forms that characterize paramodular forms”. *Abh. Math. Semin. Univ. Hambg.* 83.1 (2013).

[KM08] W. Kohnen and Y. Martin. “Products of two Eisenstein series and spaces of cusp forms of prime level”. *J. Ramanujan Math. Soc.* 23.4 (2008).

[KM86] S. S. Kudla and J. Millson. “The theta correspondence and harmonic forms. I”. *Math. Ann.* 274.3 (1986).

[KM87] S. S. Kudla and J. Millson. “The theta correspondence and harmonic forms. II”. *Math. Ann.* 277.2 (1987).

[KM90] S. S. Kudla and J. Millson. “Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables”. *Inst. Hautes Études Sci. Publ. Math.* 71 (1990).

[Kud97] S. S. Kudla. “Algebraic cycles on Shimura varieties of orthogonal type”. *Duke Math. J.* 86.1 (1997).

[KZ84] W. Kohnen and D. B. Zagier. “Modular forms with rational periods”. *Modular forms (Durham, 1983)*. Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res. Horwood, Chichester, 1984.

[Liu11] Y. Liu. “Arithmetic theta lifting and L-derivatives for unitary groups, I”. *Algebra Number Theory* 5.7 (2011). ISSN: 1937-0652. DOI: 10.2140/ant.2011.5.849.

[LL21] C. Li and Y. Liu. “Chow groups and L-derivatives of automorphic motives for unitary groups” (2021). arXiv: 2006.06139.

[Maa79a] H. Maass. “Über eine Spezialschar von Modulformen zweiten Grades”. *Invent. Math.* 52.1 (1979).

[Maa79b] H. Maass. “Über eine Spezialschar von Modulformen zweiten Grades. II”. *Invent. Math.* 53.3 (1979).

[Maa79c] H. Maass. “Über eine Spezialschar von Modulformen zweiten Grades. III”. *Invent. Math.* 53.3 (1979).

[Miy89] T. Miyake. *Modular forms*. Translated from the Japanese by Yoshitaka Maeda. Springer-Verlag, Berlin, 1989.

[Möl79] H. M. Möller. “Lower Bounds for the Number of Nodes in Cubature Formulae”. *Numerische Integration: Tagung im Mathematischen Forschungsinstitut Oberwolfach vom 1. bis 7. Oktober 1978*. Ed. by G. Hämmerlin. 1979. DOI: 10.1007/978-3-0348-6288-2_17.

[NS88] A. Neumaier and J. J. Seidel. “Discrete measures for spherical designs, eutactic stars and lattices”. *Indagationes Mathematicae (Proceedings)* 91.3 (1988). DOI: 10.1016/S1385-7258(88)80011-8.

[Pac05] C. Pache. “Shells of selfdual lattices viewed as spherical designs”. *Internat. J. Algebra Comput.* 15.5-6 (2005). ISSN: 0218-1967. DOI: 10.1142/S0218196705002797.

[Pia66] I. I. Piatetsky-Shapiro. *Géométrie des domaines classiques et théorie des fonctions automorphes*. Traduit du Russe par A. W. Golovanoff. Travaux et Recherches Mathématiques, No. 12. Paris: Dunod, 1966.

[RB91] P. Rabau and B. Bajnok. "Bounds for the number of nodes in Chebyshev type quadrature formulas". *Journal of Approximation Theory* 67.2 (1991). DOI: 10.1016/0021-9045(91)90018-6.

[RS09] A. Roy and A. J. Scott. "Unitary designs and codes". *Des. Codes Cryptogr.* 53.1 (2009). ISSN: 0925-1022. DOI: 10.1007/s10623-009-9290-2.

[SZ84] P. D. Seymour and T. Zaslavsky. "Averaging sets: A generalization of mean values and spherical designs". *Advances in Mathematics* 52.3 (1984). DOI: 10.1016/0001-8708(84)90022-7.

[Ven84] B. B. Venkov. "Even unimodular extremal lattices". *Trudy Matematicheskogo Instituta imeni VA Steklova* 165 (1984).

[Wag91] G. Wagner. "On averaging sets". *Monatshefte für Mathematik* 111.1 (1991). DOI: 10.1007/BF01299278.

[Wan20] Y. Wang. *Theta Functions and Hermitian Jacobi Forms over Imaginary Quadratic Fields*. Thesis (Ph.D.)–Northwestern University. ProQuest LLC, Ann Arbor, MI, 2020.

[Wes15] M. Westerholt-Raum. "Formal Fourier Jacobi expansions and special cycles of codimension two". *Compos. Math.* 151.12 (2015).

[Wes17] M. Westerholt-Raum. "Products of vector valued Eisenstein series". *Forum Math.* 29.1 (2017).

[YZZ09] X. Yuan, S.-W. Zhang, and W. Zhang. "The Gross-Kohnen-Zagier theorem over totally real fields". *Compos. Math.* 145.5 (2009). ISSN: 0010-437X. DOI: 10.1112/S0010437X08003734.

[Zag81] D. Zagier. "Sur la conjecture de Saito-Kurokawa (d'après H. Maass)". *Seminar on Number Theory, Paris 1979–80*. Vol. 12. Progr. Math. Birkhäuser, Boston, Mass., 1981.

[Zha09] W. Zhang. "Modularity of Generating Functions of Special Cycles on Shimura Varieties". PhD thesis. Columbia University, 2009.