Vector-valued Modular Forms

Computational Considerations

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Abstract

In the following thesis we give a thorough self-contained introduction to vector-valued modular forms with an eye to representation theoretic aspects. We also examine the mathematical details of an implementation that we provide for an algorithm that computes bases of certain spaces of vector-valued modular forms in terms of a theorem due to Raum and Xià.

Our argument consists of the following key steps. Firstly, we compute Fourier series for vector-valued Eisenstein series, including the case of quasi-modular Eisenstein series of weight 2 (chapter 3.1-3.2). Secondly, we connect the spaces that occur in work of Raum and Xià in a precise and computable way to the spaces of vector-valued modular forms under consideration. This is done by means of tracing out all relevant mappings in a large commutative diagram which also showcases the effective complexity of our approach.

We provide an implementation `ModularForms.jl` in the programming language Julia that incorporates the algorithms described in this thesis to determine spaces of vector-valued modular forms. This package includes related auxiliary functionality, thus extending the `Hecke.jl`/`Nemo.jl` ecosystem, which we built our work upon.

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Notation

$\mathbb{H}$ – the upper half plane, consisting of points $z \in \mathbb{C}$ with $\Im(z) > 0$.

$\mathrm{SL}_2(\mathbb{Z})$ – the special linear group of degree 2, consisting of matrices with integer entries and determinant equal to one. It is assumed that the reader knows that $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle$ where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we write $a(\gamma) = \gamma_{1,1}$, $b(\gamma) = \gamma_{1,2}$, $c(\gamma) = \gamma_{2,1}$ and $d(\gamma) = \gamma_{2,2}$.

$\mathrm{Mp}_1(\mathbb{Z})$ – the metaplectic group, or double cover of $\mathrm{SL}_2(\mathbb{Z})$, consisting of elements (γ, ω) where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\omega : \mathbb{H} \to \mathbb{C}$ is a function with $\omega(\tau)^2 = c(\gamma)\tau + d(\gamma)$ for $\tau \in \mathbb{H}$. The metaplectic group surjects onto $\mathrm{SL}_2(\mathbb{Z})$ and is generated by $\tilde{T} = (T, 1)$ and $\tilde{S} = (S, \tau \mapsto \sqrt{\tau})$ where $\sqrt{\cdot}$ is the principal square root.

$\mathrm{GL}(V)$ – the general linear group of the vector space V, consisting of linear automorphisms $V \to V$. For vector spaces V over $\mathbb{C}$, we identify $\mathrm{GL}(V)$ with the corresponding matrices in a basis, the choice of which will be clear from the context.

$\Gamma(N)$ – the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices that are congruent to the identity matrix modulo N.

$\Gamma_0(N)$ – the subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ satisfying $c(\gamma) \equiv_N 0$ for every $\gamma \in G$.

$\Gamma_1(N)$ – the subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ satisfying $c(\gamma) \equiv_N 0$ and $a(\gamma) \equiv_N d(\gamma) \equiv_N 1$ for every $\gamma \in G$.

$\widetilde{\Gamma}(N)$, $\widetilde{\Gamma}_0(N)$, and $\widetilde{\Gamma}_1(N)$ – the preimages of $\Gamma(N)$, $\Gamma_0(N)$, and $\Gamma_1(N)$, respectively, with respect to the canonical projection $\mathrm{Mp}_1(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$.

Γ_∞ – the subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ generated by $\pm T$.

$\mathbb{Q}^{\mathrm{ab}}$ – the universal cyclotomic field, that is the maximal abelian extension of $\mathbb{Q}$.

Chapter 1

Introduction

1.1 What are you reading?

This is an expository thesis on vector-valued modular forms with a focus on computational aspects. It consists of four interdependent parts.

The first part contains the basic definitions and fundamental results of vector-valued modular forms. In particular we define arithmetic types, permutation types, and Weil types, which are finite-dimensional complex representations of the special linear group of degree 2, $\mathrm{SL}_2(\mathbb{Z})$ (in the case of integral weight), or its double cover $\mathrm{Mp}_1(\mathbb{Z})$ (in the case of half-integral weight). We also restate some definitions coming from representation theory, such as for example induced representations, which we use to define certain crucially important permutation types that feature heavily in the second part. We also give a formula for the dimension and a Sturm bound for spaces of vector-valued modular forms. Both of these already existed in the literature, but are of vital importance when computing bases for any space of modular forms, vector-valued modular forms being no exception, and thus deserve to be restated in the context of computational aspects.

The second part contains results on vector-valued Eisenstein series, and in particular the Fourier series expansions for all vector-valued Eisenstein series of positive integer weights. This builds on previous work by Xià in his licentiate thesis [Xià19], Raum in his paper [Wes17], and the last chapter of Miyake's book [Miy89]. In the case of weight 2, one obtains a quasi-modular vector-valued modular form, and therefore some extra work is carried out to remove the non-modular part.

The third part contains the most significant contribution of this thesis. It describes, by way of many interdependent commutative diagrams, how to leverage a theorem by Raum and Xià [RX19] computationally. Specifically, it describes how one can translate computations of invariant spaces of tensor products of generic arithmetic types and permutation types into corresponding computations on spaces of modular forms. The theorem by Raum and Xià, states precisely that the latter spaces in fact span the entire space of vector-valued modular forms with a given weight.

The fourth part is devoted to invariant theory, and shows how the aforementioned invariant spaces are computed in practice. They turn out to be isomorphic to the zeroth cohomology groups of tensor products of generic arithmetic types with two or more permutation types. We show how one can combine standard methods from linear algebra, such as row reduction, together with an orbit-stabilizer algorithm to obtain bases for these spaces. It deserves to be mentioned that there is room for improvement on how to more effectively compute these spaces.

Raum and the author of this thesis provide an implementation called `ModularForms.jl` in the programming language Julia that incorporates the algorithms laid out in this thesis to determine bases of spaces of vector-valued modular forms. This package includes related auxiliary functionality, thus extending the `Hecke.jl`/`Nemo.jl` ecosystem, which the work was built upon.

1.2 Why does this thesis have to exist?

Computational aspects are of considerable influence in the study of modular forms in general and vector-valued modular forms in particular. While the author has been fascinated by this impact of explicit methods, there are numerous, intrinsic reasons to examine the theory of vector-valued modular forms.

For instance, the well-explored realm of modular forms for congruence subgroups can profit from contributions made to the theory of vector-valued modular forms. The L-functions and Modular Forms Database [LMFDB] is a sizeable project funded by, among others, the National Science Foundation of the USA and the Simons Foundation. It collects data around L-functions and all related objects, usually obtained through expansive computational efforts. However, as Assaf pointed out in a recent preprint [Ass20], there is a significant gap in this data, connected to modular forms for congruence subgroups that correspond to non-split Cartan subgroups of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$. While he suggested to mitigate the situation by an approach that is based on the theory of modular symbols, vector-valued modular forms provide an alternative path to follow.

One of the key contributions of this thesis is therefore to explore vector-valued modular forms as a computational device for modular forms for arbitrary congruence subgroups, including the ones corresponding to non-split Cartan subgroups that challenged Assaf.

Assaf's approach relies on generalizing the theory of modular symbols to the case of an arbitrary congruence subgroup. This is a natural idea, as the theory of modular symbols is well-explored, having been initiated in the 1970s by Birch, and later developed by Manin and Mazur, and then yet later by Merel (see [Ste07]). Most of Assaf's results are connected to the work of Merel. Merel's work is explained thoroughly in chapters 8 and 9 in Stein's book [Ste07], but he restricts to the congruence subgroups $\Gamma(N)$, $\Gamma_0(N)$, and $\Gamma_1(N)$. From a mathematical point of view, modular symbols (or to be more accurate, cuspidal modular symbols) are natural to consider in the sense that they are the Hecke module dual to the space of cusp forms. They also have a geometric interpretation in connection with the compactified modular curve $X(\Gamma)$ where Γ is an arbitrary congruence subgroup, especially in the case of weight 2 in which case they represent homology classes of paths between the cusps.

This thesis shows that modular forms for arbitrary congruence subgroups occur as the special case of vector-valued modular forms with respect to an induced type. To give the reader a feeling for the difference of the approach between vector-valued modular forms and modular symbols, we briefly explain how the space of cusp forms of weight 2 and level $\Gamma_{\mathrm{ns}}(11)$ can be computed using the two different methods.

With the method of vector-valued modular forms, we first rely on the elementary but crucial fact that $M_k(\Gamma) \cong M_k(\mathrm{Ind}_{\Gamma}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1})$ for any finite-index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. This can be seen by defining the map

$$\mathrm{Ind} : M_k(\Gamma) \to M_k(\mathrm{Ind}_{\Gamma}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1})$$

by $\mathrm{Ind}(f) = \sum_{[\gamma]\in\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})}(f|_k\gamma)\mathfrak{e}_\gamma$. Here, the $\mathfrak{e}_\gamma$ are a basis for the representation space $V(\mathrm{Ind}_{\Gamma}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1})$, which will be defined in full detail later. An inverse to the map Ind is given by extracting the coefficient of the basis vector $\mathfrak{e}_I$, where I is the identity matrix. In our implementation, this coefficient will always be the first entry in an array representing the modular form. As we shall see when we define the induced representation, the index of the chosen entry is in fact unimportant, as long as it remains fixed.

To compute $S_2(\Gamma_{\mathrm{ns}}(11))$ using `ModularForms.jl`, we run the following code.

```
using ModularForms
rho = induction(TrivialArithmeticType(GammaNS(11)))
wgt = SL2RCoverWeight(2)
mfs = ModularFormsSpace(wgt, rho, QQab)
```

At this point we have computed a basis for $M_2(\mathrm{Ind}_{\Gamma_{\mathrm{ns}}(11)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1})$ and it can be viewed by running `basis(mfs)`. To find the cusp space $S_2(\Gamma_{\mathrm{ns}}(11))$, we compute the kernel of the map

$$\phi : M_2(\mathrm{Ind}_{\Gamma_{\mathrm{ns}}(11)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1}) \ni f \mapsto [q^0]f \in V(\mathrm{Ind}_{\Gamma_{\mathrm{ns}}(11)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1}),$$

where $[q^r]g$ extracts the coefficient of q^r in every component of g. Indeed, by virtue of Ind being a bijection, we have that

$$\{\langle f, \mathfrak{e}_I\rangle : f \in \ker(\phi)\} = S_2(\Gamma_{\mathrm{ns}}(11)).$$

where $\langle f, \mathfrak{e}_I\rangle$ denotes the coefficient of $\mathfrak{e}_I$. This follows from the fact that the components of the Fourier series expansion[1] of $\mathrm{Ind}\langle f, \mathfrak{e}_I\rangle$ are exactly the cusp expansions of $\langle f, \mathfrak{e}_I\rangle$.

Computing the kernel of ϕ is just a linear algebra problem. We solve it in Julia as follows:

[1] We identify Puiseux series with coefficients in a vector space with vectors of Puiseux series with scalar coefficients.

```
vvB = basis(mfs)
fs =  [twist_fourier_expansion(vvB[i].fourier_expansion,translation_orbits(rho),
    cyclotomic_tower(QQab)) for i=1:length(vvB)]
M = zero_matrix(QQab, dim(rho), length(vvB))
for j=1:length(vvB)
        for i=1:dim(rho)
                M[i,j] = coeff(fs[j][i],0)
        end
end
nullspace(M)
```

Specifically, we represent ϕ as a matrix and compute its nullspace. We find that $S_2(\Gamma_{\text{ns}}(11))$ is of dimension 4, which following Assaf [Ass20] gives yet another verification of the result from Dose, Fernández, González, and Schoof concerning the compactified modular curve $X_{\text{ns}}(11)$ [Dos+14].

Our method is in fact more powerful than computing Fourier series expansions, and for example also returns a polynomial expression in terms of Eisenstein series for every modular form.

With Assaf's approach, one first decomposes the space $\mathbb{S}_2(\Gamma_{\text{ns}}(11))$ of cuspidal modular symbols of weight 2 associated to $\Gamma_{\text{ns}}(11)$ into so-called irreducible Hecke-modules. It will however be beyond the scope of this thesis to go into details on Hecke-module theory. Assaf then provides an algorithm that can be used to obtain a Fourier series expansion of a so-called eigenform for each of these irreducible modules. By virtue of the duality of symbols and forms, the eigenforms are modular forms belonging to the space $S_2(\Gamma_{\text{ns}}(11))$. The decomposition into irreducible submodules also means that the eigenforms form a basis for $S_2(\Gamma_{\text{ns}}(11))$. As Assaf has implemented his algorithms in MAGMA, which is proprietary, we refer to his paper for details on how to carry out the above method using his package.

Other than being a computational device for studying classical modular forms, vector-valued modular forms also appear in the context of Jacobi modular forms. As Raum points out in [Rau16], there is a bijective correspondence between weakly holomorphic Jacobi forms of weight k and index L and weakly holomorphic vector-valued modular forms of the dual of the Weil type associated to L. The correspondence goes by way of θ-series, and is currently being developed to be a part of `ModularForms.jl`.

Finally, while outside the scope of this thesis, it deserves to be mentioned that vector-valued modular forms are at the foundation of contemporary study for noncongruence subgroups, initiated by Atkin and Swinnerton-Dyer in experimental work from the 1960's, see [AS71]. Of central importance in their work is a conjecture often known as the unbounded denominator conjecture, stating that the denominators of the Fourier coefficients of modular forms with respect to a noncongruence subgroup, grow without bound. Mason, Gottesman, Marks, and Franc, (see [Got20] and references therein) have studied vector-valued modular forms in this context, but would benefit from having more computational tools at their disposal. To satisfy this need, `ModularForms.jl` is planned to have noncongruence subgroup functionality in a much later release.

1.3 How do we do what we do?

Let us briefly describe how the results of the second and third parts are obtained. In what follows, ρ and σ will denote arithmetic types with finite index kernels, k and l will denote positive integers, and $\mathbf{1}$ will denote the 1-dimensional trivial representation.

In the second part of the thesis, the end goal is to describe the following space

$$A = H^0(\mathcal{E}_k[\rho_N^\times] \otimes \rho) + H^0(\mathcal{E}_l[\rho_{N_0}^\times] \cdot \mathcal{E}_{k-l}[\rho_{N_0}^\times] \otimes \rho),$$

where N_0 is an integer bounded by some large number, in a way that is suitable for implementation. The spaces $\mathcal{E}_k[\sigma]$ are components of vector-valued Eisenstein series of type σ and weight k, treated as finite-dimensional complex representations of $\mathrm{SL}_2(\mathbb{Z})$ through the slash-action $|_k$, and the representations $\rho_N^\times$ are the induced types $\mathrm{Ind}_{\Gamma_1(N)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1}$. For a representation ρ, the space $H^0(\rho)$ denotes the space of invariants, that is the vectors $v \in V(\rho)$ satisfying $\rho(\gamma)v = v$ for every γ.

The reason for considering this space is that the aforementioned theorem by Raum and Xià, states exactly that A is equal to the space of vector-valued modular forms of type ρ and weight k. We note that it is easy to see that $A \subseteq M_k(\rho)$ and thus the main result of Raum and Xià is that $A \supseteq M_k(\rho)$.

Equipped with a description of A that is suitable for implementation, or as will turn out to be more appropriate, a space that surjects onto A, we can formulate the algorithm that we have implemented in `ModularForms.jl` to compute a basis for $M_k(\rho)$.

Making the space suitable for implementation consists of two parts:

(i) Obtaining Fourier series expansions of elements of the spaces $\mathcal{E}_k[\sigma]$.

(ii) Finding a "nice" space that surjects onto A, and use this surjection together with the Fourier series expansions to span A.

Since we can, due to the existence of a Sturm bound for vector-valued modular forms in $M_k(\rho)$, truncate Fourier series expansions without losing information, we can treat the elements in A as elements in $K^{s\cdot\dim(\rho)}$ where s is the Sturm bound, and where K is the coefficient field (in this thesis, we will have $K = \mathbb{Q}^{\mathrm{ab}}$, but the approach can be generalized to handle other fields as well). This is the raison d'être behind part (i).

Given that we can find modular forms that span A, we can reduce them to a basis using row reduction. Since the bound N_0 is large, it will from the perspective of space complexity be necessary to have a decomposition of A into smaller pieces, that can be used to construct A iteratively. We use the following decomposition:

$$A = H^0(\mathcal{E}_k[\rho_N^\times] \otimes \rho) + \sum_{\substack{N \mid N_1 \leq N_0 \\ N_2 \mid N_1 \\ 1 \leq l \leq k}} H^0((\mathcal{E}_l[\rho_{N_1}^\times] \cdot \mathcal{E}_{k-l}[\rho_{N_2}^\times]) \otimes \rho),$$

Part (ii) is rather intricate, and requires an argument that spans several commutative diagrams to be carried out in full detail. What follows is a condensed version. Firstly, we notice that the spaces $H^0(*)$ are as the notation suggests nothing else than the zeroth cohomology groups where we identify complex $\mathrm{SL}_2(\mathbb{Z})$-representations with $\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$-modules. Next we use the fact that there is a surjection

$$(\rho_N^\times)^\vee \twoheadrightarrow \mathcal{E}_k[\rho_N^\times]$$

for any integer k. From group cohomology we then know that there is an induced surjection

$$H^0((\rho_N^\times)^\vee \otimes \rho) \twoheadrightarrow H^0(\mathcal{E}_k[\rho_N^\times] \otimes \rho)$$

Similarly, we have for any pair of positive integers (l, k) with $l - k \geq 1$, a surjection of the form

$$H^0((\rho_{N_1}^\times)^\vee \otimes (\rho_{N_2}^\times)^\vee \otimes \rho) \twoheadrightarrow H^0(\mathcal{E}_l[\rho_{N_1}^\times] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^\times] \otimes \rho).$$

From this we obtain the following monster of a surjection

$$\begin{aligned}\psi : H^0((\rho_N^\times)^\vee \otimes \rho) \oplus \bigoplus_{\substack{N \mid N_1 \leq N_0 \\ N_2 \mid N_1 \\ 1 \leq l \leq k}} H^0((\rho_{N_1}^\times)^\vee \otimes (\rho_{N_2}^\times)^\vee \otimes \rho) \\ \longrightarrow\!\!\!\!\!\rightarrow H^0(\mathcal{E}_k[\rho_N^\times] \otimes \rho) \oplus \bigoplus_{\substack{N \mid N_1 \leq N_0 \\ N_2 \mid N_1 \\ 1 \leq l \leq k}} H^0(\mathcal{E}_l[\rho_{N_1}^\times] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^\times] \otimes \rho),\end{aligned}$$

whose co-domain surjects onto A. The key point here is that we can use tools from invariant theory to compute bases of the spaces

$$H^0((\rho_N^\times)^\vee \otimes \rho) \text{ and } H^0((\rho_{N_1}^\times)^\vee \otimes (\rho_{N_2}^\times)^\vee \otimes \rho),$$

and thus also a basis of the domain of ψ.

Therefore, the algorithm, roughly speaking, consists of iteratively constructing a basis of the domain of ψ, in every step mapping it through ψ, computing Fourier series expansions of the results, and then performing row reduction on the truncated Fourier series expansions. Since ψ is a surjection, we eventually obtain a spanning set of A. We know when in the iteration we have a spanning set, and therefore a basis, by comparing the rank with the dimension of $M_k(\rho)$, which is known a priori.

In third part of the thesis, we consider the problem of how to compute the spaces $H^0((\rho_N^\times)^\vee \otimes \rho)$ and $H^0((\rho_{N_1}^\times)^\vee \otimes (\rho_{N_2}^\times)^\vee \otimes \rho)$. Specifically we modify the standard orbit-stabilizer theorem to show that we can compute bases of spaces on the form

$$H^0(\rho_1 \otimes \rho_2)$$

where ρ_1 is a generic arithmetic type, and ρ_2 is a twisted permutation type, by computing orbits of an $\mathrm{SL}_2(\mathbb{Z})$-action. In the case of trivially twisted permutation types, the correspondence is easy to see. Indeed, say that we have trivially twisted permutation type ρ and a basis $\mathcal{B} = \{e_1, \dots, e_d\}$ of $V(\rho)$. Then $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{B}$ by $\gamma.e_i = \rho(\gamma)e_i$ and thus we obtain a partition

$$\mathcal{B} = \bigsqcup_{j=1}^{n} \mathrm{SL}_2(\mathbb{Z})h_j$$

for some representatives h_j. Write now $\mathrm{SL}_2(\mathbb{Z})h_j = \{e_{\alpha_j(1)}, \dots, e_{\alpha_j(n_j)}\}$ for every j, where α_j is a subsequence of $(1, 2, \dots, n)$. It is now easy to see that the set

$$\Big\{ \sum_{i=1}^{n_j} e_{\alpha_j(i)} : 1 \leq j \leq n \Big\}$$

forms a basis for $H^0(\rho)$.

We show that the same approach works for the case $H^0(\rho_1 \otimes \rho_2)$ as well, but with the caveat that one needs to keep track of the twists of elements in the stabilizers.

But that is it for an introduction. We hope that the reader will find the reading experience pleasurable.

Chapter 2

Preliminaries

2.1 Introductory examples

To start on familiar grounds, let us recall what a classical scalar-valued modular form with respect to a Dirichlet character is.

In this chapter we will not only work with the special linear group $\mathrm{SL}_2(\mathbb{Z})$ of degree 2, but also with its double cover, the metaplectic group $\mathrm{Mp}_1(\mathbb{Z})$, so that it makes sense to talk about modular forms with half-integral weight. Readers unfamiliar with this are advised to consult appendix A.

First we recall the slash action.

Definition 2.1 (Slash action)**.** Let k be a half-integer, and let $f : \mathbb{H} \to \mathbb{C}$ be a function. Then, for $(\gamma, \omega) \in \mathrm{Mp}_1(\mathbb{Z})$ we define

$$\left(f|_k(\gamma,\omega)\right)(\tau) = \omega(\tau)^{-2k} f(\gamma.\tau),$$

where the dot . denotes the Möbius action, so $\gamma.\tau = (a(\gamma)\tau + b(\gamma))/(c(\gamma)\tau + d(\gamma))$.

Definition 2.2 (Scalar-valued modular form with respect to a Dirichlet character)**.** Let N be a positive integer, let χ be a Dirichlet character modulo N, and let k be a half-integer. Then a function $f : \mathbb{H} \to \mathbb{C}$ is said to be a modular form of weight k and character χ on $\widetilde{\Gamma}_0(N)$ if the following conditions are satisfied:

(a) $f|_k\tilde{\gamma} = \chi(d(\gamma))f$ for every $\tilde{\gamma} = (\gamma, \omega) \in \widetilde{\Gamma}_0(N)$

(b) $f|_k\tilde{\gamma}$ extends to a holomorphic function on $\mathbb{H} \cup \{i\infty\}$ for all $\tilde{\gamma} \in \mathrm{Mp}_1(\mathbb{Z})$.

For vector-valued modular forms we want to mimic this definition. Since a character is nothing else but a 1-dimensional representation, a natural generalization is to replace χ, which can be seen as a character $\chi : \widetilde{\Gamma}_0(N) \to \mathbb{C}^\times$, with a representation $\rho : \widetilde{\Gamma}_0(N) \to \mathrm{GL}(V(\rho))$. Or more generally, with a representation $\rho : G \to \mathrm{GL}(V(\rho))$ for some sufficiently nice subgroup $G \subseteq \mathrm{Mp}_1(\mathbb{Z})$.

To make our investigations meaningful, we put a few sensible restrictions on the kind of representations that we are interested in.

Definition 2.3 (Arithmetic type, congruence type, level)**.** Let G be a finite index subgroup of $\mathrm{Mp}_1(\mathbb{Z})$ or $\mathrm{SL}_2(\mathbb{Z})$. Then an arithmetic type ρ of G is a finite-dimensional complex representation $\rho : G \to \mathrm{GL}(V(\rho))$.

Furthermore, if in addition ρ satisfies that $\Gamma(N) \subseteq \ker(\rho)$ for some positive integer N, it is called a congruence type. The smallest such N is called the level of ρ.

For convenience, we modify the slash action.

Definition 2.4 (Slash action for vector-valued functions)**.** Let $G \subseteq \mathrm{Mp}_1(\mathbb{Z})$ be a subgroup, and let k be a half-integer. Let ρ be an arithmetic type of G. Let $f : \mathbb{H} \to V(\rho)$ be a function.

For $\tilde{\gamma} = (\gamma, \omega) \in G$ we put

$$(f|_{k,\rho}\tilde{\gamma})(\tau) = \omega(\tau)^{-2k}\rho(\tilde{\gamma})^{-1} f(\gamma.\tau).$$

The definition of a vector-valued modular form (vvmf) is now a natural generalization of scalar-valued modular forms with a character.

Definition 2.5 (Vector-valued modular form)**.** Let $G \subseteq \mathrm{Mp}_1(\mathbb{Z})$ be a subgroup, and let ρ be an arithmetic type of G. Let k be a half-integer, and let $f : \mathbb{H} \to V(\rho)$ be a function. We say that f is a modular form of type ρ and weight k if the following conditions are satisfied:

(a) $f|_{k,\rho}\gamma = f$ for every $\gamma \in G$,

(b) $f|_{k,\rho}\gamma$ extends to a holomorphic function on $\mathbb{H} \cup \{i\infty\}$ for all $\gamma \in \mathrm{Mp}_1(\mathbb{Z})$.

The set of all vector-valued modular forms of type ρ and weight k forms a $\mathbb{C}$-vector space, denoted by $M_k(\rho)$.

Here are a few concrete examples.

Example 2.1 (A 1-dimensional vector-valued modular form)**.** As said before, vector-valued modular forms constitute a generalization of scalar-valued modular forms with respect to a character, such as for instance Dedekind's η-function.

As a vector-valued modular form, we can describe it as follows: let $\rho_\eta : \mathrm{Mp}_1(\mathbb{Z}) \to \mathbb{C}^\times$ be the representation defined by its images

$$\rho_\eta(\tilde{S}) = e^{7\pi i/4}, \text{ and } \rho_\eta(\tilde{T}) = e^{\pi i/12}.$$

Then

$$\eta|_{1/2,\rho_\eta}\tilde{\gamma} = \eta,$$

for all $\tilde{\gamma} \in \mathrm{Mp}_1(\mathbb{Z})$, and it is well-known that η extends to a holomorphic function on $\mathbb{H} \cup \{i\infty\}$. Hence η is a weight $1/2$ modular form of type ρ_η.

Example 2.2 (Jacobi's θ-functions)**.** Let us recall Jacobi's theta functions. For $\tau \in \mathbb{H}$ we put

$$\begin{aligned}\theta_{0,0}(\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2/2},\\ \theta_{0,1}(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}, \text{ and}\\ \theta_{1,0}(\tau) &= \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2}.\end{aligned}$$

Let now $\rho_\theta : \mathrm{Mp}_1(\mathbb{Z}) \to \mathrm{GL}(V)$ be the representation defined by $V(\rho_\theta) = \mathbb{C}^3$ and

$$\rho_\theta(\tilde{S}) = e^{-i\pi/4}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \text{ and } \rho_\theta(\tilde{T}) = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & e^{i\pi/4}\end{pmatrix}.$$

Then the vector-valued function $\theta = (\theta_{0,0}, \theta_{0,1}, \theta_{1,0})^T$ is a vvmf of type ρ_θ and weight $1/2$. For a proof see for example [CS17, p. 39].

Example 2.3. Vector-valued modular forms features prominently in the study of vertex operator algebras (VOA).

For example, the characters of C^2-cofinite vertex operator algebras form a weight 0 vvmf of $\mathrm{SL}_2(\mathbb{Z})$-type, conformal blocks on the sphere can be interpreted as vvmfs with respect to a $\Gamma(2)$-type, and 1-point functions on a torus can be identified with vvmfs of $\mathrm{SL}_2(\mathbb{Z})$-type.

It is beyond the scope of this thesis to delve further into VOAs, and thus we refer the interested reader to [Gan14], [Miy04], and [Zhu96].

The next example is crucial. We will use products of components of vector-valued Eisenstein series to span the spaces of vvmfs that we are interested in.

Example 2.4 (Vector-valued Eisenstein series)**.** Let ρ be an arithmetic type with a finite index kernel and let $k > 2$ be an even integer. For $v \in V(\rho)$ we define

$$\mathrm{Stab}(v) = \{\gamma \in \mathrm{Mp}_1(\mathbb{Z}) : \rho(\gamma)v = v\}, \text{ and } \Gamma_\infty(v) = \mathrm{Stab}(v) \cap \Gamma_\infty.$$

We now define the Eisenstein series of weight k and type ρ associated to v as the series

$$E_{k,v}(\tau) = \frac{1}{[\Gamma_\infty : \Gamma_\infty(v)]} \sum_{\Gamma_\infty(v)\gamma \in \Gamma_\infty(v)\backslash \mathrm{Mp}_1(\mathbb{Z})} (v|_{k,\rho}\gamma)(\tau),$$

where the sum runs over the cosets of $\Gamma_\infty(v)\backslash\mathrm{Mp}_1(\mathbb{Z})$.

It holds, see for example [Wes17], that

(i) $\Gamma_\infty(v)$ has finite index in Γ_∞,

(ii) the series does not depend on the choice of representatives,

(iii) the series converges absolutely and locally uniformly on $\mathbb{H}$,

(iv) and that consequently $E_{k,v}$ is a modular form of weight k and type ρ.

This formulation of the definition of a vector-valued Eisenstein series is due to Raum, see [Wes17].

Let us now focus on developing the representation-theoretic tools that we need to compute with vvmfs.

2.2 A trichotomy of arithmetic types

In this thesis we will mainly look at three differents kinds of arithmetic types. Namely:

1. Trivially twisted permutation types,
2. Non-trivially twisted permutation types, and
3. Weil types.

Of these 1. is the same as representations whose image consist of permutation matrices, 2. is the same as representations whose image consist of generalized permutation matrices, and 3. is the same as the Weil representation. Tensor products of these types will also feature prominently, but they can be studied through their factors.

Types 1 and 2 can be constructed from "simple" representations of $\Gamma(n)$, $\Gamma_1(n)$ and $\Gamma_0(n)$, or more generally an arbitrary congruence subgroup, using the so-called induced representation. Type 3 on the other hand, is constructed by means of the discriminant form of a lattice.

To familiarize the reader with permutation types, we create one without using induction.

Example 2.5. Let G be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (or $\mathrm{Mp}_1(\mathbb{Z})$). Put for convenience $d = [\mathrm{SL}_2(\mathbb{Z}) : G]$. Find coset-representatives for $G\backslash\mathrm{SL}_2(\mathbb{Z})$ and fix an ordering of them, so that we have

$$G\backslash\mathrm{SL}_2(\mathbb{Z}) = \{G\gamma_1, G\gamma_2, \dots, G\gamma_d\},$$

for some representatives γ_i. Now, $\mathrm{SL}_2(\mathbb{Z})$ acts on this quotient from the right, and in particular if $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ is arbitrary, then there is a permutation $\sigma_\alpha \in S_d$ depending on α, such that

$$(G\gamma_1\alpha, G\gamma_2\alpha, \dots, G\gamma_d\alpha) = (G\gamma_{\sigma_\alpha(1)}, G\gamma_{\sigma_\alpha(2)}, \dots, G\gamma_{\sigma_\alpha(d)}).$$

Hence, we can define a representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{C})$ by

$$\rho(\alpha) = P_{\sigma_\alpha},$$

where P_π denotes the (column) permutation matrix associated to a permutation π.

It remains to verify that ρ indeed is a representation. To this end, select $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Z})$ and consider $\sigma_{\alpha\beta}$. We have that

$$\begin{aligned}(G\gamma_1\alpha\beta, \dots, G\gamma_d\alpha\beta) &= (G\gamma_{\sigma_\alpha(1)}\beta, \dots, G_{\sigma_\alpha(d)}\beta) \\ &= (G\gamma_{\sigma_\beta(\sigma_\alpha(1))}, \dots, G\gamma_{\sigma_\beta(\sigma_\alpha(d))}),\end{aligned}$$

and so $\sigma_{\alpha\beta} = \sigma_\beta \circ \sigma_\alpha$. We conclude that

$$\rho(\alpha\beta) = P_{\sigma_\beta\circ\sigma_\alpha} = P_{\sigma_\alpha}P_{\sigma_\beta} = \rho(\alpha)\rho(\beta).$$

The above example can be carried out for any subgroup G of finite index, but explicit considerations rely on being able to compute coset representatives of $G\backslash\mathrm{SL}_2(\mathbb{Z})$ in an effective way. This can be done using Farey symbols, for details we refer the reader to [Ste07].

We will always work under the assumption that arithmetic types under our consideration are unitary. The following proposition shows that this incurs no loss of generality.

Proposition 2.1 (Weyl's unitarity trick)**.** Let ρ be an arithmetic type with finite index kernel. Then there exists an inner product[1] $\langle\cdot,\cdot\rangle_\rho$ such that ρ is unitary with respect to $\langle\cdot,\cdot\rangle_\rho$.

Proof. Let $\langle\cdot,\cdot\rangle$ be any inner product on $V(\rho)$. Let $v, w \in V(\rho)$ be arbitrary and put

$$\langle v, w\rangle_\rho = \sum_{\ker(\rho)\gamma\in\ker(\rho)\backslash \mathrm{SL}_2(\mathbb{Z})} \langle\rho(\gamma)v, \rho(\gamma)w\rangle.$$

Let us first verify that this is well-defined. To this end, select $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ with $\ker(\rho)\gamma_1 = \ker(\rho)\gamma_2$. This means that for some $\alpha \in \ker(\rho)$ we have that $\gamma_1 = \alpha\gamma_2$, but then

$$\rho(\gamma_1) = \rho(\alpha\gamma_2) = \rho(\alpha)\rho(\gamma_2) = I\rho(\gamma_2) = \rho(\gamma_2),$$

and well-definedness follows. It is immediate that it is an inner product.

Let now $\beta \in \mathrm{SL}_2(\mathbb{Z})$ be arbitrary. Then we have that

$$\langle\rho(\beta)v, \rho(\beta)w\rangle_\rho = \sum_{\ker(\rho)\gamma\in\ker(\rho)\backslash \mathrm{SL}_2(\mathbb{Z})} \langle\rho(\gamma\beta)v, \rho(\gamma\beta)w\rangle.$$

Evidently $\ker(\rho)\gamma \mapsto \ker(\rho)\gamma\beta$ is a well-defined and bijective map on cosets, and hence

$$\sum_{\ker(\rho)\gamma\in\ker(\rho)\backslash \mathrm{SL}_2(\mathbb{Z})} \langle\rho(\gamma\beta)v, \rho(\gamma\beta)w\rangle = \sum_{\ker(\rho)\gamma\in\ker(\rho)\backslash \mathrm{SL}_2(\mathbb{Z})} \langle\rho(\gamma)v, \rho(\gamma)w\rangle = \langle v, w\rangle_\rho,$$

and so since β was arbitrary, we conclude that ρ is unitary with respect to $\langle v, w\rangle_\rho$. □

As stated above, we hereafter assume for every arithmetic type ρ under consideration that $V(\rho)$ is equipped with an inner product making ρ unitary. We shall denote this inner product by $\langle\cdot,\cdot\rangle_\rho$ and the corresponding norm by $\|\cdot\|_\rho$.

Now we bring in the induced representation.

Definition 2.6 (Induced representation)**.** Let $\Gamma, \Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$ (or $\mathrm{Mp}_1(\mathbb{Z})$), with $\Gamma \subseteq \Gamma'$ and $|\Gamma\backslash\Gamma'| < \infty$, be subgroups. Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation. We then define the induced representation of ρ from Γ to Γ', denoted by $\mathrm{Ind}_\Gamma^{\Gamma'}\rho$, by

- $V(\mathrm{Ind}_\Gamma^{\Gamma'}(\rho)) = V \otimes \mathbb{C}[B]$, where B is a system of coset representatives of $\Gamma\backslash\Gamma'$ containing I, and
- $(\mathrm{Ind}_\Gamma^{\Gamma'}\rho)(\gamma)(v \otimes \mathfrak{e}_\beta) = \rho(I_\beta(\gamma^{-1})^{-1})(v) \otimes \mathfrak{e}_{\beta\gamma^{-1}}$,

  where $\gamma \in \Gamma'$ and $\beta \in B$,

  $\mathfrak{e}_\beta = 1\beta$ when $\beta \in B$ and $\mathfrak{e}_\delta = \mathfrak{e}_{\beta'}$ with $\beta' \in B$ being the representative of $\Gamma\delta$, for $\delta \in \Gamma'$,

  and I_β is defined below.
- For other vectors $w \in V \otimes \mathbb{C}[B]$ we define $\mathrm{Ind}_\Gamma^{\Gamma'}\rho$ by linearity.

To define I_β, write first $\Gamma' = \bigsqcup_{\beta\in B}\Gamma\beta$. Pick then $\beta \in B$ and $\gamma \in \Gamma'$ arbitrary. Evidently $\beta\gamma \in \Gamma\beta'$ for some unique $\beta' \in B$. Thus we have that $\beta\gamma = g\beta'$ for some unique element $g \in \Gamma$. As our definition, we put $I_\beta(\gamma) = g$.

For $\delta \in \Gamma'$ we put $I_\delta = I_\beta$ where $\beta \in B$ is the (necessarily unique) element satisfying $\Gamma\delta = \Gamma\beta$.

It is not obvious that the above definition indeed gives a representation. That is the content of the following proposition.

Proposition 2.2 (Definition 2.6 yields a representation)**.** Let ρ, Γ, Γ', and B be as above. Then $\mathrm{Ind}_\Gamma^{\Gamma'}\rho$ is a representation.

[1] That is, a conjugate-symmetric and positive definite bilinear form $V(\rho) \times V(\rho) \to \mathbb{C}$.

Proof. Let $\gamma \in \Gamma'$ be arbitrary, and write for convenience $\sigma = \mathrm{Ind}_\Gamma^{\Gamma'} \rho$. Since σ is defined by linearity, and since $\rho(\delta)$ for $\delta \in \Gamma$ is linear, it is immediate that $\sigma(\gamma)$ is a linear map.

Since $\sigma(\gamma)$ is an endomorphism, it is enough to show that it is injective to show that it is an isomorphism. This too is almost immediate – let $(v_i)_i$ be a basis for V, and let $B = \{\beta_i\}_i$, with the β_i all different.

Notice that $(\mathfrak{e}_{\beta_j\gamma^{-1}})_j$ is a basis for $\mathbb{C}[B]$. Therefore $(v_i \otimes \mathfrak{e}_{\beta_j\gamma^{-1}})_{i,j}$ is a basis for $V \otimes \mathbb{C}[B]$. Since for each j, we have that $(\rho(I_{\beta_j}(\gamma^{-1})^{-1})(v_i))_i = (v_{\pi_j(i)})_i$ for some permutation π depending on j, we have that

$$(\rho(I_{\beta_j}(\gamma^{-1})^{-1})(v_i) \otimes \mathfrak{e}_{\beta_j\gamma^{-1}})_{i,j},$$

is a basis for $V \otimes \mathbb{C}[B]$. We then have that if

$$\sigma(\gamma)(\sum_{i,j} c_{ij} v_i \otimes \mathfrak{e}_{\beta_j}) = 0,$$

we must have that

$$\sum_{i,j} c_{ij} \rho(I_{\beta_j}(\gamma^{-1})^{-1})(v_i) \otimes \mathfrak{e}_{\beta_j\gamma^{-1}} = 0.$$

However, because of the basis argument above, we must have that $c_{ij} = 0$ for all i, j. Injectivity follows.

We now show that σ is a homomorphism. Let therefore $\gamma_1, \gamma_2 \in \Gamma'$, $\beta \in B$, and $v \in V$ be arbitrary. We see that

$$\sigma(\gamma_1\gamma_2)(v \otimes \mathfrak{e}_\beta) = \rho(I_\beta(\gamma_2^{-1}\gamma_1^{-1})^{-1})(v) \otimes \mathfrak{e}_{\beta\gamma_2^{-1}\gamma_1^{-1}},$$

and

$$\begin{aligned}\sigma(\gamma_1)\sigma(\gamma_2)(v \otimes \mathfrak{e}_\beta) &= \rho(I_{\beta\gamma_2^{-1}}(\gamma_1^{-1})^{-1})\rho(I_\beta(\gamma_2^{-1})^{-1})(v) \otimes \mathfrak{e}_{\beta\gamma_2^{-1}\gamma_1^{-1}} \\ &= \rho(I_{\beta\gamma_2^{-1}}(\gamma_1^{-1})^{-1} I_\beta(\gamma_2^{-1})^{-1})(v) \otimes \mathfrak{e}_{\beta\gamma_2^{-1}\gamma_1^{-1}}\end{aligned}$$

Thus we are done if we can show that

$$I_\beta(\gamma_2^{-1}\gamma_1^{-1}) = I_\beta(\gamma_2^{-1}) I_{\beta\gamma_2^{-1}}(\gamma_1^{-1}).$$

Since γ_1 and γ_2 are arbitrary, it is equivalent to prove

$$I_\beta(\gamma_1\gamma_2) = I_\beta(\gamma_1) I_{\beta\gamma_1}(\gamma_2).$$

To this end, we let $\beta' \in B$ be the element satisfying $\beta(\gamma_1\gamma_2) \in \Gamma\beta'$ and then observe that $I_\beta(\gamma_1\gamma_2)\beta' = \beta(\gamma_1\gamma_2)$. We let $\beta'' \in B$ be the element satisfying $\beta\gamma_1 \in \Gamma\beta''$ and then observe that $I_\beta(\gamma_1)\beta'' = \beta\gamma_1$. For $I_{\beta\gamma_1}$ notice that $\Gamma\beta\gamma_1 = \Gamma\beta''$ and so $I_{\beta\gamma_1} = I_{\beta''}$. Furthermore, we have that $\Gamma\beta''\gamma_2 = \Gamma\beta\gamma_1\gamma_2 = \Gamma\beta'$. Since $I \in \Gamma$, this means that $\beta''\gamma_2 \in \Gamma\beta'$. Thus we have that $I_{\beta''}(\gamma_2)\beta' = \beta''\gamma_2$.

In conclusion we have that

$$I_\beta(\gamma_1) I_{\beta\gamma_1}(\gamma_2) = I_\beta(\gamma_1) I_{\beta''}(\gamma_2) = \beta\gamma_1\beta''^{-1}\beta''\gamma_2\beta'^{-1} = \beta\gamma_1\gamma_2\beta'^{-1} = I_\beta(\gamma_1\gamma_2), \tag{2.1}$$

and so σ is indeed a representation. □

Observation 2.1. The property in equation (2.1) is referred to as a 1-cocycle property, and we say that $I_\bullet(\bullet)$ is a 1-cocycle. This terminology originates in cohomology.

With this definition we will see that the permutation representation created in example 2.5 is isomorphic to an induced representation. But first, let us agree on what we mean by an isomorphism, or more generally a homomorphism, between representations.

Definition 2.7 (Homomorphisms of representations)**.** Let G be a group and let V_1, V_2 be vector spaces. Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be representations. A homomorphism of representations from ρ_1 to ρ_2 is a linear map $f : V_1 \to V_2$ such that for all $\gamma \in G$ it holds that

$$f \circ \rho_1(\gamma) = \rho_2(\gamma) \circ f.$$

Furthermore, if f is invertible, we say that it is an isomorphism of representations.

The set of all homomorphisms from ρ_1 to ρ_2 is a vector space, and is denoted by $\mathrm{Hom}(\rho_1, \rho_2)$ or $\mathrm{Hom}_G(\rho_1, \rho_2)$ if the underlying group needs to be recalled.

When we later consider spans of products of Eisenstein series, the Hom-spaces will fill the crucial role of preserving modular invariance (that is, invariance with respect to the slash-action).

As one expects from the name, inducing an arithmetic type ρ to a larger subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (or $\mathrm{Mp}_1(\mathbb{Z})$) essentially preserves the structure of ρ, in the following sense.

Proposition 2.3. Let Γ, Γ' be subgroups of $\mathrm{SL}_2(\mathbb{Z})$ (or $\mathrm{Mp}_1(\mathbb{Z})$), and let $\rho : \Gamma \to \mathrm{GL}(V(\rho))$ be an arithmetic type. Let $\sigma = \mathrm{Ind}_{\Gamma}^{\Gamma'} \rho$. Then the restriction $\sigma|_\Gamma$ is isomorphic to ρ. Furthermore, if ρ is a congruence type of level N, then so is σ.

Proof. For the first part, let $\gamma \in \Gamma$. Then

$$\sigma(\gamma)(v \otimes \mathfrak{e}_I) = \rho(I_I(\gamma^{-1})^{-1})(v) \otimes \mathfrak{e}_{I\gamma^{-1}} = \rho(\gamma)(v) \otimes \mathfrak{e}_I,$$

whence the isomorphism is clear.

As for the second part, let $V(\rho)$ have the basis $\{e_1, \ldots, e_d\}$ for some integer d. Then we have that

$$\ker(\sigma) = \{\gamma \in \Gamma' : \rho(I_\beta(\gamma^{-1})^{-1})(e_i) \otimes \mathfrak{e}_{\beta\gamma^{-1}} = e_i \otimes \mathfrak{e}_\beta \text{ for all } \beta \in \Gamma' \text{ and } 1 \leq i \leq d\}.$$

Hence, if $\gamma \in \ker(\sigma)$ we have for all $\beta \in \Gamma'$ that

$$\Gamma\beta\gamma^{-1} = \Gamma\beta,$$

and $\rho(I_\beta(\gamma^{-1})^{-1}) = I$. The former yields that $I_\beta(\gamma^{-1}) = \beta\gamma^{-1}\beta^{-1}$, so that by picking $\beta = I$ we have that $\rho(\gamma) = I$. This means precisely that $\gamma \in \ker(\rho)$.

Let now $v \in V(\rho)$ and $\beta \in B$. Then we have for $\gamma \in \ker(\rho)$ that

$$\sigma(\beta^{-1}\gamma\beta)(v \otimes \mathfrak{e}_\beta) = \rho(I_\beta(\beta^{-1}\gamma^{-1}\beta)^{-1})(v) \otimes \mathfrak{e}_{\beta\beta^{-1}\gamma^{-1}\beta} = \rho(\gamma)(v) \otimes \mathfrak{e}_\beta = v \otimes \mathfrak{e}_\beta.$$

Thus we have that $\beta^{-1}\ker(\rho)\beta \subseteq \mathrm{Stab}(v \otimes \mathfrak{e}_\beta)$. Taking intersections, we find that

$$\bigcap_{\beta \in B} \beta^{-1}\ker(\rho)\beta \subseteq \bigcap_{\substack{1 \leq i \leq d \\ \beta \in B}} \mathrm{Stab}(e_i \otimes \mathfrak{e}_\beta) = \ker(\sigma).$$

We now make use of the fact that $\Gamma(N)$ is a normal subgroup. Take an arbitrary element $\gamma \in \Gamma(N)$. Then for any $\beta \in B$ there is an element $\gamma_\beta \in \Gamma(N)$ satisfying $\gamma = \beta^{-1}\gamma_\beta\beta$. We have however that $\gamma_\beta \in \ker(\rho)$ for all β, so $\gamma \in \beta^{-1}\ker(\rho)\beta$ for all β. Hence $\Gamma(N) \subseteq \ker(\sigma)$.

This N is minimal because if it were not, then there would exist a positive integer $M < N$ satisfying $\Gamma(M) \subseteq \ker(\sigma)$. As shown above, we have that $\ker(\sigma) \subseteq \ker(\rho)$, and therefore we would have that $\Gamma(M) \subseteq \ker(\rho)$, contradicting the minimality of N. □

Example 2.6 (Induced representations can be used to define permutation representations)**.** Denote the representation from example 2.5 by ρ_G, but keep the rest of the notation.

We now see that ρ_G is isomorphic as a representation to $\rho' = \mathrm{Ind}_G^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1}$. Indeed, for $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ arbitrary we have that

$$\rho'(\alpha)(c \otimes \mathfrak{e}_{\gamma_i}) = \mathbf{1}(I_{\gamma_i}(\alpha^{-1})^{-1})(c) \otimes \mathfrak{e}_{\gamma_i\alpha^{-1}} = c \otimes \mathfrak{e}_{\gamma_i\alpha^{-1}}.$$

Let $\tilde{e}_i = 1 \otimes \mathfrak{e}_{\gamma_i}$ for $1 \leq i \leq d$, and e_i be the canonical basis for $\mathbb{C}^d$. We then see that $P_{\sigma_\alpha} e_i = e_{\sigma_\alpha^{-1}(i)}$ and $\rho'(\alpha)(\tilde{e}_i) = \tilde{e}_{\sigma_{\alpha^{-1}}(i)} = \tilde{e}_{\sigma_\alpha^{-1}(i)}$.

This leads us to define $f : \mathbb{C} \otimes \mathbb{C}[\gamma_1, \ldots, \gamma_d] \to \mathbb{C}^d$ via $f(\tilde{e}_i) = e_i$ and elsewhere by linearity. It is clear that f is invertible, so we only have to verify compatiblity. But obviously we have that

$$f(\rho'(\alpha)(\tilde{e}_i)) = f(\tilde{e}_{\sigma_\alpha^{-1}(i)}) = e_{\sigma_\alpha^{-1}(i)} = P_{\sigma_\alpha} e_i = \rho_G(\alpha) f(\tilde{e}_i),$$

and so we are done.

Notation 1. When there is no risk of misunderstanding we will denote pure tensors in tensor products of the form $W = \mathbb{C} \otimes V$ where V is a $\mathbb{C}$-vector space by juxtaposition. That is, we make the identification $c \otimes v = cv$ for arbitrary $c \in \mathbb{C}$ and $v \in V$.

For the sake of demonstration, let us also compute a space of homomorphisms of representations.

Example 2.7. Put $\rho = \mathrm{Ind}_{\Gamma_0(2)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1}$ and $\sigma = \rho \otimes \rho$ with the same identifications as in example 2.6. Using one's favorite algorithm to compute coset representatives, one sees that $V(\rho) = \mathbb{C}^3$ and

$$\rho(S) = P_{321}, \text{ and } \rho(T) = P_{132}.$$

This yields that $V(\sigma) = \mathbb{C}^3 \otimes \mathbb{C}^3$, and

$$\sigma(S) = P_{321} \otimes P_{321} = P_{987654321}.$$

and

$$\sigma(T) = P_{132} \otimes P_{132} = P_{132798465}.$$

Since every element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ can be written $\gamma = T^{k_1}ST^{k_2}ST^{k_3}\dots ST^{k_n}$ for some integers k_i, we see that every $f \in \mathrm{Hom}(\rho, \sigma)$ is completely determined by the constraints

$$f \circ \rho(S) = \sigma(S) \circ f, \text{ and } f \circ \rho(T) = \sigma(T) \circ f.$$

In other words, we just have to find all matrices $f \in \mathrm{Mat}_{9\times 3}(\mathbb{C})$ satisfying

$$\begin{cases} fP_{321} = P_{987654321}f, \\ fP_{132} = P_{132798465}f. \end{cases}$$

This is a simple, if tedious, linear algebra problem. Using one's favorite linear algebra system, one easily finds that

$$\begin{aligned} f = {} & r_1(e_{11}|e_{22}|e_{33}) \\ & + r_2(e_{22} + e_{33}|e_{11} + e_{33}|e_{11} + e_{22}) \\ & + r_3(e_{23} + e_{32}|e_{13} + e_{31}|e_{12} + e_{21}) \\ & + r_4(e_{12} + e_{13}|e_{21} + e_{23}|e_{31} + e_{32}) \\ & + r_5(e_{21} + e_{31}|e_{12} + e_{32}|e_{13} + e_{23}), \end{aligned}$$

where $e_{ij} = e_i \otimes e_j$ and the $r_i \in \mathbb{C}$ are arbitrary coefficients. For future reference, we denote the matrix with coefficient r_i by R_i. Evidently the R_i are linearly independent and so form a basis and $\mathrm{Hom}(\rho, \sigma) = \mathrm{span}_{\mathbb{C}}\{R_1, R_2, R_3, R_4, R_5\}$.

For computational purposes, the above method is too inefficient, and instead we later translate the problem of computing $\mathrm{Hom}(\mathbf{1}, \rho)$ for an arithmetic type ρ into the problem of computing orbits of a basis for $V(\rho)$ under the action of $\rho(S)$ and $\rho(T)$. We will elaborate on this in chapter 5.

Using the so-called dual representation, we can compute all spaces of homomorphisms in essentially the same way.

Definition 2.8 (Dual representation)**.** Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation. Then the dual representation $\rho^\vee : G \to \mathrm{GL}(V^\vee)$ is defined by

$$\rho^\vee(g) = {}^t\rho(g^{-1}),$$

where ${}^t\cdot$ denotes transpose as a linear map.

Proposition 2.4. Let ρ and σ be linear representations of a group G. Then

$$\mathrm{Hom}(\rho, \sigma) \cong \mathrm{Hom}(\mathbf{1}, \sigma \otimes \rho^\vee).$$

Proof. Let $\{e_i\}_{i=1}^n$ be a basis for $V(\rho)$ and $\{e_i^\vee\}_{i=1}^n$ be the corresponding dual basis for $V(\rho)^\vee$. Similarly, let $\{f_i\}_{i=1}^m$ be a basis for $V(\sigma)$.

Consider now the right hand side. Select an element $\phi \in \mathrm{Hom}(\mathbf{1}, \sigma \otimes \rho^\vee)$. Then ϕ is completely determined by $\phi(1)$, and we have

$$\phi(1) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} f_i \otimes e_j^\vee.$$

This can be identified with a linear map from $V(\rho) \to V(\sigma)$ by the mapping

$$\Phi(\phi) = V(\rho) \ni v \mapsto \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} f_i \otimes e_j^\vee(v) \mapsto \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} e_j^\vee(v) f_i \in V(\sigma).$$

We claim that this mapping is a linear isomorphism. The linearity is clear, and hence we only have to show that Φ is injective and surjective. Since the dimensions of the domain and the co-domain of Φ are equal, it suffices to show injectivity.

To this end, let $\phi \in \ker(\Phi)$. Then for every $v \in V(\rho)$ and for every $1 \leq i \leq m$, we have that

$$\sum_{j=1}^{n} c_{ij} e_j^\vee(v) = 0.$$

In particular, this is true for $v = e_k$ with k arbitrary, and hence for all $1 \leq i \leq m$ and $1 \leq j \leq n$ we have that $c_{ij} = 0$. Therefore $\phi(1) = 0$, so that by linearity $\phi = 0$.

It remains to verify that Φ preserves the homomorphism property. Select $\phi \in \mathrm{Hom}(\mathbf{1}, \sigma \otimes \rho^\vee)$, then by definition we have for $g \in G$ arbitrary, that

$$\phi(1) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} \sigma(g)(f_i) \otimes (e_j^\vee \circ \rho(g^{-1})).$$

Since $\sigma(g)$ and $\rho(g^{-1})$ are invertible, we obtain

$$\Phi(\phi)(v) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} e_j^\vee(\rho(g^{-1})(v))\sigma(g)(f_i),$$

and thus the property of ϕ being a homomorphism can be restated as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} e_j^\vee(v) f_i = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} e_j^\vee(\rho(g^{-1})(v))\sigma(g)(f_i),$$

for all $v \in V(\rho)$ and $g \in G$. We want to use this to show that for all $v \in V(\rho)$, it holds that $\Phi(\phi) \in \mathrm{Hom}(\rho, \sigma)$, or in other words that

$$\Phi(\phi)(\rho(g)(v)) = \sigma(g)(\Phi(\phi)(v)),$$

for all $v \in V(\rho)$ and $g \in G$. This is clear, because indeed it holds that

$$\begin{aligned}\Phi(\phi)(\rho(g)(v)) &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} e_j^\vee(v)\sigma(g)(f_i) \\ &= \sigma(g)\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} e_j^\vee(v) f_i\Big) = \sigma(g)(\Phi(\phi)(v)),\end{aligned}$$

and we are done. □

Now that we have a firm grasp of induced representations, let us introduce notation for some of the most commonly occuring arithmetic types.

Notation 2. For a positive integer $N \geq 1$, and a Dirichlet character χ modulo N, we use the notation

$$\rho_N^\times = \mathrm{Ind}_{\Gamma_1(N)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1}, \text{ and } \rho_\chi^\times = \mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_2(\mathbb{Z})}\chi.$$

Here χ is defined under definition 2.2.

In essence, $\rho_N^\times$ can be viewed as the archetypical trivially twisted permutation type, and $\rho_\chi^\times$ as the archetypical twisted permutation type.

Using the orthogonality relations of Dirichlet characters, we can in fact decompose $\rho_N^\times$ as a direct sum of $\rho_\chi^\times$ for appropriate characters χ.

But before we do so, let us define what is meant by a direct sum of representations.

Definition 2.9 (Direct sum of representations)**.** Let G be a group, and let V_1, V_2 be vector spaces. Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be representations. Then $\rho_1 \oplus \rho_2$ is defined by

(i) $V(\rho_1 \oplus \rho_2) = V_1 \oplus V_2$, and

(ii) for each $g \in G$ and $(v_1, v_2) \in V_1 \oplus V_2$ we put

$$(\rho_1 \oplus \rho_2)(g)(v_1, v_2) = (\rho_1(g)(v_1), \rho_2(g)(v_2)).$$

Proposition 2.5 (Decomposition of $\rho_N^\times$)**.** Let N be a positive integer. Then

$$\rho_N^\times = \bigoplus_{\chi \bmod N} \rho_\chi^\times,$$

where the direct sum goes through all Dirichlet characters modulo N.

Note: Just as in definition 2.9, we treat this as an external direct sum.

Proof. The following proof is a compact rewriting of lemmas 3.9, 8.19, 8.21, 8.22, and 8.23, of [Xià19].

As stated before, the decomposition follows from the orthogonality relations of Dirichlet characters. They can be stated as follows:

$$\sum_{\chi \bmod N} \chi(d) = \begin{cases} \phi(N) & \text{if } d \equiv_N 1 \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

where ϕ is Euler's totient function. Similarly, we have that

$$\sum_{d \bmod N} \chi(d) = \begin{cases} \phi(N) & \text{if } \chi = \mathbf{1} \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

where $\mathbf{1}$ denotes the trivial Dirichlet character modulo N.[2]

For convenience, we let B_1 and B_0 denote a set of representatives of $\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$, respectively. Both including the identity.

We now define a function

$$\begin{aligned} f : V(\rho_N^\times) &\to \bigoplus_{\chi \bmod N} V(\rho_\chi^\times), \\ \mathfrak{e}_\beta &\mapsto \frac{1}{\sqrt{\phi(N)}} (\chi(I_I(\beta)^{-1})\mathfrak{e}_\beta)_{\chi \bmod N}, \end{aligned}$$

and elsewhere by linearity. Similarly, put

$$\begin{aligned} g : \bigoplus_{\chi \bmod N} V(\rho_\chi^\times) &\to V(\rho_N^\times), \\ (\delta_\chi \mathfrak{e}_{\beta_\chi})_{\chi \bmod N} &\mapsto \frac{1}{\sqrt{\phi(N)}} \sum_{\chi \bmod N} \delta_\chi \sum_{\substack{\gamma \in B_1 \\ \Gamma_0(N)\gamma = \Gamma_0(N)\beta_\chi}} \chi(I_I(\gamma))\mathfrak{e}_\gamma \end{aligned}$$

where $\delta_\chi \in \mathbb{C}$, and elsewhere by linearity. In both of these definitions, $I_\bullet(\bullet)$ is the 1-cocycle coming from $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$. Notice also that if $\gamma \in \Gamma_0(N)$, then $\gcd(d(\gamma), N) = 1$, and hence $\chi(\gamma)$ is non-zero.

We now claim that

(i) $f \in \mathrm{Hom}(\rho_N^\times, \bigoplus_{\chi \bmod N} \rho_\chi^\times)$, that

(ii) $g \in \mathrm{Hom}(\bigoplus_{\chi \bmod N} \rho_\chi^\times, \rho_N^\times)$, that

(iii) $f \circ g = \mathrm{id}$, and that

(iv) $g \circ f = \mathrm{id}$.

[2] That is, $\mathbf{1}(d) = 1$ for d coprime with N, and $\mathbf{1}(d) = 0$ otherwise.

Of these (i) and (ii) follow from the cocycle property of $I_\bullet(\bullet)$. Indeed, since f and g are linear, we only have to prove that they intertwine with the action. To this end, let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be arbitrary. In the case of claim (i), we want to show that

$$f \circ \rho_N^\times(\gamma) = (\bigoplus_{\chi \bmod N} \rho_\chi^\times)(\gamma) \circ f.$$

By linearity, it is sufficient to verify that this holds on a basis $\{\mathfrak{e}_\beta\}_{B_1}$.

As for the left-hand side, we obtain for $\beta \in B_1$ the following

$$\begin{aligned} f \circ \rho_N^\times(\gamma)(\mathfrak{e}_\beta) &= f(\mathfrak{e}_{\beta\gamma^{-1}}) \\ &= \frac{1}{\sqrt{\phi(N)}}(\chi(I_I(\beta\gamma^{-1})^{-1})\mathfrak{e}_{\beta\gamma^{-1}})_{\chi \bmod N}. \end{aligned}$$

The right-hand side becomes in turn

$$\begin{aligned} (\bigoplus_{\chi \bmod N} \rho_\chi^\times)(\gamma) \circ f(\mathfrak{e}_\beta) &= (\bigoplus_{\chi \bmod N} \rho_\chi^\times)(\gamma)(\frac{1}{\sqrt{\phi(N)}}(\chi(I_I(\beta)^{-1})\mathfrak{e}_\beta)_{\chi \bmod N}) \\ &= \frac{1}{\sqrt{\phi(N)}}(\chi(I_I(\beta)^{-1})\chi(I_\beta(\gamma^{-1})^{-1})\mathfrak{e}_{\beta\gamma^{-1}})_{\chi \bmod N}. \end{aligned}$$

Now we use the co-cycle property of $I_\bullet(\bullet)$. We have that

$$I_I(\beta)I_\beta(\gamma^{-1}) = I_I(\beta\gamma^{-1}),$$

and therefore

$$\begin{aligned} \chi(I_I(\beta)^{-1})\chi(I_\beta(\gamma^{-1})^{-1}) &= \chi(I_\beta(\gamma^{-1})^{-1})\chi(I_I(\beta)^{-1}) \\ &= \chi((I_I(\beta)I_\beta(\gamma^{-1}))^{-1}) \\ &= \chi(I_I(\beta\gamma^{-1})^{-1}). \end{aligned}$$

Now we consider claim (ii). Consider, as above, an arbitrary basis vector $v = (\delta_\chi \mathfrak{e}_{\beta_\chi})_{\chi \bmod N}$, where $\delta_\chi \in \mathbb{C}$ and $\beta_\chi \in B_0$ for every χ. Then we have that the left-hand side equals

$$\begin{aligned} g \circ (\bigoplus_{\chi \bmod N} \rho_\chi^\times)(\gamma)(v) &= g\big((\delta_\chi \chi(I_{\beta_\chi}(\gamma^{-1}))^{-1}\mathfrak{e}_{\beta_\chi\gamma^{-1}})_{\chi \bmod N}\big) \\ &= \frac{1}{\sqrt{\phi(N)}} \sum_{\chi \bmod N} \delta_\chi \sum_{\substack{\gamma' \in B_1 \\ \Gamma_0(N)\gamma' = \Gamma_0(N)\beta_\chi\gamma^{-1}}} \chi(I_I(\gamma'))\chi(I_{\beta_\chi}(\gamma^{-1}))^{-1}\mathfrak{e}_{\gamma'}. \end{aligned}$$

And similarly, we obtain for the right-hand side

$$\rho_N^\times(\gamma) \circ g(v) = \frac{1}{\sqrt{\phi(N)}} \sum_{\chi \bmod N} \delta_\chi \sum_{\substack{\gamma' \in B_1 \\ \Gamma_0(N)\gamma' = \Gamma_0(N)\beta_\chi}} \chi(I_I(\gamma'))\mathfrak{e}_{\gamma'\gamma^{-1}}.$$

In the innermost sum of the right-hand side, we let $\gamma'' \in B_1$ be the representative of $\gamma'\gamma^{-1}$. Then we have that $\Gamma_0(N)\gamma' = \Gamma_0(N)\beta_\chi$ if and only if $\Gamma_0(N)\gamma'' = \Gamma_0(N)\beta_\chi\gamma^{-1}$. Furthermore, we have that $\gamma' = g\gamma''\gamma$ for some $g \in \Gamma_1(N)$, and this leads to

$$I_I(\gamma') = I_I(g\gamma''\gamma) = I_I(g)I_g(\gamma''\gamma).$$

But $I_I(g) = g$ since $g \in \Gamma_1(N) \subseteq \Gamma_0(N)$. Hence, since $\chi(g) = 1$, we obtain

$$\chi(I_I(\gamma')) = \chi(gI_g(\gamma''\gamma)) = \chi(I_g(\gamma''\gamma)) = \chi(I_I(\gamma''\gamma)),$$

where the step follows from the definition of $I_\bullet(\bullet)$. Now, let β'_χ be the representative of γ'' in B_0. Then $I_I(\gamma'') = \gamma''(\beta'_\chi)^{-1}$ and $I_{\beta_\chi}(\gamma^{-1}) = \beta_\chi\gamma^{-1}(\beta'_\chi)^{-1}$, and consequently

$$I_I(\gamma'')I_{\beta_\chi}(\gamma^{-1})^{-1} = \gamma''\gamma\beta_\chi^{-1} = I_I(\gamma''\gamma),$$

where the last equality follows from $\Gamma_0(N)\gamma''\gamma = \Gamma_0(N)\beta_\chi$ and $\beta_\chi \in B_0$.

Putting all this together, we obtain

$$\begin{aligned}\sum_{\substack{\gamma'\in B_1\\ \Gamma_0(N)\gamma'=\Gamma_0(N)\beta_\chi}} \chi(I_I(\gamma'))\mathfrak{e}_{\gamma'\gamma^{-1}} &= \sum_{\substack{\gamma''\in B_1\\ \Gamma_0(N)\gamma''=\Gamma_0(N)\beta_\chi\gamma^{-1}}} \chi(I_I(\gamma''\gamma))\mathfrak{e}_{\gamma''}\\ &= \sum_{\substack{\gamma''\in B_1\\ \Gamma_0(N)\gamma''=\Gamma_0(N)\beta_\chi\gamma^{-1}}} \chi(I_I(\gamma'')I_{\beta_\chi}(\gamma^{-1})^{-1})\mathfrak{e}_{\gamma''}\\ &= \sum_{\substack{\gamma'\in B_1\\ \Gamma_0(N)\gamma'=\Gamma_0(N)\beta_\chi\gamma^{-1}}} \chi(I_I(\gamma'))\chi(I_{\beta_\chi}(\gamma^{-1}))^{-1}\mathfrak{e}_{\gamma'},\end{aligned}$$

and so the left-hand and right-hand sides are equal. This concludes the proof.

Concerning (iii) and (iv), it is on account of linearity, enough to prove that the statements hold on a basis.

As for (iii), we have

$$f(g((\delta_\chi \mathfrak{e}_{\beta_\chi})_{\chi \bmod N})) = \frac{1}{\phi(N)} \sum_{\chi \bmod N} \delta_\chi\Big(\sum_{\substack{\gamma\in B_1\\ \Gamma_0(N)\gamma=\Gamma_0(N)\beta_\chi}} \frac{\chi}{\chi'}(I_I(\gamma))\mathfrak{e}_\gamma\Big)_{\chi' \bmod N}.$$

One can easily show that[3] for $\beta \in \mathrm{SL}_2(\mathbb{Z})$ that the map

$$\{I_I(\gamma) : \gamma \in B_1, \gamma\beta^{-1} \in \Gamma_0(N)\} \ni \alpha \mapsto \mathrm{rem}(d(\alpha), N) \in \{1 \le d \le N : \gcd(d, N) = 1\},$$

where $\mathrm{rem}(a, b)$ denotes the remainder modulo b of a, is a bijection and thus we conclude that

$$\begin{aligned}&\frac{1}{\phi(N)} \sum_{\chi \bmod N} \delta_\chi\Big(\sum_{\substack{\gamma\in B_1\\ \Gamma_0(N)\gamma=\Gamma_0(N)\beta_\chi}} \frac{\chi}{\chi'}(I_I(\gamma))\mathfrak{e}_\gamma\Big)_{\chi' \bmod N}\\ &= \frac{1}{\phi(N)} \sum_{\chi \bmod N} \delta_\chi\Big(\sum_{\substack{\gamma\in B_1\\ \Gamma_0(N)\gamma=\Gamma_0(N)\beta_\chi}} \frac{\chi}{\chi'}(I_I(\gamma))\mathfrak{e}_{\beta_\chi}\Big)_{\chi' \bmod N}\\ &= \frac{1}{\phi(N)}\Big(\sum_{\chi \bmod N} \delta_\chi\Big(\sum_{\gcd(d,N)=1} \frac{\chi}{\chi'}(d)\Big)\mathfrak{e}_{\beta_\chi}\Big)_{\chi' \bmod N}\\ &= \frac{1}{\phi(N)} \Big(\phi(N)\delta_{\chi'}\mathfrak{e}_{\beta_{\chi'}}\Big)_{\chi' \bmod N}\\ &= \Big(\delta_{\chi'}\mathfrak{e}_{\beta_{\chi'}}\Big)_{\chi' \bmod N},\end{aligned}$$

where in the last step we used equation 2.3.

As for (iv), we have that

$$\begin{aligned}g(f(\mathfrak{e}_\beta)) &= \frac{1}{\phi(N)} \sum_{\chi \bmod N} \sum_{\substack{\gamma\in B_1\\ \Gamma_0(N)\gamma=\Gamma_0(N)\beta}} \chi(I_I(\gamma))\chi(I_I(\beta))^{-1}\mathfrak{e}_\gamma\\ &= \frac{1}{\phi(N)} \sum_{\chi \bmod N} \sum_{\substack{\gamma\in B_1\\ \Gamma_0(N)\gamma=\Gamma_0(N)\beta}} \chi(I_I(\gamma))\chi(I_I(\beta))^{-1}\mathfrak{e}_\beta\\ &= \frac{1}{\phi(N)} \sum_{\substack{\gamma\in B_1\\ \Gamma_0(N)\gamma=\Gamma_0(N)\beta}} \Big(\sum_{\chi \bmod N} \chi(I_I(\gamma)I_I(\beta)^{-1})\Big)\mathfrak{e}_\beta.\end{aligned}$$

[3]Since $I_I(\gamma) = \gamma\beta_r^{-1}$ for the representative $\beta_r \in B_0$ of β, we have that injectivity is immediate. As for surjectivity let $1 \le d \le N$ be coprime with N. Then for some integers a, k we have that $ad - kN = 1$ and so $\alpha = (a, k; N, d) \in \Gamma_0(N)$. Let now $\gamma \in B_1$ satisfy $\Gamma_1(N)\gamma = \Gamma_1(N)\alpha\beta$. Then $\gamma\beta^{-1} = A\alpha$ for some $A \in \Gamma_1(N)$. Then $d(\gamma\beta^{-1}) = d(A\alpha) = d$ and also $\gamma\beta^{-1} \in \Gamma_0(N)$, so we are done.

Now since $\Gamma_0(N)\gamma = \Gamma_0(N)\beta$ in the sum, we have that there exists some $g \in B_0$ for which $I_I(\gamma) = \gamma g^{-1}$ and $I_I(\beta) = \beta g^{-1}$, and so $I_I(\gamma)I_I(\beta)^{-1} = \gamma\beta^{-1}$. This yields that

$$\sum_{\chi \bmod N} \chi(I_I(\gamma)I_I(\beta)^{-1}) = \sum_{\chi \bmod N} \chi(\gamma\beta^{-1}) = \begin{cases} \phi(N) & \text{if } d(\gamma\beta^{-1}) \equiv_N 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now, if $d(\gamma\beta^{-1}) \equiv_N 1$ and $\gamma\beta^{-1} \in \Gamma_0(N)$, we immediately have that $\gamma\beta^{-1} \in \Gamma_1(N)$. But then, since $\gamma, \beta \in B_1$, we have that $\gamma = \beta$. From this we conclude that

$$\frac{1}{\phi(N)} \sum_{\substack{\gamma \in B_1 \\ \Gamma_0(N)\gamma = \Gamma_0(N)\beta}} \Big(\sum_{\chi \bmod N} \chi(\gamma\beta^{-1}) \Big) \mathfrak{e}_\beta = \mathfrak{e}_\beta,$$

as desired. □

Since we have yet to look at a concrete example of a non-trivially twisted permutation type, let us do so now.

Example 2.8 (Non-trivially twisted permutation type)**.** Let χ be the Dirichlet character modulo 5 given by

$$\chi(0) = 0, \chi(1) = 1, \chi(2) = i, \chi(3) = -i, \text{ and } \chi(4) = -1.$$

Using one's favorite algorithm to find (right) coset-representatives of $\Gamma_0(5)$, one obtains

$$B = \left\{ I, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \right\}.$$

For convenience, we denote the elements of B by γ_1 through γ_6, with the same order as above.

After some computation, one finds that

$$\rho_\chi^\times(S) = \operatorname{diag}(-1, -i, -1, i, 1, 1)P_{526413}, \text{ and}$$
$$\rho_\chi^\times(T) = \operatorname{diag}(1, 1, 1, i, 1, -1)P_{135264}.$$

Which again, completely determines the representation.

We now introduce Weil types. They are more intricate to define than twisted permutation types, and therefore we devote a subsection to the definition.

2.2.1 Discriminant forms and the Weil type

A Weil representation is a representation of $\mathrm{Mp}_1(\mathbb{Z})$ associated to an object referred to as a finite quadratic module, that in the following sense captures every irreducible congruence type.

Theorem 1 (Skoruppa)**.** Let ρ be an irreducible congruence type with respect to $\mathrm{Mp}_1(\mathbb{Z})$ or $\mathrm{SL}_2(\mathbb{Z})$, then ρ is isomorphic to an irreducible component of a Weil type.

Proof. See [CS17, p. 580]. □

Let us begin by defining what a finite quadratic module is.

Definition 2.10. Let D be a finite abelian group and let $Q : D \to \mathbb{Q}/\mathbb{Z}$ be a quadratic form such that with $B(x, y) = Q(x+y) - Q(x) - Q(y)$ we have that $Q(ax) = a^2Q(x)$ for all $a \in \mathbb{Z}$ and $x \in D$ and such that $D \ni x \mapsto B(x, \cdot) \in \mathrm{Hom}(D, \mathbb{Q}/\mathbb{Z})$ is a linear isomorphism. Then we call (D, Q) a finite quadratic module.

An important special case of a finite quadratic module is a discriminant form, which we define in terms of lattices.

Definition 2.11. Let A be an m-by-n matrix with entries in $\mathbb{R}$ and let $q : A\mathbb{Z}^n \to \mathbb{R}$ be a quadratic form. Then the pair $(A\mathbb{Z}^n, q)$ is called a lattice.

Lattices carry a natural notion of isomorphism.

Definition 2.12. Let (A, q_1) and (B, q_2) be two lattices, and let $\phi : A \to B$ be a group homomorphism. Then ϕ is a homomorphism of lattices if in addition it holds that $q_1 = q_2 \circ \phi$.

The same notion of isomorphism is appropriate also for finite quadratic modules. In the sequel we shall often identify lattices and finite quadratic modules with their first component.

Since any quadratic form $q : A \to \mathbb{R}$ where A is a torsion-free abelian group can be written as $q(x) = x^T Qx$ for some square matrix Q, we can for sufficiently nice quadratic forms, choose a canonical way of representing a lattice.

Proposition 2.6. Let $L = (A\mathbb{Z}^n, q)$ be a lattice, with q a non-degenerate quadratic form. Then there exists a matrix Q such that

$$L \cong (\mathbb{Z}^n, x \mapsto \frac{1}{2}x^T Qx).$$

Proof. Express q in terms of a matrix, and make the appropriate change of variables. □

In the sequel, we refer to the matrix Q above as the Gram matrix of the lattice.

We now define a few technical notions that are needed to introduce discriminant forms.

Definition 2.13. Let $L = (A, q)$ be a lattice. We say that L is even if $q(x) \in \mathbb{Z}$ for every $x \in A$. And we say that L is non-singular if its Gram matrix is non-singular.

Definition 2.14 (Dual lattice)**.** Let $L = (A, q)$ be a lattice and write $b(x, y) = q(x+y)-q(x)-q(y)$. Define

$$B = \{y \in \mathbb{R}^n : b(x, y) \in \mathbb{Z} \text{ for all } x \in A\},$$

then $L^* = (B, q)$ is called the dual lattice of L.

A discriminant form can now be defined as follows.

Definition 2.15 (Discriminant form)**.** Let $L = (A, q)$ be a non-singular even lattice, and let $L^* = (B, q)$ be its dual. Then the lattice

$$L^*/L = (B/A, x + B \mapsto q(x) + \mathbb{Z}),$$

is a finite quadratic module, called the discriminant form of L.

By virtue of the following proposition, no loss of generality is incurred by restricting ourselves to discriminant forms.

Proposition 2.7. Let A be a finite quadratic module, then there exists a lattice L such that $A \cong L^*/L$.

Proof. See [CS17, p. 581]. □

Before we can introduce Weil representations we need to define another property of lattices and discriminant forms – the signature.

Proposition-Definition 1 (Signature)**.** Let $\mathcal{L} = (L, q)$ be a nonsingular even lattice. Then there exists a unique integer $0 \leq k < 8$ such that

$$\sum_{\gamma \in \mathcal{L}^*/\mathcal{L}} e(q(\gamma)) = \sqrt{|L'/L|}e(k/8).$$

We write $k = \text{sign}(\mathcal{L})$ and refer to this as the signature of $\mathcal{L}$.

If A is a finite quadratic module, then we define the signature of A as the signature of any lattice whose discriminant form is isomorphic to A.

Proof. This is referred to as Milgram's formula. See [Bor00, p. 323] or [CS17, p. 581]. □

We can now define the Weil representation associated to a discriminant form, or equivalently, the Weil representation associated to a lattice.

Definition 2.16 (Weil representation)**.** Let A be a finite quadratic module, say isomorphic to the discriminant form of the lattice $\mathcal{L} = (L, q)$. Write as usual $b(x, y) = q(x+y)-q(x)-q(y)$. Let $\mathbb{C}[A]$ be the group ring of A, with standard basis $\mathfrak{e}_\gamma$ for $\gamma \in A$. We then define the Weil representation associated to A, denoted by $\rho_A : \text{Mp}_1(\mathbb{Z}) \to \text{GL}(\mathbb{C}[A])$, by

$$\rho_A(\tilde{S})(\mathfrak{e}_\gamma) = \frac{e(-\text{sign}(A)/8)}{\sqrt{|A|}} \sum_{\delta \in A} e(-b(\gamma, \delta))\mathfrak{e}_\delta$$
$$\rho_A(\tilde{T})(\mathfrak{e}_\gamma) = e(q(\gamma))\mathfrak{e}_\gamma.$$

As for the Weil representation actually being a representation, see [CS17, p. 548]. The weight that a modular form of type ρ_A can have is connected to the signature of A.

Proposition 2.8. Let A be a finite quadratic module. Then if $\text{sign}(A)$ is even, we have that ρ_A restricts to a representation of $\text{SL}_2(\mathbb{Z})$.

Proof. See [CS17, p. 584]. □

2.3 Dimensions of spaces of modular forms and the Sturm bound

Note that we can by using induced representations restrict ourselves to arithmetic types associated to $\text{Mp}_1(\mathbb{Z})$ or $\text{SL}_2(\mathbb{Z})$.

The following theorem is a corollary of a theorem of Borcherds, see [Bor00, p. 335].

Theorem 2. Let ρ be an arithmetic type associated to $\text{Mp}_1(\mathbb{Z})$ and let $k \in \frac{1}{2}\mathbb{Z}$ with $k > 2$. Then

$$\dim(M_k(\rho)) = \frac{1}{2}\sum_{j=0}^{3} e(jk/2)\psi(k,\rho,j),$$

where

$$\begin{aligned}\psi(k,\rho,j) = \frac{k-1}{12}\text{Tr}_\rho(Z^j) &+ \delta_3(e(1/6)\rho(R),\rho(Z^j)) \\ &+ \delta_2(e(1/4)\rho(\tilde{S}),\rho(Z^j)) \\ &+ \delta_0(\rho(\tilde{T}^{-1}),\rho(Z^j)),\end{aligned}$$

where $R = \tilde{T}\tilde{S}$, and

$$\delta_N(X,g) = \frac{1}{N}\sum_{j=1}^{N-1}\frac{\text{Tr}(X^j g)}{1-e(j/N)},$$

for $N > 0$ and

$$\delta_0(X,g) = \frac{\text{Tr}(g)}{2M} + \frac{1}{M}\sum_{j=1}^{M-1}\frac{\text{Tr}(X^j g)}{1-e(j/M)},$$

where M is the smallest positive integer such that $X^M = \text{Id}$.

Proof. The result follows by noting that there are two elliptic fixed points in $\text{Mp}_1(\mathbb{Z})\backslash\mathbb{H}$, namely i and $e^{\pi i/3}$, with order 2 and 3, respectively. Their corresponding primitive elliptic elements are $\tilde{S}$ and $\tilde{T}\tilde{S}$, respectively. Since the genus of $\text{Mp}_1(\mathbb{Z})\backslash\mathbb{H}$ is zero, we furthermore get that $\omega(\text{Mp}_1(\mathbb{Z})\backslash\mathbb{H}) = \pi/3$.

Inserting this into Borcherds' corollary 7.4 [Bor00, p. 335], we obtain the theorem. □

Knowing the dimension a priori is a crucial ingredient in what allows us to compute a basis for $M_k(\rho)$ given a spanning set with a sufficiently nice decomposition and whose elements have known Fourier series expansions.

Another vital ingredient is the property of vector-valued modular forms having a Sturm bound, which tells us that only a finite number of Fourier coefficients are needed to fully characterize a vector-valued modular form.

Proposition 2.9 (Sturm bound)**.** Let k be a positive integer and let ρ be an arithmetic type of level N. Let $F \in M_k(\rho)$. Put $b = \lfloor k/12 \rfloor + 1$. Then if for all $v \in V(\rho)$ and all $n \in \mathbb{Q}$ with $0 \leq n \leq bN - 1$ it holds that

$$c(v^\vee \circ F; n) = 0,$$

we have that F is identically zero.

Proof. This follows from proposition 1.4 of [BW15] by setting $g = 1$. □

Chapter 3

Vector-valued Eisenstein series

The goal of this chapter is to introduce the necessary theory about vector-valued Eisenstein series needed to use the following theorem by Raum and Xià.

Theorem 3. Let k, l and N be positive integers. Then there exists a positive integer N_0 such that

$$M_{k+l}(\Gamma(N)) \subseteq \mathcal{E}_{k+l}(N) + \mathcal{E}_k(N_0) \cdot \mathcal{E}_l(N_0).$$

Here $\mathcal{E}_k(N)$ denotes the space of Fourier series expansions at any cusp of all Eisenstein series of weight k and level N. We omit the precise definition, as we will only need to use their vector-valued counterparts.

Reformulated to the setting of vector-valued modular forms, the theorem instead becomes the following.

Theorem 4. Let k, l and N be positive integers, and let ρ be a congruence type of level N. Then there exists a positive integer N_0 such that

$$M_{k+l}(\rho) = \mathrm{Hom}(\mathbf{1}, \mathcal{E}_k[\rho_N^\times] \otimes \rho) + \mathrm{Hom}(\mathbf{1}, (\mathcal{E}_l[\rho_{N_0}^\times] \cdot \mathcal{E}_k[\rho_{N_0}^\times]) \otimes \rho). \tag{3.1}$$

Let us start by understanding the unfamiliar components in the right-hand side of equation (3.1). Namely the $\mathcal{E}_k[\rho]$ for an arithmetic type ρ.

3.1 Vector-valued Eisenstein series

Let $k \geq 1$ be an integer, and ρ be an arithmetic type. Then by $\mathcal{E}_k[\rho]$ we mean components of a space of vector-valued Eisenstein series, which we denote by $E_k(\rho)$, treated as a representation through the slash action $|_k$.

Given that we understand the space $E_k(\rho)$, it is relatively easy to grasp $\mathcal{E}_k[\rho]$, especially when $\rho = \rho_N^\times$.

In order to have a definition of $E_k(\rho)$ that is valid for all positive integral weights, we need to define the elements as limits of real-analytic Eisenstein series. Specifically, we use the following definition.

Definition 3.1. Let $k \geq 1$ be an integer, and let ρ be an arithmetic type. Then for $v \in V(\rho)$ we define for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $k + 2\mathrm{Re}(s) > 2$, the real-analytic Eisenstein series by

$$E_{k,v}(\tau, s) = \frac{1}{[\Gamma_\infty : \Gamma_\infty(v)]} \sum_{[\gamma] \in \Gamma_\infty(v) \backslash \mathrm{SL}_2(\mathbb{Z})} y^s v|_{k,\rho}\gamma, \tag{3.2}$$

where $y = \mathrm{Im}(\tau)$, as in the introduction $\Gamma_\infty(v) = \Gamma_\infty \cap \mathrm{Stab}(v)$, and $\mathrm{Stab}(v) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \rho(\gamma)v = v\}$. It holds that $E_{k,v}(\tau, s)$ has a meromorphic continuation in s to the whole s-plane, holomorphic at $s = 0$. For s satisfying $k + 2\mathrm{Re}(s) \leq 2$, we define $E_{k,v}(\tau, s)$ as the meromorphic continuation of the series (3.2).

On account of $E_{k,v}(\tau, s)$ being holomorphic at $s = 0$, we define

$$\begin{aligned} E_k(\rho, s) &= \mathrm{span}_{\mathbb{C}}\{E_{k,v} : v \in V(\rho)\}, \text{ and} \\ E_k(\rho) &= M_k(\rho) \cap \lim_{s \to 0} E_k(\rho, s), \end{aligned} \tag{3.3}$$

where the limit is taken for every element.

When $k \geq 3$, the above definition of $E_k(\rho)$ coincides with the definition in the introduction. As we shall see later, the intersection with $M_k(\rho)$ in (3.3) is only necessary when $k = 2$.

Notice that for s satisfying $k + 2\mathrm{Re}(s) > 2$ we have that $E_{k,v}(\tau, s)$ converges absolutely and locally uniformly. From this we see that for fixed s it holds that $E_{k,v}|_{k,\rho}\gamma = E_{k,v}$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and so the elements in $E_k(\rho, s)$ are modular invariant in τ. By analytic continuation, the modular invariance extends to all s for which $E_{k,v}(\tau, s)$ is holomorphic.

In the next section we will show how to find the meromorphic continuation alluded to in definition 3.1, in the special case of $\rho = \rho_N^{\times}$.

As for the space of components, we use the following notation.

Notation 3. Let ρ be an arithmetic type. Then we write

$$\mathcal{E}_k[\rho, s] = \mathrm{span}_{\mathbb{C}}\{f \circ E : f \in V(\rho)^{\vee}, E \in E_k(\rho, s)\},$$

and similarly for $\mathcal{E}_k[\rho]$.

In the case of $v \in V(\rho_N^{\times})$, the expression for $E_{k,v}$ simplifies considerably.

Proposition 3.1. For s with $k + 2\mathrm{Re}(s) > 2$, it holds that

$$\mathcal{E}_k[\rho_N^{\times}, s] = \mathrm{span}_{\mathbb{C}}\{E_{k,N,c,d} : c, d \in \mathbb{Z}/N\mathbb{Z}, \gcd(c, d, N) = 1\},$$

where

$$E_{k,N,c,d}(\tau, s) = \sum_{\substack{(c',d')\in\mathbb{Z}^2 \\ \gcd(c',d')=1 \\ c'\equiv_N c, d'\equiv_N d}} \frac{y^s}{(c\tau + d)^k |c\tau + d|^{2s}}.$$

The proof of proposition 3.1 depends on the following lemma.

Lemma 3.1. The right cosets $\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})$ are in bijection with the set $\{(c, d) \in \mathbb{Z}^2 : \gcd(c, d) = 1; c > 0 \text{ or } (c, d) = (0, 1)\}$. The coset containing a matrix $(a, b; c, d)$ corresponds to $(\mathrm{sgn}(c)c, \mathrm{sgn}(c)d)$ where $\mathrm{sgn}(0) = \mathrm{sgn}(d)$.

Proof. Recall that if $\gcd(c, d) = 1$ then there exists $x_0, y_0 \in \mathbb{Z}$ such that $cx_0 + dy_0 = 1$ and all other solutions (x, y) to $cx + dy = 1$ are on the form $x = x_0 - kd$ and $y = y_0 + kc$ for some $k \in \mathbb{Z}$. Hence we conclude that $\Gamma_\infty^+\gamma_1 = \Gamma_\infty^+\gamma_2$ if and only if $c(\gamma_1) = c(\gamma_2)$ and $d(\gamma_1) = d(\gamma_2)$. Since $-I \in \Gamma_\infty$ we get $\Gamma_\infty\gamma_1 = \Gamma_\infty\gamma_2$ if and only if $c(\gamma_1) = \pm c(\gamma_2)$ and $d(\gamma_1) = \pm d(\gamma_2)$.

The map is given explicitly by $\phi(\Gamma_\infty(a, b; c, d)) = (\mathrm{sgn}(c)c, \mathrm{sgn}(c)d)$. From the above, it is immediate that ϕ is well-defined, injective, and surjective. □

Proof of proposition 3.1. This proof is only a slight generalization of lemma 5.10 in [Xià19].

Let us first show that for every $c, d \in \mathbb{Z}/N\mathbb{Z}$ with $\gcd(c, d, N) = 1$, we have that $E_{k,N,c,d} \in \mathcal{E}_k[\rho_N^{\times}, s]$. To this end, let $v_0 = \mathfrak{e}_{(0,1)} + (-1)^k\mathfrak{e}_{(0,-1)}$, where $\mathfrak{e}_{(c,d)}$ is shorthand notation for

$$\mathfrak{e}_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}.$$

We remind the reader that the choice of a and b does not matter. To save space, we shall use the notation $(a, b; c, d)$ for the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

We have that $\rho_N^{\times}(T)(v_0) = \rho_N^{\times}(-I)(v_0) = v_0$, and so $\Gamma_\infty(v_0) = \Gamma_\infty$. This, together with lemma 3.1 yields that

$$E_{k,v_0}(\tau, s) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c>0 \text{ or } c=0,d=1}} y^s v_0|_{k,\rho_N^{\times}} = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c>0 \text{ or } c=0,d=1}} \frac{y^s(\mathfrak{e}_{(c,d)} + (-1)^k\mathfrak{e}_{(-c,-d)})}{(c\tau + d)^k|c\tau + d|^{2s}}.$$

We next inspect E_{k,v_0}. To this end, fix $0 \leq c', d' < N$ with $\gcd(c', d', N) = 1$ and consider $\mathfrak{e}_{(c',d')}^{\vee} \circ E_{k,v_0}$. We have that

$$\begin{aligned}
&\mathfrak{e}_{(c',d')}^{\vee} \circ E_{k,v_0}(\tau, s) \\
&= \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c>0 \text{ or } c=0,d=1}} \frac{y^s}{(c\tau + d)^k|c\tau + d|^{2s}}\big([(c, d) \equiv_N (c', d')] + (-1)^k[(-c, -d) \equiv_N (c', d')]\big),
\end{aligned}$$

where $(a_1, b_1) \equiv_N (a_2, b_2)$ if and only if $a_1 \equiv_N a_2$ and $b_1 \equiv_N b_2$, and where $[\cdot]$ denotes Iverson's bracket. That is, $[P] = 1$ if the proposition P is true, and $[P] = 0$ otherwise.

Let us first assume that $N \geq 3$. Then the congruence conditions are mutally exclusive, and we obtain

$$\begin{aligned}
\mathfrak{e}^{\vee}_{(c',d')} \circ E_{k,v_0}(\tau, s) &= \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c>0 \text{ or } c=0,d=1 \\ (c,d)\equiv_N(c',d') \text{ or } (-c,-d)\equiv_N(c',d')}} \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}} \\
&= \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c>0 \text{ or } c=0,d=1 \\ (c,d)\equiv_N(c',d')}} \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}} + (-1)^k \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c<0 \text{ or } c=0,d=-1 \\ (c,d)\equiv_N(c',d')}} (-1)^k \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}} \\
&= E_{k,N,c',d'}(\tau, s).
\end{aligned}$$

For $N = 2$ the same result follows by letting $v_0 = \mathfrak{e}_{(0,1)}$, and for $N = 1$ it follows by noting that any two numbers are congruent modulo 1.

By linearity, we now conclude that

$$\operatorname{span}_{\mathbb{C}}\{E_{k,N,c,d} : c, d \in \mathbb{Z}/N\mathbb{Z}, \gcd(c, d, N) = 1\} \subseteq \mathcal{E}_k[\rho_N^{\times}, s].$$

For the other direction, we first apply a technique to eliminate the $\Gamma_\infty(v)$. Essentially we apply the third isomorphism theorem (the "freshman" theorem) in the following sense.

Given $v \in V(\rho_N^{\times})$, then there exists a bijection

$$\Gamma_\infty(v)\backslash \mathrm{SL}_2(\mathbb{Z}) \to (\Gamma_\infty(v)\backslash\Gamma_\infty) \times (\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})).$$

The proof goes by way of constructing maps in both directions and showing that they are inverse to each other. Fix first a set of representatives for $\Gamma_\infty(v)\backslash \mathrm{SL}_2(\mathbb{Z})$, $\Gamma_\infty(v)\backslash\Gamma_\infty$ and $\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})$, call them R_1, R_2 and R_3, respectively. Then define

$$f : R_1 \ni \beta \mapsto (\alpha', \beta') \in R_2 \times R_3,$$

where $\beta' \in R_3$ is the representative of $\Gamma_\infty\beta$, so that $\beta = \alpha\beta'$ for a unique $\alpha \in \Gamma_\infty$, and where $\alpha' \in R_2$ is the representative of $\Gamma_\infty(v)\alpha$. Define similarly

$$g : R_2 \times R_3 \ni (\alpha, \beta) \mapsto \beta' \in R_1,$$

where β' is the representative of $\Gamma_\infty(v)\alpha\beta$. It is straightforward to show that $f \circ g$ and $g \circ f$ are identity maps, and thus the bijection follows.

Since $|_{k,\rho_N^{\times}}$ is an action, we now obtain

$$\begin{aligned}
E_{k,v}(\tau, s) &= \frac{1}{[\Gamma_\infty : \Gamma_\infty(v)]} \sum_{[\gamma]\in\Gamma_\infty(v)} \operatorname{Im}(\tau)^s v|_{k,\rho_N^{\times}}\gamma \\
&= \sum_{[\gamma']\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \Big([\Gamma_\infty : \Gamma_\infty(v)]^{-1} \sum_{[\gamma'']\in\Gamma_\infty(v)\backslash\Gamma_\infty} \operatorname{Im}(\tau)^s v|_{k,\rho_N^{\times}}\gamma''\Big)|_{k,\rho_N^{\times}}\gamma'.
\end{aligned}$$

Notice now that because $c(\pm T^k) = 0$ and $d(\pm T^k) = \pm 1$, we get

$$\operatorname{Im}(\tau)^s v|_{k,\rho_N^{\times}}\gamma'' = \operatorname{Im}(\tau)^s [d(\gamma'')^{-k}\rho_N^{\times}(\gamma'')^{-1}v].$$

Therefore, we define

$$w = [\Gamma_\infty : \Gamma_\infty(v)]^{-1} \sum_{[\gamma'']\in\Gamma_\infty(v)\backslash\Gamma_\infty} d(\gamma'')^{-k}\rho_N^{\times}(\gamma'')^{-1}v,$$

so that

$$E_{k,v}(\tau, s) = \sum_{[\gamma']\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \operatorname{Im}(\tau)^s w|_{k,\rho_N^{\times}}\gamma',$$

and we have eliminated $\Gamma_\infty(v)$.

Let now $v \in V(\rho_N^\times)$ and $(c', d') \in \mathbb{Z}/N\mathbb{Z}$ with $\gcd(c', d', N) = 1$, be arbitrary. Let also w be as above. By linearity, it suffices to show that $\mathfrak{e}^\vee_{(c,d)} \circ E_{k,v}(\tau, s)$ is a linear combination of the $E_{k,N,c',d'}(\tau, s)$.

We obtain

$$E_{k,v}(\tau, s) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c\geq 0 \text{ or } c=0, d=1}} \mathrm{Im}(\tau)^s w|_{k,\rho_N^\times}(a, b; c, d).$$

To simplify this further, notice that

$$\sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \mathrm{Im}(\tau)^s w|_{k,\rho_N^\times}(a, b; c, d) = \Big(\sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c>0}} + \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c<0}} + \sum_{\substack{c=0 \\ d\in\{1,-1\}}} \Big) \mathrm{Im}(\tau)^s w|_{k,\rho_N^\times}(a, b; c, d).$$

Since

$$\begin{aligned} \mathrm{Im}(\tau)^s w|_{k,\rho_N^\times}(-a, -b; -c, -d) &= \frac{\mathrm{Im}(\tau)^s}{|c\tau + d|^{2s}} (w|_{k,\rho_N^\times}(-a, -b; -c, d)) \\ &= \frac{\mathrm{Im}(\tau)^s}{|c\tau + d|^{2s}} (w|_{k,\rho_N^\times}(a, b; c, d)) \\ &= \mathrm{Im}(\tau)^s w|_{k,\rho_N^\times}(a, b; c, d), \end{aligned}$$

we obtain

$$\Big(\sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c>0}} + \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c<0}} + \sum_{\substack{c=0 \\ d\in\{1,-1\}}} \Big) \mathrm{Im}(\tau)^s w|_{k,\rho_N^\times}(a, b; c, d) = 2 \sum_{\substack{(c,d)\in\gcd(c,d)=1 \\ c>0 \text{ or } c=0, d=1}} \mathrm{Im}(\tau)^s w|_{k,\rho_N^\times}(a, b; c, d),$$

which means that

$$E_{k,v}(\tau, s) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{\mathrm{Im}(\tau)^s}{(c\tau + d)^k |c\tau + d|^{2s}} \rho_N^\times((a, b; c, d)^{-1}) w.$$

It is now straightforward to find the components of this sum. Indeed, write

$$w = \sum_{\substack{(e,f)\in(\mathbb{Z}/N\mathbb{Z})^2 \\ \gcd(e,f,N)=1}} c_{(e,f)} \mathfrak{e}_{(e,f)},$$

then

$$\rho_N^\times((a, b; c, d)^{-1})(w) = \sum_{\substack{(e,f)\in(\mathbb{Z}/N\mathbb{Z})^2 \\ \gcd(e,f,N)=1}} c_{(e,f)(a,b;c,d)^{-1}} \mathfrak{e}_{(e,f)},$$

so that

$$\mathfrak{e}^\vee_{(c',d')}(\rho_N^\times((a, b; c, d)^{-1})(w)) = c_{(c',d')(a,b;c,d)^{-1}}.$$

Since $\rho_N^\times(\gamma) w = w$ for $\gamma \in \Gamma_1(N)$, we get that $c_{(c',d')(*,*;c_1,d_1)^{-1}} = c_{(c',d')(*,*;c_2,d_2)^{-1}}$ whenever $(c_1, d_1) \equiv_N (c_2, d_2)$ and $\gcd(c_i, d_i, N) = 1$. Therefore we have that

$$\mathfrak{e}^\vee_{(c',d')} \circ E_{k,v}(\tau, s) = \frac{1}{2} \sum_{\substack{(e,f)\in(\mathbb{Z}/N\mathbb{Z})^2 \\ \gcd(e,f,N)=1}} c_{(c',d')(*,*;e,f)^{-1}} E_{k,N,e,f}(\tau, s),$$

whence we are done. □

To ease further discussion, we define a variant $G_{k,N,c,d}(\tau, s)$ of $E_{k,N,c,d}(\tau, s)$ that does not require c and d to be coprime.

Proposition 3.2. Let k and N be positive integers. Let $c', d' \in \mathbb{Z}/N\mathbb{Z}$ satisfy $\gcd(c', d', N) = 1$. Define for s with $k + 2\mathrm{Re}(s) > 2$ the series

$$G_{k,N,c',d'}(\tau, s) = \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (c,d)\equiv_N(c',d')}} \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}}.$$

Then we have for $k + 2\mathrm{Re}(s) > 2$ that

$$\begin{aligned}&\mathrm{span}\{E_{k,N,c,d} : c, d \in \mathbb{Z}/N\mathbb{Z}, \gcd(c, d, N) = 1\} \\ &= \mathrm{span}\{G_{k,N,c,d} : c, d \in \mathbb{Z}/N\mathbb{Z}, \gcd(c, d, N) = 1\}.\end{aligned}$$

Proof. Recall that $\mu * \mathbf{1} = e$ where $e(n) = \delta_{1,n}$ for positive integers n, and where $*$ denotes Dirichlet convolution. Inserting this into $E_{k,N,c',d'}(\tau, s)$ yields

$$\begin{aligned}E_{k,N,c',d'}(\tau, s) &= \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\equiv_N(c',d')}} \sum_{\substack{n|c \\ n|d}} \mu(n) \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}} \\ &= \sum_{\substack{n\geq 1 \\ \gcd(n,N)=1}} \frac{\mu(n)}{n^{k+2s}} \sum_{\substack{(k,l)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (nk,nl)\equiv_N(c',d')}} \frac{y^s}{(k\tau+l)^k|k\tau+l|^{2s}} \\ &= \sum_{\substack{n\geq 1 \\ \gcd(n,N)=1}} \frac{\mu(n)}{n^{k+2s}} G_{k,N,n^{-1}c',n^{-1}d'}(\tau, s) \\ &= \sum_{\substack{1\leq m\leq N \\ \gcd(m,N)=1}} G_{k,N,m^{-1}c',m^{-1}d'}(\tau, s) \sum_{\substack{n\geq 1 \\ n\equiv_N m}} \frac{\mu(n)}{n^{k+2s}}.\end{aligned}$$

We note that we could deduce that $\gcd(n, N) = 1$ from n being a common divisor of c and d, and that $\gcd(c, d, N) = \gcd(c', d', N) = 1$.

We now conclude that for $k + 2\mathrm{Re}(s) > 2$ it holds that

$$\mathfrak{e}^\vee_{(c',d')} \circ E_{k,v_0}(\tau, s) \in \mathrm{span}_{\mathbb{C}}\{G_{k,N,c,d}(\tau, s) : c, d \in \mathbb{Z}/N\mathbb{Z}, \gcd(c, d, N) = 1\},$$

and so we are done with the first inclusion.

As for the second inclusion, we have that

$$\begin{aligned}G_{k,N,c',d'}(\tau, s) &= \sum_{\substack{n\geq 1 \\ \gcd(n,N)=1}} \sum_{\substack{(k_1,k_2)\in\mathbb{Z}^2 \\ \gcd(k_1,k_2)=1 \\ n(k,l)\equiv_N(c',d')}} \frac{y^s}{(nk_1\tau+nk_2)^k|nk_1\tau+nk_2|^{2s}} \\ &= \sum_{\substack{n\geq 1 \\ \gcd(n,N)=1}} \frac{1}{n^{k+2s}} E_{k,N,n^{-1}c',n^{-1}d'}(\tau, s) \\ &= \sum_{\substack{1\leq m\leq N \\ \gcd(m,N)=1}} E_{k,N,m^{-1}c',m^{-1}d'}(\tau, s) \sum_{\substack{n\geq 1 \\ n\equiv_N m}} \frac{1}{n^{k+2s}}.\end{aligned}$$

We have that $\sum_{n\geq 1, n\equiv_N m} n^{-k-2s} = \zeta(k + 2s, m/N)N^{-k-2s}$, where ζ is the Hurwitz ζ-function. Note that this function has a simple pole of order 1 at $k + 2s = 1$, and is holomorphic everywhere else. □

We now endow $\mathcal{E}_k[\rho_N^\times, s]$ with the structure of a representation through the slash action.

Proposition 3.3. Let ρ be an arithmetic type. Then $|_k$ acts on $\mathcal{E}_k[\rho, s]$ and

$$(\mathcal{E}_k[\rho, s], \mathrm{SL}_2(\mathbb{Z}) \ni \gamma \mapsto (f \mapsto f|_k\gamma^{-1})),$$

is a linear representation.

Proof. Follows since the slash action is linear. □

Proposition 3.1 and 3.2 show that finding a meromorphic continuation in s of the $E_{k,v}(\tau, s)$ for $v \in V(\rho_N^\times)$ is equivalent to finding a meromorphic continuation of $G_{k,N,c',d'}(\tau, s)$ in s, for any $c, d \in \mathbb{Z}/N\mathbb{Z}$ with $\gcd(c, d, N) = 1$. As expected, it is a fact that such a meromorphic continuation exists.

Theorem 5. Let k and N be positive integers, and let $c, d \in \mathbb{Z}/N\mathbb{Z}$. Then the series defining $G_{k,N,c,d}(\tau, s)$ has a meromorphic continuation in s to the whole s-plane, holomorphic at $s = 0$.

On account of this theorem, we extend our definition of $G_{k,N,c,d}(\tau, s)$ and $E_{k,N,c,d}(\tau, s)$ to all $s \in \mathbb{C}$.

Definition 3.2. Let k, N, c, and d be as in theorem 5. Then for s satisfying $k + 2\mathrm{Re}(s) \leq 2$, we define $G_{k,N,c,d}(\tau, s)$ and $E_{k,N,c,d}(\tau, s)$ by the meromorphic continuation alluded to in theorem 5.

When $G_{k,N,c,d}(\tau, s)$ is holomorphic at $s = 0$, we define the function

$$G_{k,N,c,d}(\tau) = \lim_{s \to 0} G_{k,N,c,d}(\tau, s),$$

and likewise for $E_{k,N,c,d}(\tau)$.

In the next section we will show how theorem 5 follows from work of Shimura, retold in [Miy89, Chapter 7], and simultaneously find a Fourier series expansion of $G_{k,N,c,d}(\tau, s)$ in τ.

It is important to note that for s satisfying that the elements in $E_k[\rho, s]$ are holomorphic in s, we have a surjection $(\rho_N^\times)^\vee \twoheadrightarrow \mathcal{E}_k[\rho_N^\times, s]$.

Proposition 3.4. Let k be a positive integer and s be a complex number at which the $E_{k,N,c,d}(\tau, s)$ are holomorphic s. Let also

$$E_{k,N}(\tau, s) = \sum_{(c,d) \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} E_{k,N,c,d}(\tau, s) \mathfrak{e}_{(c,d)}.$$

Then

$$\phi : (\rho_N^\times)^\vee \ni f \mapsto f \circ E_{k,N} \ni \mathcal{E}_k[\rho_N^\times, s],$$

is a surjection.

Proof. For an arbitrary $(c, d) \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})$, we have that

$$\phi(\mathfrak{e}_{(c,d)}^\vee) = E_{k,N,c,d}(\tau, s).$$

Since ϕ is linear, the result follows. □

3.2 From congruences to characters

Let L and M be positive integers, and let χ and ψ be Dirichlet characters modulo L and modulo M, respectively. Following Miyake we define for s satisfying $k + 2\mathrm{Re}(s) > 2$ the series

$$E_k(\tau, s; \chi, \psi) = \sum_{(m,n) \in \mathbb{Z}^2 \backslash \{(0,0)\}} \frac{\chi(m)\psi(n)}{(m\tau + n)^k |m\tau + n|^{2s}}.$$

Using the M-test, we see that $E_k(\tau, s; \chi, \psi)$ converges absolutely and locally uniformly for s satisfying $k + 2\mathrm{Re}(s) > 2$ and thus defines a holomorphic function in that half-plane.

Miyake shows in [Miy89, Chapter 7] how to analytically continue $E_k(\tau, s; \chi, \psi)$ to a meromorphic function in s, holomorphic at $s = 0$. He also provides the Fourier series expansion. Hence, all we need to do is relate our series $G_{k,N,c',d'}(\tau, s)$ to Miyake's $E_k(\tau, s; \chi, \psi)$.

On [Miy89, p. 289], Miyake defines a series which is nearly identical to our $G_{k,N,c',d'}(\tau, s)$. For $k + 2\mathrm{Re}(s) > 2$, and integers $0 \leq \mu, \nu < N$ he gives the definition

$$E_k(\tau, s; \mu, \nu; N) = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \backslash \{(0,0)\} \\ (m,n) \equiv_N (\mu,\nu)}} \frac{1}{(m\tau + n)^k |m\tau + n|^{2s}}.$$

Clearly we have, for $k + 2\mathrm{Re}(s) > 2$, that $G_{k,N,c',d'}(\tau, s) = y^s E_k(\tau, s; c', d'; N)$, where we identify a class $a \in \mathbb{Z}/N\mathbb{Z}$ with its unique representative $0 \leq a' < N$. Clearly, if we have a meromorphic continuation of $E_k(\tau, s; c', d'; N)$, holomorphic at $s = 0$, it holds that $G_{k,N,c',d'}(\tau) = \lim_{s\to 0} E_k(\tau, s; c', d'; N)$. For brevity, we shall henceforth write $E_k(\tau; c', d'; N)$ for $\lim_{s\to 0} E_k(\tau; s; c', d'; N)$, when the limit exists.

The answer to the question on how to relate $G_{k,N,c',d'}(\tau, s)$ to $E_k(\tau, s; \chi, \psi)$ is thus given by [Miy89, Theorem 7.2.15], which shows how $E_k(\tau, s; c', d'; N)$ relates to $E_k(\tau, s; \chi, \psi)$. The proof of the theorem omits details that are necessary for our exposition, and thus we recreate the proof below.

Proposition 3.5. Let k and N be positive integers. Let

$$A = \{(\chi, \psi, u, v) : \chi \in D(L), \psi \in D(M), 0 < uL \mid N, 0 < vM \mid N \text{ for some } L \mid N \text{ and } M \mid N\},$$

where for R a positive integer $D(R)$ denotes the set of Dirichlet characters modulo R.

Then

$$\mathrm{span}_{\mathbb{C}}\{E_k(\tau, s; \mu, \nu; N) : 0 \leq \mu, \nu < N\} = \mathrm{span}_{\mathbb{C}}\{E_k(\frac{u}{v}\tau, s; \chi, \psi) : (\chi, \psi, \mu, \nu) \in A\},$$

for $k + 2\mathrm{Re}(s) > 2$.

Proof. We first show the right-hand side is included in the left-hand side. It follows by a clever arrangement of the summands. Let $(\chi, \psi, u, v) \in A$, we then have that

$$\begin{aligned}
E_k(\frac{u}{v}\tau, s; \chi, \psi) &= \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\chi(m)\psi(n)}{(muv^{-1}\tau + n)^k |muv^{-1}\tau + n|^{2s}} \\
&= v^{k+2s} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\chi(m)\psi(n)}{(mu\tau + nv)^k |mu\tau + nv|^{2s}} \\
&= v^{k+2s} \sum_{0\leq\mu,\nu<N} \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (m,n)\equiv_N(\mu,\nu) \\ u\mid m \\ v\mid n}} \frac{\chi(m/u)\psi(n/v)}{(m\tau + n)^k |m\tau + n|^{2s}} \qquad (3.4) \\
&= v^{k+2s} \sum_{\substack{0\leq\mu,\nu<N \\ u\mid\mu \\ v\mid\nu}} \chi(\mu/u)\psi(\nu/v) \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (m,n)\equiv_N(\mu,\nu)}} \frac{1}{(m\tau + n)^k |m\tau + n|^{2s}} \qquad (3.5) \\
&= v^{k+2s} \sum_{\substack{0\leq\mu,\nu<N \\ u\mid\mu \\ v\mid\nu}} \chi(\mu/u)\psi(\nu/v) E_k(\tau, s; \mu, \nu; N).
\end{aligned}$$

Step (3.4) to (3.5) follows from the fact that given that $m \equiv_N \mu$ then $u \mid m$ if and only if $u \mid \mu$, and likewise for n, ν, and v.

Indeed, say $u \mid m$, then $m = um'$ but also $m = Nk + \mu$ for some integer k and therefore $\mu = um' - Nk$ so that $\mu/v = m' - Nk/u$. However, it holds that $uL \mid N$ so $N = uLk'$ for some integer k', and thus $\mu/v = m' - Nkk'$. The other direction is almost identical, and of course the same argument works for n, ν, and v.

From the above argument we also see that $m/u \equiv_L \mu/u$ and $n/v \equiv_L \nu/v$, which is what allows to factor out the character product to the outermost sum.

We now show that the left-hand side is included in the right-hand side. Let first $0 \leq \mu, \nu < N$ and define $u = \gcd(\mu, N)$ and $v = \gcd(\nu, N)$. Define also $\mu' = \mu/u$, $\nu' = \nu/v$, $L = N/u$, and $M = N/v$. Recall now that

$$\sum_{\chi\in D(L)} \overline{\chi(\mu')}\chi(m) = \phi(L)[m \equiv_L \mu'].$$

We remind the reader that Iverson's bracket is defined by $[P] = 1$ if P is true and $[P] = 0$ otherwise. Consequently it holds that

$$\sum_{\chi\in D(L),\psi\in D(M)} \overline{\chi(\mu')\psi(\nu')}\chi(m)\psi(n) = \phi(L)\phi(M)[m \equiv_L \mu' \text{ and } n \equiv_M \nu'].$$

We now obtain

$$E_k(\tau,s;\mu,\nu;N) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}\\(m,n)\equiv_N(\mu,\nu)}} \frac{1}{(m\tau+n)^k|m\tau+n|^{2s}} \tag{3.6}$$

$$= \sum_{(m',n')\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{[m'\equiv_L\mu'][n'\equiv_M\nu']}{(m'u\tau+n'v)^k|m'u\tau+n'v|^{2s}} \tag{3.7}$$

$$= \frac{1}{\phi(L)\phi(M)} \sum_{(m',n')\in\mathbb{Z}^2\setminus\{(0,0)\}} \sum_{\substack{\chi\in D(L)\\\psi\in D(M)}} \frac{\overline{\chi(\mu')\psi(\nu')}\chi(m')\psi(n')}{(m'u\tau+n'v)^k|m'u\tau+n'v|^{2s}}$$

$$= \frac{v^{-k-2s}}{\phi(L)\phi(M)} \sum_{\substack{\chi\in D(L)\\\psi\in D(M)}} \overline{\chi(\mu')\psi(\nu')} \sum_{(m',n')\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\chi(m')\psi(n')}{(m'uv^{-1}\tau+n')^k|m'uv^{-1}\tau+n'|^{2s}}$$

$$= \frac{v^{-k-2s}}{\phi(L)\phi(M)} \sum_{\substack{\chi\in D(L)\\\psi\in D(M)}} \overline{\chi(\mu')\psi(\nu')} E_k(\frac{u}{v}\tau,s;\chi,\psi).$$

Step (3.6) to (3.7) follows, analogous to the argument from the inclusion of the right-hand side into the left-hand side, from $m\equiv_N\mu$ implying that $u\mid m$ and $m'=m/u\equiv_L\mu/u=\mu'$, and vice versa. □

Notice now that $E_k(\tau,s;\chi,\psi)$ is entire in s when $k\geq 3$, and in that case no analytic continuation is necessary. Miyake provides through [Miy89, Theorem 7.1.3] the Fourier series expansion of $E_k(\tau;\chi,\psi)=\lim_{s\to 0}E_k(\tau,s;\chi,\psi)$ for $\chi\in D(L)$ and $\psi\in D(M)$ satisfying $\chi(-1)\psi(-1)=(-1)^k$. The latter is no restriction because if $\chi(-1)\psi(-1)\neq(-1)^k$ then $E_k(\tau,s;\chi,\psi)=0$, see [Miy89, p. 269].

When $k\leq 2$, Miyake provides through [Miy89, Theorem 7.2.9] a meromorphic continuation of $E_k(\tau,s;\chi,\psi)$ for $\chi\in D(L)$ and $\psi\in D(M)$ satisfying $\chi(-1)\psi(-1)=(-1)^k$ to the whole s-plane, together with its Fourier series expansion. Miyake shows with [Miy89, Corollary 7.2.10] that the meromorphic continuation is holomorphic at $s=0$.

Proposition 3.5 therefore gives us a meromorphic continuation of $E_k(\tau,s;c',d';N)$, holomorphic at $s=0$, and thus a proof of theorem 5. After some work, it also gives us Fourier series expansions of $G_{k,N,c',d'}(\tau,s)$ for any positive integer weight k.

When $k\neq 2$, it holds that $E_k(\tau;\chi,\psi)$ is a modular form, and when $k=2$ it is potentially a non-holomorphic function. In the following sections we give all the Fourier series expansions, and show how to recover holomorphicity in the case of $k=2$.

3.3 Fourier series expansion

For any Dirichlet character χ, we let m_χ be its conductor, and χ^0 be the primitive character inducing χ. We let

$$W(\chi)=\sum_{a=0}^{m_\chi-1}\chi(a)e(a/m_\chi),$$

be the Gauß sum of χ. We also let $L(s,\chi)$ be the Dirichlet L-function associated to χ. We recall that for R a positive integer $D(R)$ denotes the set of Dirichlet characters modulo R.

We now restate Miyake's results [Miy89, Theorem 7.1.3, 7.2.9, Corollary 7.2.10, Corollary 7.2.14] on the meromorphic continuation of $E_k(\tau,s;\chi,\psi)$ and the Fourier series expansions of $E_k(\tau;\chi,\psi)$ for $\chi\in D(L)$ and $\psi\in D(M)$ satisfying $\chi(-1)\psi(-1)=(-1)^k$, and then specialize the Fourier series expansions to the case of $G_{k,N,c,d}(\tau)$.

Theorem 6 (Miyake [Miy89], following Shimura [Shi82; Shi83; Shi85a; Shi85b])**.** Let k be a positive integer. Let L and M be positive integers. Let $\chi\in D(L)$ and $\psi\in D(M)$ be Dirichlet characters satisfying $\chi(-1)\psi(-1)=(-1)^k$. Then $E_k(\tau,s;\chi,\psi)$ has a meromorphic continuation in s, holomorphic at $s=0$.

In particular, for $k \geq 3$ it holds that $E_k(\tau; \chi, \psi)$ has the following Fourier series expansion

$$E_k(\tau; \chi, \psi) = C + A \sum_{n \geq 1} a_n e(\frac{n\tau}{M}),$$

where

$$\begin{aligned} A &= \frac{2(-2\pi i)^k W(\psi^0)}{M^k (k-1)!}, \\ C &= 2L(k, \psi)[\chi = \chi_0], \\ a_n &= \sum_{c|n} \chi(n/c) c^{k-1} \sum_{d | \gcd(l,c)} d\mu(l/d) \psi^0(l/d) \overline{\psi^0(c/d)}, \end{aligned}$$

and where χ_0 denotes the principal character[1], and where $l = M/m_\psi$.

For $k = 2$ it holds that

$$E_2(\tau; \chi, \psi) = C + \frac{D}{y} + A \sum_{n \geq 1} a_n e(\frac{n\tau}{M}),$$

where A, C, and a_n are as above, and where

$$D = \Big(- \pi \prod_{p|M} (1 - p^{-1}) \prod_{p|L} (1 - p^{-1}) \Big) [\chi = \mathbf{1}_L \text{ and } \psi = \mathbf{1}_M],$$

where for R a positive integer $\mathbf{1}_R$ denotes the trivial character modulo R, and where p denotes prime numbers.

For $k = 1$ it holds that

$$E_1(\tau; \chi, \psi) = C + D + A \sum_{n \geq 1} a_n e(\frac{n\tau}{M}),$$

where A, C, and a_n are as above, and where

$$D = \Big(- 2\pi i L(0, \chi) \prod_{p|M} (1 - p^{-1}) \Big) [\psi = \mathbf{1}_M].$$

In all cases, the series $\sum_{n \geq 1} a_n e(\frac{n\tau}{M})$ converges absolutely and locally uniformly for $\tau \in \mathbb{H}$.

Since $1/\mathrm{Im}(\tau)$ is not holomorphic, we see that $E_2(\tau; \mathbf{1}_L, \mathbf{1}_M)$ is not holomorphic. As we shall see in the next section, $E_2(\tau; \mathbf{1}_L, \mathbf{1}_M) - D/y$ is in fact a so-called quasi-modular form of depth 1, see [Bru+08, p. 58].

We now sum over pairs of characters to obtain Fourier series expansions of $G_{k,N,c,d}(\tau)$, or equivalently of $E_k(\tau; c, d; N)$.

Theorem 7. Let k and N be positive integers and let $0 \leq \mu, \nu < N$. Let also $u = \gcd(\mu, N)$, $v = \gcd(\nu, N)$, $\mu' = \mu/u$, $\nu' = \nu/v$, $L = N/u$, and $M = N/v$.

Then for $k \geq 3$ it holds that

$$G_{k,N,\mu,\nu}(\tau) = C + \sum_{n \geq 1} a_n e(\frac{n\tau}{L}),$$

where

$$C = \frac{[L = 1]}{N^k} \times \begin{cases} \zeta(k, \frac{\nu'}{M}) + (-1)^k \zeta(k, \frac{M - \nu'}{M}) & \text{if } \nu' > 0 \\ \zeta(k)(1 + (-1)^k) & \text{if } \nu' = 0. \end{cases}$$

and where

$$a_n = \frac{(-2\pi i)^k}{N^k (k-1)!} \sum_{c|n} c^{k-1} ([\frac{n}{c} \equiv_L \mu'] e(\frac{\nu' c}{M}) + (-1)^k [\frac{n}{c} \equiv_L -\mu'] e(\frac{-\nu' c}{M})).$$

[1] In this section, we use Miyake's terminology, in which the principal character is the unique element of $D(1)$.

For $k = 2$ it holds that

$$G_{2,N,\mu,\nu}(\tau) = C + \frac{D}{y} + \sum_{n\geq 1} a_n e(\frac{n\tau}{L}),$$

where C and a_n are as above, and where

$$D = -\frac{\pi}{v^2 LM}.$$

For $k = 1$ it holds that

$$G_{1,N,\mu,\nu}(\tau) = C + D + \sum_{n\geq 1} a_n e(\frac{n\tau}{L}),$$

where a_n is as above, and where

$$C = \frac{[L = 1 \text{ and } \nu' > 0]}{N}\Big(\frac{\Gamma'}{\Gamma}(\frac{M-\nu'}{M}) - \frac{\Gamma'}{\Gamma}(\frac{\nu'}{M})\Big),$$

and

$$D = \frac{-2\pi i}{N}\frac{L - 2\mu'}{L}[\mu' > 0].$$

In all cases, it holds that $\sum_{n\geq 1} a_n e(\frac{n\tau}{L})$ converges absolutely and locally uniformly for $\tau \in \mathbb{H}$.

Proof. From the proof of proposition 3.5 we obtain

$$E_k(\tau;\mu,\nu;N) = \frac{v^{-k}}{\phi(L)\phi(M)} \sum_{\chi\in D(L),\psi\in D(M)} \overline{\chi(\mu')\psi(\nu')} E_k(\frac{u}{v}\tau;\chi,\psi).$$

Since the sum is finite, the above equality shows immediately that the last statement of theorem is true.

When $k \geq 3$, we obtain

$$E_k(\tau;\mu,\nu;N) = \frac{v^{-k}}{\phi(L)\phi(M)} \sum_{\substack{\chi\in D(L),\psi\in D(M)\\ \chi\psi(-1)=(-1)^k}} \overline{\chi(\mu')\psi(\nu')}\Big(C_{\chi,\psi} + \sum_{n\geq 1} A_\psi a_n^{\chi,\psi} e(\frac{n\tau}{L})\Big), \tag{3.8}$$

where $C_{\chi,\psi}$, A_ψ, and $a_n^{\chi,\psi}$ are the same coefficients as in theorem 6 but with the dependence on the characters χ and ψ emphasized. We also used the fact that $u\tau/(vM) = \tau/(vM/u) = \tau/L$.

We focus first on the sum of the terms $C_{\chi,\psi}$, which we see corresponds to the constant term in the Fourier series expansion.

$$\begin{aligned}\sum_{\substack{\chi\in D(L),\psi\in D(M)\\ \chi\psi(-1)=(-1)^k}} \overline{\chi(\mu')\psi(\nu')} 2L(k,\psi)[\chi = \chi_0] &= [L=1] \sum_{\substack{\psi\in D(M)\\ \psi(-1)=(-1)^k}} \overline{\psi(\nu')} 2L(k,\psi)\\ &= [L=1] \sum_{\psi\in D(M)} \overline{\psi(\nu')}(1 + (-1)^k\psi(-1))L(k,\psi).\end{aligned}$$

The last sum equals

$$\sum_{\psi\in D(M)} \overline{\psi(\nu')} L(k,\psi) + (-1)^k \sum_{\psi\in D(M)} \overline{\psi(-\nu')} L(k,\psi).$$

Since $k \geq 3 > 1$, the L-functions converge absolutely and we can move in the sums over the characters to obtain for any integer a coprime with M

$$\sum_{\psi\in D(M)} \overline{\psi(a)} L(k,\psi) = \sum_{n\geq 1} \frac{\phi(M)[n \equiv_M a]}{n^k} = \frac{\phi(M)}{M^k} \times \begin{cases} \zeta(k, \frac{a}{M}) & \text{if } a > 0\\ \zeta(k) & \text{if } a = 0\\ \zeta(k, \frac{M-a}{M}) & \text{if } a < 0. \end{cases}$$

Hence we obtain for the constant term of the Fourier series expansion

$$C = \frac{[L=1]}{N^k}\Big(\zeta(k, \frac{\nu'}{M}) + (-1)^k \zeta(k, \frac{M-\nu'}{M})\Big),$$

when $\nu' > 0$ and

$$C = \frac{[L=1]}{N^k}\zeta(k)(1+(-1)^k),$$

when $\nu' = 0$.

We now consider the sum of the terms $\sum_{n\geq 1} A_\psi a_n^{\chi,\psi} e(n\tau/L)$. By absolute convergence, we can move in the sum over the characters and thus we consider the sum

$$\sum_{\substack{\chi\in D(L),\psi\in D(M)\\ \chi\psi(-1)=(-1)^k}} \overline{\chi(\mu')\psi(\nu')} A_\psi a_n^{\chi,\psi}. \tag{3.9}$$

Similar to what we used above, we observe that

$$\frac{1}{2}(1+(-1)^k\chi\psi(-1)) = [\chi\psi(-1) = (-1)^k].$$

Therefore, we have that (3.9) equals

$$\frac{1}{2}\sum_{\chi\in D(L),\psi\in D(M)} \overline{\chi(\mu')\psi(\nu')} A_\psi a_n^{\chi,\psi} + \frac{(-1)^k}{2}\sum_{\chi\in D(L),\psi\in D(M)} \overline{\chi(-\mu')\psi(-\nu')} A_\psi a_n^{\chi,\psi}\Big).$$

Let us consider the first of these sums. It is equal to

$$\frac{(-2\pi i)^k}{M^k(k-1)!}\sum_{\psi\in D(M)} \overline{\psi(\nu')} W(\psi^0) \sum_{\chi\in D(L)} \overline{\chi(\mu')} \sum_{c|n} \chi(\frac{n}{c}) c^{k-1} \sum_{d|\gcd(l,c)} d\mu(\frac{l}{d})\psi^0(\frac{l}{d})\overline{\psi^0(\frac{c}{d})}. \tag{3.10}$$

We interchange the sum over characters $\chi \in D(L)$ and the sum over divisors $c \mid n$ to obtain that (3.10) equals

$$\phi(L)\frac{(-2\pi i)^k}{M^k(k-1)!}\sum_{\psi\in D(M)} \overline{\psi(\nu')} W(\psi^0) \sum_{c|n} [\frac{n}{c} \equiv_L \mu'] c^{k-1} \sum_{d|\gcd(l,c)} d\mu(\frac{l}{d})\psi^0(\frac{l}{d})\overline{\psi^0(\frac{c}{d})}. \tag{3.11}$$

We factor in the Gauß sum and move the sum over characters $\psi \in D(M)$ inside the sum over divisors $c \mid n$ to obtain that (3.11) equals

$$\phi(L)\frac{(-2\pi i)^k}{M^k(k-1)!}\sum_{c|n} [\frac{n}{c} \equiv_L \mu'] c^{k-1} \sum_{\psi\in D(M)} \overline{\psi(\nu')} W(\psi^0) \sum_{d|\gcd(l,c)} d\mu(\frac{l}{d})\psi^0(\frac{l}{d})\overline{\psi^0(\frac{c}{d})}. \tag{3.12}$$

By [Miy89, Lemma 3.1.3], we have that

$$W(\psi^0)\sum_{d|\gcd(l,c)} d\mu(\frac{l}{d})\psi^0(\frac{l}{d})\overline{\psi^0(\frac{c}{d})} = \sum_{a=0}^{M-1}\psi(a)e(\frac{ac}{M}).$$

Whence we obtain

$$\sum_{\psi\in D(M)} \overline{\psi(\nu')} W(\psi^0) \sum_{d|\gcd(l,c)} d\mu(\frac{l}{d})\psi^0(\frac{l}{d})\overline{\psi^0(\frac{c}{d})} = \phi(M)\sum_{a=0}^{M-1}[a \equiv_M \nu'] e(\frac{ac}{M}).$$

We now notice that since $0 \leq \nu' < M$, we obtain

$$\sum_{a=0}^{M-1}[a \equiv_M \nu'] e(\frac{ac}{M}) = e(\frac{\nu' c}{M}).$$

Hence (3.12) is equal to

$$\phi(L)\phi(M)\frac{(-2\pi i)^k}{M^k(k-1)!}\sum_{c|n}[\frac{n}{c} \equiv_L \mu'] c^{k-1} e(\frac{\nu' c}{M}).$$

From this we obtain that (3.9) is equal to

$$\phi(L)\phi(M)\frac{(-2\pi i)^k}{M^k(k-1)!}\sum_{c|n} c^{k-1}([\frac{n}{c}\equiv_L \mu']e(\frac{\nu' c}{M})+(-1)^k[\frac{n}{c}\equiv_L -\mu']e(\frac{-\nu' c}{M})).$$

Therefore we can conclude that the positive Fourier indices are equal to

$$\begin{aligned} a_n &= \frac{v^{-k}}{\phi(L)\phi(M)}\sum_{\substack{\chi\in D(L),\psi\in D(M)\\ \chi\psi(-1)=(-1)^k}} \overline{\chi(\mu')\psi(\nu')}A_\psi a_n^{\chi,\psi} \\ &= \frac{(-2\pi i)^k}{N^k(k-1)!}\sum_{c|n} c^{k-1}([\frac{n}{c}\equiv_L \mu']e(\frac{\nu' c}{M})+(-1)^k[\frac{n}{c}\equiv_L -\mu']e(\frac{-\nu' c}{M})). \end{aligned}$$

Let us now consider the case of $k=2$. Since the L-function value $L(2,\psi)$ is still given by the series which converges absolutely for $k=2$, and since the Fourier series converges absolutely and locally uniformly, we see that C and a_n remains unchanged, as claimed. Let $D_{\chi,\psi}$ be the same coefficient as in theorem 6, with the dependence on χ and ψ emphasized. Then we obtain that the coefficient of $1/y$ in the Fourier series expansion of $G_{k,N,\mu,\nu}(\tau)$ is equal to

$$D=\frac{v^{-2}}{\phi(L)\phi(M)}\sum_{\substack{\chi\in D(L),\psi\in D(M)\\ \chi\psi(-1)=1}} \overline{\chi(\mu')\psi(\nu')}D_{\chi,\psi}.$$

But $D_{\chi,\psi}$ is non-zero only when χ and ψ are trivial. Hence we obtain

$$D=\frac{v^{-2}}{\phi(L)\phi(M)}\overline{\mathbf{1}_L(\mu')\mathbf{1}_M(\nu')}\Big(-\pi\prod_{p|M}(1-p^{-1})\prod_{p|L}(1-p^{-1})\Big).$$

However, for R a positive integer it holds $\prod_{p|R}(1-p^{-1})=\phi(R)/R$. Therefore we obtain

$$D=-\frac{\pi}{v^2LM}.$$

Let us now consider the case of $k=1$. Just as for $k=2$, the Fourier coefficients a_n remain unchanged, but the L-function value $L(1,\psi)$ is no longer given by the series, and instead by analytic continuation. This requires us to change our approach slightly.

Let $C_{\chi,\psi}$ be given as in the theorem. Just as before we obtain for the constant term C

$$C=\frac{v^{-1}}{\psi(L)\phi(M)}\sum_{\substack{\chi\in D(L),\psi\in D(M)\\ \chi\psi(-1)=-1}} \overline{\chi(\mu')\psi(\nu')}C_{\chi,\psi}=\frac{v^{-1}[L=1]}{\phi(M)}\sum_{\substack{\psi\in D(M)\\ \psi(-1)=-1}} \overline{\psi(\nu')}2L(1,\psi).$$

Notice that if $\psi(-1)=-1$, we have that ψ is a non-trivial character, and then $L(s,\psi)$ is non-zero and finite at $s=1$.

We now obtain

$$\begin{aligned} C &= \frac{v^{-1}[L=1]}{\phi(M)}\lim_{s\to 1^+}\Big(\sum_{\psi\in D(M)}\overline{\psi(\nu')}L(s,\psi)-\sum_{\psi\in D(M)}\overline{\psi(-\nu')}L(s,\psi)\Big) \\ &= \frac{[L=1]}{N}\times\begin{cases}0 & \text{if } \nu'=0\\ \lim_{s\to1^+}\Big(\zeta(s,\frac{\nu'}{M})-\zeta(s,\frac{M-\nu'}{M})\Big) & \text{if } \nu'>0\end{cases} \\ &= \frac{[L=1 \text{ and } \nu'>0]}{N}\Big(\frac{\Gamma'}{\Gamma}(\frac{M-\nu'}{M})-\frac{\Gamma'}{\Gamma}(\frac{\nu'}{M})\Big). \end{aligned}$$

where the limit is taken along the real axis from the right.

As for D, we let $D_{\chi,\psi}$ be given as in the theorem. We then obtain

$$\begin{aligned} D &= \frac{v^{-1}}{\phi(L)\phi(M)} \sum_{\substack{\chi \in D(L), \psi \in D(M) \\ \chi\psi(-1)=-1}} \overline{\chi(\mu')\psi(\nu')} D_{\chi,\psi} \\ &= \frac{-2\pi i}{\phi(L)N} \sum_{\substack{\chi \in D(L) \\ \chi(-1)=-1}} \overline{\chi(\mu')} L(0,\chi) \\ &= \frac{-2\pi i}{\phi(L)N} \Big(\sum_{\chi \in D(L)} \overline{\chi(\mu')} L(0,\chi) - \sum_{\chi \in D(L)} \overline{\chi(-\mu')} L(0,\chi) \Big). \end{aligned} \tag{3.13}$$

We see that if $\mu' = 0$, then (3.13) is equal to zero. From [Apo76, p. 249], we have that $L(0,\chi) = \sum_{r=1}^{L} \chi(r)\zeta(0, r/L)$. Inserting this relation into (3.13), we obtain

$$D = \frac{-2\pi i}{N} \Big(\zeta(0, \frac{\mu'}{L}) - \zeta(0, \frac{L-\mu'}{L}) \Big) [\mu' > 0].$$

However, from [Apo76, p. 268], we have that $\zeta(0,a) = (1-2a)/2$ for real numbers $0 < a \leq 1$, and thus we obtain

$$D = \frac{-2\pi i}{N} \frac{L - 2\mu'}{L} [\mu' > 0].$$

□

When $k \geq 2$, the term C can be simplified by using the discrete Fourier transform and the functional equation for the Hurwitz ζ-function.

Proposition 3.6. Let $k \geq 2$ be an integer and let M be a positive integer. Then for $1 \leq \nu' < M$ it holds that

$$\zeta(k, \frac{\nu'}{M}) + (-1)^k \zeta(k, \frac{M-\nu'}{M}) = -\frac{(2\pi i)^k M^{k-1}}{k!} \sum_{r=0}^{k} \binom{k}{k-r} B_{k-r} \sum_{d=1}^{M} e(\frac{-d\nu'}{M}) (d/M)^r,$$

and for k even it holds that

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2(k!)},$$

where B_k is the kth Bernoulli number.

Proof. The latter statement is well-known, see for example [Apo76, p. 266]. Therefore we focus on the former.

We have that for $1 \leq d \leq M$ it holds that

$$\zeta(1-k-2s, \frac{d}{M}) = \frac{\Gamma(k+2s)}{(2\pi M)^{k+2s}} \sum_{r=1}^{M} 2\mathrm{Re}(e(\frac{k+2s}{4} - \frac{dr}{M})) \zeta(k+2s, \frac{r}{M}),$$

see for example [Apo76, Theorem 12.8]. Applying the discrete Fourier transform to the sequence $f = (\zeta(1-k-2s, \frac{d}{M}))_{d=1}^{M}$ we now find

$$\begin{aligned} \mathcal{F}\{f\}(\nu') &= \sum_{d=1}^{M} e(-\frac{d\nu'}{M}) \zeta(1-k-2s, \frac{d}{M}) \\ &= \frac{\Gamma(k+2s)}{(2\pi M)^{k+2s}} \sum_{d=1}^{M} \sum_{r=1}^{M} \left(e(\frac{k+2s}{4} - \frac{d(\nu'+r)}{M}) + e(\frac{-k-2s}{4} - \frac{d(\nu'-r)}{M}) \right) \zeta(k+2s, \frac{r}{M}). \end{aligned}$$

We interchange the order of summation and get

$$\begin{aligned}\sum_{d=1}^{M}\sum_{r=1}^{M}\zeta(k+2s,\frac{r}{M})e(-\frac{d(\nu'+r)}{M}) &= \sum_{n\geq 0}\sum_{d=1}^{M}\left(\sum_{r=1}^{M}(n+\frac{r}{M})^{-k-2s}e^{-\frac{2\pi ird}{M}}\right)e^{-\frac{2\pi id\nu'}{M}}\\ &= \sum_{n\geq 0}\sum_{d=1}^{M}\left(\sum_{r=1}^{M}(n+\frac{r}{M})^{-k-2s}e^{-\frac{2\pi ird}{M}}\right)e^{\frac{2\pi id(M-\nu')}{M}}\\ &= \sum_{n\geq 0}M\mathcal{F}^{-1}\{\mathcal{F}\{(n+\frac{r}{M})^{-k-2s}\}\}(M-\nu')\\ &= M\sum_{n\geq 0}(n+\frac{M-\nu'}{M})^{-k-2s} = M\zeta(k+2s,\frac{M-\nu'}{M}),\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{d=1}^{M}\sum_{r=1}^{M}\zeta(k+2s,\frac{r}{M})e(-\frac{d(\nu'-r)}{M}) &= \sum_{n\geq 0}\sum_{d=1}^{M}\left(\sum_{r=1}^{M}(n+\frac{r}{M})^{-k-2s}e^{\frac{2\pi ird}{M}}\right)e^{-\frac{2\pi id\nu'}{M}}\\ &= \sum_{n\geq 0}\mathcal{F}\{M\mathcal{F}^{-1}\{(n+\frac{r}{M})^{-k-2s}\}\}(\nu')\\ &= M\zeta(k+2s,\frac{\nu'}{M}),\end{aligned}$$

and therefore we have that

$$\sum_{d=1}^{M}e(-\frac{d\nu'}{M})\zeta(1-k-2s,\frac{d}{M}) \tag{3.14}$$

$$= \frac{M\Gamma(k+2s)}{(2\pi M)^{k+2s}}\left(e(\frac{k+2s}{4})\zeta(k+2s,\frac{M-\nu'}{M})+e(\frac{-k-2s}{4})\zeta(k+2s,\frac{\nu'}{M})\right). \tag{3.15}$$

Since $k\geq 2$, we can without any issue let $s\to 0$, whence we obtain

$$\sum_{d=1}^{M}e(\frac{-d\nu'}{M})\zeta(1-k,\frac{d}{M}) = \frac{\Gamma(k)}{(2\pi i)^k M^{k-1}}\left(\zeta(k,\frac{\nu'}{M})+(-1)^k\zeta(k,\frac{M-\nu'}{M})\right).$$

From [Apo76, Theorem 12.13] we recall that $\zeta(-n,x) = -B_{n+1}(x)/(n+1)$, where $n\geq 0$ is an integer, and $B_n(x)$ denotes the nth Bernoulli polynomial, which we recall satisfies

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}.$$

Combining all of this, we conclude that when $1\leq\nu'<M$, it holds that

$$\begin{aligned}\zeta(k,\frac{\nu'}{M})+(-1)^k\zeta(k,\frac{M-\nu'}{M}) &= \frac{(2\pi i)^k M^{k-1}}{\Gamma(k)}\sum_{d=1}^{M}e(\frac{-d\nu'}{M})(-\frac{B_k(d/M)}{k})\\ &= -\frac{(2\pi i)^k M^{k-1}}{k!}\sum_{d=1}^{M}\sum_{r=0}^{k}e(\frac{-d\nu'}{M})\binom{k}{k-r}B_{k-r}(d/M)^r\\ &= -\frac{(2\pi i)^k M^{k-1}}{k!}\sum_{r=0}^{k}\binom{k}{k-r}B_{k-r}\sum_{d=1}^{M}e(\frac{-d\nu'}{M})(d/M)^r.\end{aligned}$$

□

3.4 The weight 2 case

The function $G_{2,N,c,d}$ is as we have remarked earlier, an almost holomorphic modular form, see [Bru+08, p. 58]. Specifically, since $G_{2,\rho_N^\times} = \sum_{(c,d)\in\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbb{Z})}G_{2,N,c,d}\mathfrak{c}_{(c,d)}$ is invariant with

respect to $|_{2,\rho_N^\times}$ it holds that $G_{2,\rho_N^\times}$ is vector-valued almost holomorphic modular form of weight 2 and depth 1 with respect to the type $\rho_N^\times$. Here we use the same definition as in the scalar valued case [Bru+08, p. 58], but with the action $|_{k,\rho_N^\times}$ instead of $|_k$.

Notation 4. The space of almost holomorphic vector-valued modular forms of depth p and type ρ is denoted by $\widetilde{M}_k^{[p]}(\rho)$.

With the following decomposition of $\rho_N^\times$, we can describe $\widetilde{M}_2^{[1]}(\rho_N^\times)$ along the lines of [Bru+08, p. 59, Proposition 20].

Proposition 3.7. There exists an arithmetic type ρ' such that

$$\rho_N^\times = \mathbf{1} \oplus \rho',$$

and $\mathrm{Hom}(\mathbf{1}, \rho') = 0$. Here $V(\mathbf{1})$ and $V(\rho')$ are subspaces of $V(\rho_N^\times)$, and $\mathbf{1}$ is a one-dimensional trivial representation.

Proof. This is a twist on a standard exercise in representation theory. We begin by labeling the basis $(\mathfrak{e}_{(c,d)})_{(c,d)\in\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbb{Z})}$ with numbered indices as $(\mathfrak{e}_i)_{i=1}^r$. We then define ρ' by $V(\rho') = \mathrm{span}_{\mathbb{C}}\{\mathfrak{e}_1 - \mathfrak{e}_2, \dots, \mathfrak{e}_{r-1} - \mathfrak{e}_r\}$, and $\rho'(\gamma) = \rho_N^\times(\gamma)$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Similarly, we define $\mathbf{1}$ by $V(\mathbf{1}) = \mathrm{span}_{\mathbb{C}}\{\mathfrak{e}_1 + \cdots + \mathfrak{e}_r\}$ and $\mathbf{1}(\gamma)(v) = \rho_N^\times(\gamma)v$ for every $v \in V(\mathbf{1})$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Note that since $\rho_N^\times$ is a trivially twisted permutation type, it holds that $\rho_N^\times(\gamma)v = v$ for every $v \in V(\mathbf{1})$ and every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, so that $\mathbf{1}$ is indeed a trivial representation.

Let us also verify that ρ' is well-defined. We have that $\rho_N^\times(\mathfrak{e}_i - \mathfrak{e}_{i+1}) = \mathfrak{e}_{\pi_\gamma(i)} - \mathfrak{e}_{\pi_\gamma(i+1)}$ for some permutation $\pi_\gamma \in S_r$. Without loss of generality we can assume that $\pi_\gamma(i) < \pi_\gamma(i+1)$, say with $k = \pi_\gamma(i+1) - \pi_\gamma(i)$. Then

$$\mathfrak{e}_{\pi_\gamma(i)} - \mathfrak{e}_{\pi_\gamma(i+1)} = \mathfrak{e}_{\pi_\gamma(i)} - \mathfrak{e}_{\pi_\gamma(i)+1} + \mathfrak{e}_{\pi_\gamma(i)+1} - \mathfrak{e}_{\pi_\gamma(i)+2} + \cdots + \mathfrak{e}_{\pi_\gamma(i)+k-1} - \mathfrak{e}_{\pi_\gamma(i)+k} \in V(\rho').$$

It is clear that $\{\mathfrak{e}_1 + \cdots + \mathfrak{e}_r, \mathfrak{e}_1 - \mathfrak{e}_2, \mathfrak{e}_2 - \mathfrak{e}_3, \dots, \mathfrak{e}_{r-1} - \mathfrak{e}_r\}$ is a linearly independent set and that $V(\mathbf{1}) + V(\rho') = V(\rho_N^\times)$ whence $V(\rho_N^\times) = V(\mathbf{1}) \oplus V(\rho')$. Since $\mathbf{1}$ and ρ' are just restrictions of $\rho_N^\times$ it follows that $\rho_N^\times = \mathbf{1} \oplus \rho'$.

As for the space of homomorphisms, we have by Frobenius reciprocity, see [Eti+11, p. 107], that

$$\mathrm{Hom}(\mathbf{1}, \rho_N^\times) = \mathrm{Hom}(\mathbf{1}, \mathrm{Ind}_{\Gamma_1(N)}^{\mathrm{SL}_2(\mathbb{Z})}\mathbf{1}) \cong \mathrm{Hom}(\mathbf{1}|_{\Gamma_1(N)}, \mathbf{1}),$$

whence $\dim(\mathrm{Hom}(\mathbf{1}, \rho_N^\times)) = 1$. However, it also holds that

$$\mathrm{Hom}(\mathbf{1}, \rho_N^\times) = \mathrm{Hom}(\mathbf{1}, \mathbf{1}) \oplus \mathrm{Hom}(\mathbf{1}, \rho'),$$

and therefore we must have that $\dim(\mathrm{Hom}(\mathbf{1}, \rho')) = 0$. We thus conclude that $\mathrm{Hom}(\mathbf{1}, \rho') = \{0\}$, as claimed. □

Using [Xià19, Lemma 3.13], we get

$$\widetilde{M}_2^{[1]}(\rho_N^\times) = \widetilde{M}_2^{[1]}(\mathbf{1}) \oplus \widetilde{M}_2^{[1]}(\rho').$$

Since elements in $\widetilde{M}_2^{[1]}(\mathbf{1})$ are component-wise equal at every $\tau \in \mathbb{H}$, and the components are weight 2 almost holomorphic modular forms of depth 1, we get by [Bru+08, p. 59, Proposition 20] that

$$\widetilde{M}_2^{[1]}(\mathbf{1}) = \mathbb{C} \sum_{(c,d)\in\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbb{Z})} \mathfrak{e}_{(c,d)} E_2^*,$$

where $E_2^*(\tau) = \frac{1}{C}E_2(\tau; \chi_0, \chi_0)$, that is $E_2(\tau; \chi_0, \chi_0)$ normalized to have its constant term equal to 1.

As for $\widetilde{M}_2^{[1]}(\rho')$, we employ the weight lowering operator $\xi_2 = 2iy^2\overline{\frac{\partial}{\partial\overline{\tau}}}$ to establish a correspondence with $M_0(\rho')$, see [Bri+17, p. 74].

It is easily verified that ξ_2 maps $\widetilde{M}_2^{[1]}(\rho')$ to $M_0(\overline{\rho'})$. By the maximum modulus principle[2], we have that $M_0(\overline{\rho'})$ consists of constant functions invariant with respect to $\overline{\rho'}$, and so $M_0(\rho') \cong \mathrm{Hom}(\mathbf{1}, \overline{\rho'}) \cong \mathrm{Hom}(\mathbf{1}, \rho') = \{0\}$. It follows that

$$\ker \xi_2 = \widetilde{M}_2^{[1]}(\rho').$$

[2]Stating that holomorphic functions with a local maximum in the domain are constant. The components of elements in $M_0(\rho')$ are holomorphic on a compact Riemann surface of the form $\Gamma\backslash\mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$, and thus attain a maximum in the domain. Hence the maximum modulus principle gives that the elements in $M_0(\rho')$ must be constant.

Notice now that $f \in \ker \xi_2$ if and only if $\partial_{\overline{\tau}} f = 0$ if and only if f is holomorphic. Hence $\widetilde{M}_2^{[1]}(\rho') = M_2(\rho')$. Hence we have the decomposition

$$\widetilde{M}_2^{[1]}(\rho_N^\times) = \mathbb{C}\mathfrak{e}E_2^* \oplus M_2(\rho'),$$

where $\mathfrak{e} = \sum_{(c,d)\in\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \mathfrak{e}_{(c,d)}$.

With this decomposition in mind, it is straightforward to obtain a full description of $\mathcal{E}_2[\rho_N^\times]$. We take $G_{2,\rho_N^\times} \in \widetilde{M}_2^{[1]}(\rho_N^\times)$, remove the contribution coming from $\mathbb{C}\mathfrak{e}E_2^*$, and then extract the components as we did before.

By the decomposition, we have that

$$G_{2,\rho_N^\times} = A\mathfrak{e}E_2^* + F,$$

with F modular and $A \in \mathbb{C}$ constant. We wish to determine A.

Applying $\sum_{(c,d)\in\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z}} \mathfrak{e}_{(c,d)}^\vee$, we obtain

$$\sum_{(c,d)\in\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})} G_{2,N,c,d} = AE_2^* |\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})| + F',$$

where $F' \in M_2(\mathrm{SL}_2(\mathbb{Z}))$ and so $F' = 0$. Hence, by extracting constant coefficients, we find

$$A = \frac{c(\sum_{(c,d)\in\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})} G_{2,N,c,d}; 0)}{|\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})|}. \tag{3.16}$$

We conclude that $\mathcal{E}_2[\rho_N^\times]$ is equal to the span of the components of the weight 2 and $\rho_N^\times$-type vector valued modular form

$$G_{2,\rho_N^\times} - A\mathfrak{e}E_2^*.$$

We end this section by explicitly computing A. For convenience, we use the notation from theorem 7 and the notation $C_{c,d,N}$ for the constant coefficient of $G_{2,N,c,d}(\tau)$. We also identify classes in $\mathbb{Z}/N\mathbb{Z}$ with their unique representatives in the interval $[0, N)$.

Notice first that $L = 1$ if and only if $u = N$ if and only if $c = 0$. So if $c \neq 0$, then $C_{c,d,N} = 0$.

As for $C_{0,d,N}$ we split into the cases $N = 1$ and $N > 1$. We notice that

$$C_{0,d,N} = \frac{2}{N^2}\zeta(2) = \frac{\pi^2}{3N^2},$$

only when $d = 0$, and on account of the criterion $\gcd(c, d, N) = 1$, this happens only when $N = 1$. Since all integers are congruent to 0 modulo 1, this is the only term in the sum $\sum_{(c,d)\in\Gamma_1(1)\backslash \mathrm{SL}_2(\mathbb{Z})} C_{c,d,1}$.

In the case of $N > 1$, proposition 3.6 gives us that

$$\sum_{(c,d)\in\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})} C_{c,d,N} = \frac{2\pi^2}{N} \sum_{\substack{0\le d'\le N-1 \\ \gcd(d',N)=1}} \left(\frac{1}{6}\sum_{d=1}^{N} e(-\frac{dd'}{N}) - \sum_{d=1}^{N} e(-\frac{dd'}{N})\frac{d}{N} + \sum_{d=1}^{N} e(-\frac{dd'}{N})(\frac{d}{N})^2 \right), \tag{3.17}$$

Let us assume that $N > 1$ and simplify (3.17). Noticing that $\sum_{d=1}^{N} e(-dd'/N) = 0$ and exchanging the order of summation, we obtain that (3.17) is equal to

$$-\frac{2\pi^2}{N^3}\sum_{d=1}^{N} d(N-d) \sum_{\substack{0\le d'\le N-1 \\ \gcd(d',N)=1}} e(\frac{-dd'}{N}).$$

The inner sum is sometimes known as a Ramanujan sum and it satisfies, see for example [Apo76, p. 164], the following identity

$$\sum_{\substack{0\le d'\le N-1 \\ \gcd(d',N)=1}} e(\frac{-dd'}{N}) = \phi(N)\phi(N/\gcd(d,N))^{-1}\mu(N/\gcd(d,N)).$$

Thus (3.17) is equal to

$$-\frac{2\pi^2\phi(N)}{N^3}\sum_{d=1}^{N} d(N-d)\phi(N/\gcd(d,N))^{-1}\mu(N/\gcd(d,N)),$$

and it is well-known (see for example [Kil15, pp. 22–24]) that

$$|\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})| = N^2\prod_{p|N}(1-p^{-2}),$$

whence the A from (3.16) is explicitly determined.

Before ending this chapter, we make the following important remark.

Remark 3.1. After normalization of the elements in $\mathcal{E}_k[\rho_N^{\times}]$ by $(2\pi i)^k$ the Fourier coefficients lie in $\mathbb{Q}^{\mathrm{ab}}$.

The natural question to ask at this stage is precisely how the right hand side of 3.1 relates to the space of modular forms $M_k(\rho)$. The following chapter answers this question in complete detail.

Chapter 4

Computing spaces of vector-valued modular forms

4.1 Background

In this section we will work extensively with spaces of homomorphisms between representations, and in order to easily use results from category theory and homological algebra we make the following remark.

Remark 4.1. The category of linear representations of a group G over a field K, denoted by $\mathrm{Rep}_K(G)$, is isomorphic to the category of left $K[G]$-modules, denoted by $K[G]$-**mod**.

Specifically, we define a functor $F : \mathrm{Rep}_K(G) \to K[G]$-**mod** as follows.

We let $F(\rho) = V(\rho)$, and define the action by

$$(\sum_{g\in G} a_g g)v = \sum_{g\in G} a_g \rho(g)v,$$

where $a_g \in K$, and $v \in V(\rho)$, and all but finitely many a_g are zero. If $\rho \xrightarrow{f} \sigma$ is a morphism, then we define $F(f) = f$. The latter is valid since morphims in $\mathrm{Rep}_K(G)$ correspond exactly to equivariant maps.

Inspired by the above remark, we make the following definition.

Definition 4.1. Let ρ be an arithmetic type. Then we define

$$H^0(\rho) = \mathrm{Hom}(\mathbf{1}, \rho),$$

where identify $f \in \mathrm{Hom}(\mathbf{1}, \rho)$ with $f(1)$.

Remark 4.2. The above definition is motivated by group cohomology. Indeed, let $F : \mathrm{Rep}_{\mathbb{C}}(\mathrm{SL}_2(\mathbb{Z})) \to \mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$-**mod** be the isomorphism from remark 4.1, $T(\rho) = \mathrm{Hom}(\mathbf{1}, \rho)$, and $\mathrm{R}^n FT$ be the right-derived functors of FT.

Then for a non-negative integer n we define

$$H^n(\rho) = (\mathrm{R}^n FT)(\rho).$$

We then have that

$$H^0(\rho) = \mathrm{R}^0 FT \cong FT(\rho) = F(\mathrm{Hom}(\mathbf{1}, \rho)) \cong \mathrm{Hom}(\mathbf{1}, \rho).$$

Proposition 4.1. Let ρ, ρ_1, ρ_2 be arithmetic types and let $\phi : \rho_1 \to \rho_2$ be an injective (or surjective) homomorphism. Then there exists an injection (or surjection)

$$H^0(\rho_1 \otimes \rho) \to H^0(\rho_2 \otimes \rho).$$

Proof. Without loss of generality, we assume ϕ to be injective.

Let $F : \mathrm{Rep}_{\mathbb{C}}(\mathrm{SL}_2(\mathbb{Z})) \to \mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$**-mod** be the isomorphism from remark 4.1, and let G be its inverse.

Since the trivial $\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$-module $\mathbb{C}$ has basis $\{1\}$, it is free, and therefore also projective. By [Jac89, Proposition 3.9], we therefore have that the functor $\mathrm{Hom}_{\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]}(\mathbb{C}, -)$ is exact.

We have an exact sequence of linear representations

$$0 \to \rho_1 \otimes \rho \to \rho_2 \otimes \rho,$$

which yields an exact sequence of $\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$-modules

$$0 \to F(\rho_1 \otimes \rho) \to F(\rho_2 \otimes \rho),$$

which yields an exact sequence of abelian groups

$$0 \to \mathrm{Hom}(\mathbb{C}, F(\rho_1 \otimes \rho)) \to \mathrm{Hom}(\mathbb{C}, F(\rho_2 \otimes \rho))),$$

which finally yields another exact sequence of abelian groups

$$0 \to G(\mathrm{Hom}(\mathbb{C}, F(\rho_1 \otimes \rho))) \to G(\mathrm{Hom}(\mathbb{C}, F(\rho_2 \otimes \rho)))).$$

But $G(\mathrm{Hom}(\mathbb{C}, F(\rho_i \otimes \rho))) = \mathrm{Hom}(\mathbf{1}, \rho_i \otimes \rho)$ for $i \in \{1, 2\}$, so we are done. □

We have previously shown that

$$(\rho_N^{\times})^{\vee} \to \mathcal{E}_k[\rho_N^{\times}], \text{ where } w \mapsto w \circ E_{k,N},$$

is a surjective homomorphism for any positive integer k, and thus there is a surjection

$$H^0((\rho_N^{\times})^{\vee} \otimes \rho) \twoheadrightarrow H^0(\mathcal{E}_k[\rho_N^{\times}] \otimes \rho).$$

Similarly, there is a surjection

$$H^0((\rho_{N_1}^{\times})^{\vee} \otimes (\rho_{N_2}^{\times})^{\vee} \otimes \rho) \twoheadrightarrow H^0(\mathcal{E}_l[\rho_{N_1}^{\times}] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^{\times}] \otimes \rho).$$

This ostensibly simple fact provides us with a computationally feasible way to compute a basis for $M_k(\rho)$.

Before going into specifics, let us state a few preliminaries.

Definition 4.2. Let $N, k \geq 1$ be integers. Then we define $\mathrm{FE} : M_k(\Gamma(N)) \to \mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!]$ to be the injective homomorphism which sends modular forms to their Fourier series expansions.

If ρ is a congruence type of level N with basis $\{\mathfrak{e}_i\}_{i=1}^n$, and $f \in M_k(\rho)$, we define $\mathrm{FE} : M_k(\rho) \to \mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!] \otimes V(\rho)$ by

$$\mathrm{FE}(f) = \sum_{i=1}^{n} \mathrm{FE}(\mathfrak{e}_i^{\vee} \circ f) \otimes \mathfrak{e}_i.$$

We emphasize that FE is injective.

Proposition 4.2. Let N_1, N_2, k and $l \leq k$ be positive integers and let ρ be an arithmetic type of level N.

Let $\pi : \mathcal{E}_l[\rho_{N_1}^{\times}] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^{\times}] \to \mathcal{E}_l[\rho_{N_1}^{\times}] \cdot \mathcal{E}_{k-l}[\rho_{N_2}^{\times}]$ be defined by $\pi(f \otimes g) = f \cdot g$ and for all other elements by linearity.

Then

$$\mathrm{FE} \circ \pi : H^0(\mathcal{E}_l[\rho_{N_1}^{\times}] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^{\times}] \otimes \rho) \to \mathbb{Q}^{\mathrm{ab}}[\![q^{1/\mathrm{lcm}(N_1,N_2)}]\!] \otimes V(\rho),$$

restricts on the co-domain to $\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!] \otimes V(\rho)$.

Proof. Let $f \in H^0(\mathcal{E}_l[\rho_{N_1}^{\times}] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^{\times}] \otimes \rho)$, and let as before $\{\mathfrak{e}_i\}_{i=1}^n$ be a basis for $V(\rho)$. Then

$$f = \sum_{c,d,c',d',i} A_{c,d,c',d',i} G_{l,N_1,c,d} \otimes G_{k-l,N_2,c',d'} \otimes \mathfrak{e}_i,$$

where c, d and c', d' traverses representatives of $\Gamma_1(N_1)\backslash\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_1(N_2)\backslash\mathrm{SL}_2(\mathbb{Z})$, respectively, and where $A_{c,d,c',d',i} \in \mathbb{C}$ are coefficients.

Hence we see that

$$\pi(f) = \sum_{c,d,c',d',i} A_{c,d,c',d',i}(G_{l,N_1,c,d} \cdot G_{k-l,N_2,c',d'}) \otimes \mathfrak{e}_i \in M_k(\Gamma(\mathrm{lcm}(N_1,N_2))) \otimes V(\rho).$$

Computing Fourier series expansions, we get for each i

$$\sum_{c,d,c',d'} A_{c,d,c',d',i} G_{l,N_1,c,d} \cdot G_{k-l,N_2,c',d'} = \sum_{n\geq 0} F_n^i q^{n/\mathrm{lcm}(N_1,N_2)},$$

for some Fourier coefficients $F_n^i \in \mathbb{C}$.

Invariance now gives that

$$f = \sum_{c,d,c',d',i} A_{c,d,c',d',i} G_{l,N_1,(c,d)\gamma^{-1}}|_l\gamma^{-1} \otimes G_{k-l,N_2,c',d'}|_{k-l}\gamma^{-1} \otimes \rho(\gamma)\mathfrak{e}_i.$$

In particular, this holds for $\gamma = T^{-N}$. It holds that $\rho(T^{-N})\mathfrak{e}_i = \mathfrak{e}_i$ and combining this with the Fourier series expansions, we obtain for each $n \geq 0$ that

$$e(\frac{nN}{\mathrm{lcm}(N_1,N_2)})F_n^i = F_n^i.$$

So, either $F_n^i = 0$ or $n = k \cdot \mathrm{lcm}(N_1,N_2)/N$ for some non-negative integer k. It follows that

$$\sum_{n\geq 0} F_n^i q^{n/\mathrm{lcm}(N_1,N_2)} = \sum_{k\geq 0} F^i_{\frac{k\cdot \mathrm{lcm}(N_1,N_2)}{N}} q^{k/N},$$

and thus it is clear that $\mathrm{FE}(\pi(f)) \in \mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!] \otimes V(\rho)$, as claimed. □

For computational purposes it is beneficial to decompose the term

$$\mathrm{Hom}(\mathbf{1}, (\mathcal{E}_k[\rho_{N_0}^\times] \cdot \mathcal{E}_l[\rho_{N_0}^\times]) \otimes \rho),$$

in equation (3.1) as a sum. Doing so, we obtain the following reformulation of the Raum-Xià theorem.

Theorem 8. Let N, k be positive integers and let ρ be a congruence type of level N. Then there exists a positive integer $N_{\max}$, divisible by N, such that

$$M_k(\rho) = H^0(\mathcal{E}_k[\rho_N^\times] \otimes \rho) + \sum_{\substack{N|N_1\leq N_{\max}\\ N_2|N_1\\ 1\leq l\leq k}} H^0((\mathcal{E}_l[\rho_{N_1}^\times] \cdot \mathcal{E}_{k-l}[\rho_{N_2}^\times]) \otimes \rho).$$

Let us note that the right-hand side certainly is included in $M_k(\rho)$. Indeed, we have that[1]

$$\mathcal{E}_l[\rho_{N_1}^\times] \cdot \mathcal{E}_{k-l}[\rho_{N_2}^\times] \subseteq M_k(\Gamma(\mathrm{lcm}(N_1,N_2))) = M_k(\Gamma(N_1)),$$

and that[2]

$$\mathcal{E}_k[\rho_N^\times] + \sum_{\substack{N|N_1\leq N_{\max}\\ N_2|N_1\\ 1\leq l\leq k}} \mathcal{E}_l[\rho_{N_1}^\times] \cdot \mathcal{E}_{k-l}[\rho_{N_2}^\times] \subseteq M_k(\Gamma(N_{\max})).$$

[1] Say that N_1, N_2, k are positive integers, and $1 \leq l \leq k$. Then we have that $\mathcal{E}_k[\rho_{N_1}^\times] \subseteq M_k(\Gamma(N_1))$. So if $f \in \mathcal{E}_l[\rho_{N_1}^\times]$ and $g \in \mathcal{E}_{k-l}[\rho_{N_2}^\times]$, we get for $\gamma \in \Gamma(\mathrm{lcm}(N_1,N_2))$ that $\gamma \in \Gamma(N_1)$ and $\gamma \in \Gamma(N_2)$, so

$$f(\gamma.\tau)g(\gamma.\tau) = (c(\gamma)\tau + d(\gamma))^l f(\tau)(c(\gamma)\tau + d(\gamma))^{k-l} g(\tau) = (c(\gamma)\tau + d(\gamma))^k (f\cdot g)(\tau),$$

so $f \cdot g \in M_k(\Gamma(\mathrm{lcm}(N_1,N_2)))$.

[2] Say that N_1, N_2, k are positive integers with $N_1 \mid N_2$, and $f \in M_k(\Gamma(N_1))$ and $g \in M_k(\Gamma(N_2))$. Let $\gamma \in \Gamma(N_2)$, then also $\gamma \in \Gamma(N_1)$, and so $f + g \in M_k(\Gamma(N_2))$.

Hence we have that

$$H^0(\mathcal{E}_k[\rho_N^\times] \otimes \rho) + \sum_{\substack{N \mid N_1 \leq N_{\max} \\ N_2 \mid N_1 \\ 1 \leq l \leq k}} H^0((\mathcal{E}_l[\rho_{N_1}^\times] \cdot \mathcal{E}_{k-l}[\rho_{N_2}^\times]) \otimes \rho)$$

$$\subseteq H^0((\mathcal{E}_k[\rho_N^\times] + \sum_{\substack{N \mid N_1 \leq N_{\max} \\ N_2 \mid N_1 \\ 1 \leq l \leq k}} \mathcal{E}_l[\rho_{N_1}^\times] \cdot \mathcal{E}_{k-l}[\rho_{N_2}^\times]) \otimes \rho) \subseteq H^0(M_k(\Gamma(N_{\max})) \otimes \rho) \subseteq M_k(\rho),$$

where the first inclusion comes from linearity, and the last inclusion comes from invariance and the identification $\mathbb{C} \otimes V(\rho) = V(\rho)$. Thus, the main content of the theorem is that the right-hand side actually contains $M_k(\rho)$.

4.2 A large diagram

We are now in a position to state exactly how we compute bases for $M_k(\rho)$ in terms of products of components of vector-valued Eisenstein series. Let us begin by considering figure 5. Note that we use the letter s for the Sturm bound.

Let us analyze the columns separately. The second column is analogous to the first and the fourth column can be studied together with the third. We will pay extra attention to the latter two columns.

Note that we use asterisks $*$, as a shorthand notation for the corresponding objects in figure 5.

Column 1 and 2

$$\begin{array}{ccccccc} & {\scriptstyle *}\uparrow{\scriptstyle \iota_1} & & & {\scriptstyle *}\uparrow{\scriptstyle \iota_2} & & \\ H^0((\rho_{N_1}^\times)^\vee \otimes (\rho_{N_2}^\times)^\vee \otimes \rho) & \xrightarrow{f_{l,k}} & H^0(\mathcal{E}_l[\rho_{N_1}^\times] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^\times] \otimes \rho) & \xrightarrow{\mathrm{FE}\circ\pi} & \mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!] \otimes V(\rho) & \xrightarrow{\mathrm{trunc}} & (\frac{\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!]}{q^s\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!]})^{\dim(\rho)} \end{array}$$

Figure 1: Column 1 and 2 of figure 5.

Here l is an arbitrary integer in the interval $1 \leq l \leq k$, and N_1, N_2, N are positive integers satisfying $N \mid N_1$ and $N_2 \mid N_1$.

The maps ι_1 and ι_2 are sections of the canonical projections, the map $f_{l,k}$ is the induced map coming from the surjection $(\rho_{N_1}^\times)^\vee \otimes (\rho_{N_2}^\times)^\vee \twoheadrightarrow \mathcal{E}_l[\rho_{N_1}^\times] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^\times]$ that we saw in proposition 4.1, the map $\mathrm{FE} \circ \pi$ is precisely the map discussed in proposition 4.2, and finally the map trunc. is component-wise projection

$$\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!] \ni G \mapsto G + q^s\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!],$$

(that is, truncation) together with the identification

$$(\frac{\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!]}{q^s\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!]}) \otimes V(\rho) = (\frac{\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!]}{q^s\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!]})^{\dim(\rho)}.$$

Column 2 is analogous to column 1, and thus we omit its analysis.

Columns 3 and 4

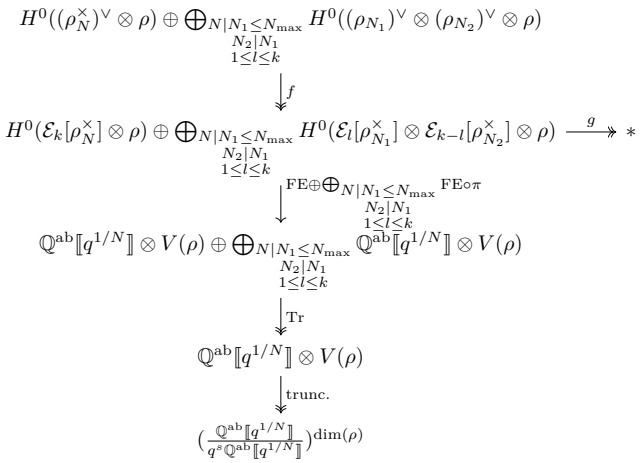

Figure 2: Column 3 and 4 of figure 5.

Here the $\oplus$ indicates an external direct sum.

The map f is given as follows

$$f = f_k \oplus \bigoplus_{\substack{N \mid N_1 \leq N_{\max} \\ N_2 \mid N_1 \\ 1 \leq l \leq k}} f_{N_1, N_2, l, k},$$

where f_k and $f_{N_1,N_2,l,k}$ are the maps corresponding to the first arrow in column 2 and column 1, respectively. The second and third map in the column are self-explanatory.

The map g factors as follows.

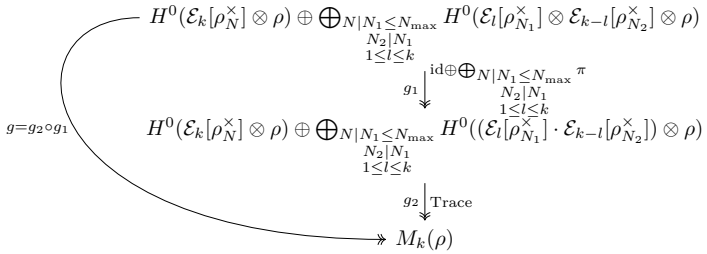

Figure 3: Factoring of the map g from figure 2.

The map π is the product described in the proof of proposition 4.2, and the trace is surjective on account of the Raum-Xià theorem. As we will use the product map again, we put

$$\Pi = \mathrm{id} \oplus \bigoplus_{\substack{N \mid N_1 \leq N_{\max} \\ N_2 \mid N_1 \\ 1 \leq l \leq k}} \pi.$$

We shall now show how the basis computation fits into the diagram. As we shall soon see, it is possible to compute bases for the spaces

$$H^0((\rho_N^\times)^\vee \otimes \rho) \text{ and } H^0((\rho_{N_1}^\times)^\vee \otimes (\rho_{N_2}^\times)^\vee \otimes \rho).$$

Let therefore $\mathcal{B}_N$ be a basis for the first of these groups, and $\mathcal{B}_{N_1,N_2}$ be a basis for the second one. Then

$$\mathcal{B} = \mathcal{B}_N \oplus \bigoplus_{\substack{N \mid N_1 \leq N_{\max} \\ N_2 \mid N_1 \\ 1 \leq l \leq k}} \mathcal{B}_{N_1, N_2},$$

where for each (N_1, N_2) in the sum there are k copies of $\mathcal{B}_{N_1,N_2}$, is a basis for the first row.

We now obtain the following extended diagram.

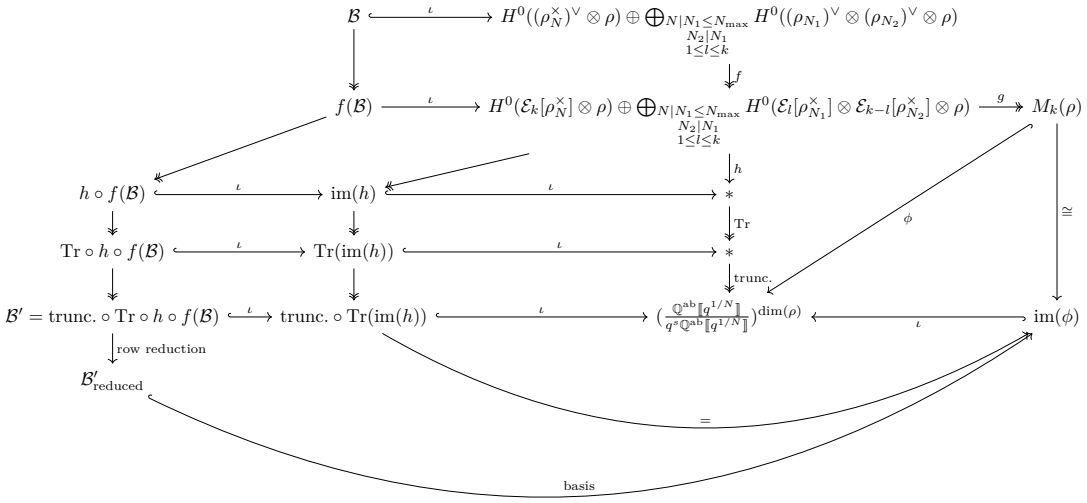

Figure 4: Figure 2 together with basis inclusions.

Here $\phi = \mathrm{trunc.} \circ \mathrm{FE}$, the ι are inclusion maps, and $h = \mathrm{FE}' \circ \Pi$ where FE' is component-wise Fourier series expansion. Let us first note that

$$\mathrm{im}(\phi) = \mathrm{trunc.} \circ \mathrm{Tr}(\mathrm{im}(h)).$$

Indeed, if $F \in \mathrm{im}(\phi)$, then $F = \mathrm{trunc.} \circ \mathrm{FE} \circ g(G)$ for some

$$G \in H^0(\mathcal{E}_k[\rho_N^\times] \otimes \rho) \oplus \bigoplus_{\substack{N \mid N_1 \leq N_{\max} \\ N_2 \mid N_1 \\ 1 \leq l \leq k}} H^0(\mathcal{E}_l[\rho_{N_1}^\times] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^\times] \otimes \rho).$$

We recall that $g = \mathrm{Tr} \circ \Pi$, where Π is the product map defined above, and so

$$F = \mathrm{trunc.} \circ \mathrm{FE} \circ \mathrm{Tr} \circ \Pi(G).$$

By uniqueness of Fourier series expansions and linearity, we obtain that

$$\mathrm{Tr} \circ \mathrm{FE}' = \mathrm{FE} \circ \mathrm{Tr}.$$

This means that

$$F = \mathrm{trunc.} \circ \mathrm{FE} \circ \mathrm{Tr} \circ \Pi(G) = \mathrm{trunc.} \circ \mathrm{Tr} \circ \mathrm{FE}' \circ \Pi(G) = \mathrm{trunc.} \circ \mathrm{Tr} \circ h(G).$$

So $F \in \mathrm{trunc.} \circ \mathrm{Tr}(\mathrm{im}(h))$. The argument is reversible, and thus we obtain the desired equality.

Now since f is linear and surjective, we have that $\mathcal{B}' = \mathrm{trunc.} \circ \mathrm{Tr} \circ h \circ f(\mathcal{B})$ is a spanning set for $\mathrm{trunc.} \circ \mathrm{Tr}(\mathrm{im}(h))$, and thus a spanning set for $\mathrm{im}(\phi)$.

Since the space

$$(\frac{\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!]}{q^s\mathbb{Q}^{\mathrm{ab}}[\![q^{1/N}]\!]})^{\dim(\rho)},$$

can be identified with $W = (\mathbb{Q}^{\mathrm{ab}})^{sN\dim(\rho)}$ we can identify $\mathrm{im}(\phi)$ as a subspace of W, and we obtain a basis of $\mathrm{im}(\phi)$, and therefore $M_k(\rho)$, by performing row reduction on $\mathcal{B}'$.

4.3 Efficiency considerations

In this section we use the same notation as in the previous section. For convenience, we also define $\psi = \mathrm{trunc.} \circ \mathrm{Tr} \circ h \circ f$.

From a computational perspective, it is inefficient to construct all of $\mathcal{B}'$ at once. After all, we might find a basis for $M_k(\rho)$ already in a much smaller set.

In our implementation, we therefore construct an "accumulating submodule" $\mathrm{span}_{\mathbb{Q}^{\mathrm{ab}}} \mathcal{B}_{\mathrm{acc}}$ of W as follows.

Algorithm 1: Computing a basis for $M_k(\rho)$ using an accumulating submodule.

function SubmoduleAcc (k, ρ);
Input : k – an integer weight
ρ – an arithmetic type of level N
Output: A basis $\{f_i\}_{i=1}^{d}$ of $M_k(\rho)$ as truncated Fourier series expansions.
$M \leftarrow 1 + k \cdot \sum_{N_1 | N \leq N_{\max}} d(N_1)$;
/* $d(N_1)$ is the number of positive divisors of N_1, so M is the number of terms in the direct sum. */
$t \leftarrow$ (first term of $\mathcal{B}) \oplus \bigoplus_{j=2}^{M} \{0\}$;
$\mathcal{B}_{\text{acc}} \leftarrow \psi(t)$;
$\mathcal{B}_{\text{acc}} \leftarrow \text{RowReduce}(\mathcal{B}_{\text{acc}})$;
$i \leftarrow 2$;
while $i \leq M$ *and* $|\mathcal{B}_{acc}| < \dim(M_k(\rho)$ **do**
  $t \leftarrow \bigoplus_{j=1}^{i} \{0\} \oplus (i\text{th term of } \mathcal{B}) \oplus \bigoplus_{j=i+1}^{M} \{0\}$;
  $\mathcal{B}_{\text{acc}} \leftarrow \mathcal{B}_{\text{acc}} \cup \psi(t)$;
  $\mathcal{B}_{\text{acc}} \leftarrow \text{RowReduce}(\mathcal{B}_{\text{acc}})$;
  $i \leftarrow i + 1$;
end
return $\mathcal{B}_{acc}$;

We record that it is easy to extend algorithm 1 to also keep track of the products of Eisenstein series $G_{k,L,c,d}$ which generate the elements of $\mathcal{B}_{\text{acc}}$. Instead of just applying ψ to the elements of $\mathcal{B}$, one can simultaneously map them, in the obvious way, to a module of Eisenstein series expressions

$$\mathbb{Q}^{\text{ab}}[S]^{\dim(\rho)},$$

where S is the set of symbols defined as

$$S = \{\mathfrak{e}_{l,L,c,d} : l, L \in \mathbb{Z}_{>0}, (c,d) \in \Gamma_1(L)\backslash \text{SL}_2(\mathbb{Z})\}.$$

One then obtains a corresponding set $\mathcal{B}_{\text{acc}}^{\text{expr}}$ in bijection with $\mathcal{B}_{\text{acc}}$, to which one applies the same transformations as one does to $\mathcal{B}_{\text{acc}}$.

4.3.1 Compression through T-orbits

In this subsection we assume $V(\rho)$ is given a basis $\{\mathfrak{e}_i\}_{i=1}^{d}$ where $d = \dim(\rho)$, such that $\rho(T^{-1})$ acts as a twisted permutation.

We can then use the term-wise invariance of the space

$$H^0(\mathcal{E}_k[\rho_N^{\times}] \otimes \rho) \oplus \bigoplus_{\substack{N | N_1 \leq N_{\max} \\ N_2 | N_1 \\ 1 \leq l \leq k}} H^0(\mathcal{E}_l[\rho_{N_1}^{\times}] \otimes \mathcal{E}_{k-l}[\rho_{N_2}^{\times}] \otimes \rho),$$

with respect to $|_k T \otimes \rho(T^{-1})$ or $|_l T \otimes |_{k-l} T \otimes \rho(T^{-1})$ (depending on the term) to obtain a space isomorphic as a $\mathbb{Q}^{\text{ab}}$-vector space to $\text{Tr}(\text{im}(h))$ but which contains fewer coefficients.

To ease understanding, we refer to this process as deflation. The deflation map is defined as follows.

Definition 4.3. Write $\rho(T^{-1})(e_i) = \epsilon_i \rho(T^{-1})_{\text{triv}}(e_i)$ where $\rho(T^{-1})_{\text{triv}}$ is a permutation and ϵ_i is a root of unity.

Let $\{\mathfrak{e}_{i_j}\}_{j=1}^{m}$ with $m \leq d$, be a complete set of representatives of $\rho(T^{-1})_{\text{triv}}\backslash\{\mathfrak{e}_i\}_{i=1}^{d}$, so that

$$\bigsqcup_{j=1}^{m} \rho(T^{-1})_{\text{triv}}\{\mathfrak{e}_{i_j}\} = \{\mathfrak{e}_i\}_{i=1}^{d}.$$

Let $V(\rho)_{\text{defl}} = \text{span}_{\mathbb{Q}^{\text{ab}}}\{e_{i_j}\}_{j=1}^{m}$. We now define a map

$$\text{defl} : \mathbb{Q}^{\text{ab}}[\![q^{1/N}]\!] \otimes V(\rho) \to \mathbb{Q}^{\text{ab}}[\![q^{1/N}]\!] \otimes V(\rho)_{\text{defl}},$$

by

$$\mathrm{defl}(\sum_{i=1}^{d}(\sum_{n\geq 0}F_n^i q^{n/N})\otimes\mathfrak{e}_i)=\sum_{j=1}^{m}(\sum_{n\geq 0}F_n^{i_j}q^{n/N})\otimes\mathfrak{e}_{i_j}.$$

From here on we will refer to the orbits $\rho(T^{-1})_{\mathrm{triv}}\backslash\{\mathfrak{e}_i\}_{i=1}^d$ as T-orbits. Note also that the deflation map depends on a choice of orbit representatives $\mathfrak{e}_{i_j}$, so from here on we assume that such a choice has been made.

Proposition 4.3. The restricted deflation map

$$\mathrm{defl}|_{\mathrm{Tr}(\mathrm{im}(h))}:\mathrm{Tr}(\mathrm{im}(h))\to\mathrm{defl}\circ\mathrm{Tr}(\mathrm{im}(h)),$$

is an isomorphism of $\mathbb{Q}^{\mathrm{ab}}$-vector spaces.

Proof. Evidently this map is linear and surjective, so we only have to construct a linear left inverse. Let us call it inflation, and denote it by infl.

We first need to introduce some notation. Let $L_j=|\rho(T^{-1})_{\mathrm{triv}}\{\mathfrak{e}_{i_j}\}|$, and for $1\leq l\leq L_j$ let $\mathfrak{e}_{i_j}^l$ and $\epsilon_{i_j}^{(l)}$ be defined by

$$\mathfrak{e}_{i_j}^l=\rho(T^{-1})_{\mathrm{triv}}(\mathfrak{e}_{i_j}^{l-1}),\text{ and }\rho(T^{-1})(\mathfrak{e}_{i_j}^{l-1})=\epsilon_{i_j}^{(l)}\mathfrak{e}_{i_j}^l,$$

and let $\mathfrak{e}_{i_j}^0=\mathfrak{e}_{i_j}$. Note that $\mathfrak{e}_{i_j}^{L_j}=\mathfrak{e}_{i_j}$.

Consider now an element $F\in\mathrm{Tr}(\mathrm{im}(h))$. Grouping by T-orbits, we obtain

$$F=\sum_{j=1}^{m}\sum_{l=0}^{L_j-1}(\sum_{n\geq 0}F_n^{i_j,l}q^{n/N})\otimes\mathfrak{e}_{i_j}^l,$$

where $F_n^{i_j,l}=(q^{n/N}\otimes\mathfrak{e}_{i_j}^l)^\vee(F)$.

As mentioned before, the domain of h consists of elements that are component-wise invariant with respect to $|_kT\otimes\rho(T^{-1})$ or $|_lT\otimes|_{k-l}T\otimes\rho(T^{-1})$ (depending on the component). Taking products (that is, applying the map g_1 from the section on columns 3 and 4 in the diagram) we obtain elements that are component-wise invariant with respect to $|_kT\otimes\rho(T^{-1})$. Taking the trace, we then obtain elements that are invariant with respect to $|_kT\otimes\rho(T^{-1})$.

As we have seen before, FE commutes with taking the trace, and thus by uniqueness of the Fourier series expansions, we finally conclude that

$$F=\sum_{j=1}^{m}\sum_{l=0}^{L_j-1}(\sum_{n\geq 0}F_n^{i_j,l}e^{2\pi in/N}q^{n/N})\otimes\rho(T^{-1})(\mathfrak{e}_{i_j}^l).$$

By definition, we have that $\rho(T^{-1})(\mathfrak{e}_{i_j}^l)=\epsilon_{i_j}^{(l+1)}\mathfrak{e}_{i_j}^{l+1}$. Comparing coefficients we obtain

$$\epsilon_{i_j}^{(r)}F_n^{i_j,r-1}e^{2\pi in/N}=F_n^{i_j,r},$$

for every $n\geq 0$ and for $1\leq r\leq L_j$. Consequently, we have that

$$\epsilon_{i_j}^{(1)}\epsilon_{i_j}^{(2)}\ldots\epsilon_{i_j}^{(r)}e^{2\pi irn/N}F_n^{i_j,0}=F_n^{i_j,r}.$$

This tells us how to define infl.

Let $G\in\mathrm{comp}\circ\mathrm{Tr}(\mathrm{im}(h))$. Then

$$G=\sum_{j=1}^{m}(\sum_{n\geq 0}G_n^{i_j}q^{n/N})\otimes\mathfrak{e}_{i_j}.$$

We put

$$\mathrm{infl}(G)=\sum_{j=1}^{m}\sum_{l=0}^{L_j-1}(\sum_{n\geq 0}e^{2\pi iln/N}G_n^{i_j}(\prod_{s=1}^{l}\epsilon_{i_j}^{(s)})q^{n/N})\otimes\mathfrak{e}_{i_j}^l,$$

where the the empty product is taken to be 1.

We have that infl is linear and it is a left inverse by construction, and thus we have the desired isomorphism. □

On account of the above isomorphism, we also have that

$$\text{trunc.} \circ \text{Tr}(\text{im}(h)) \cong \text{trunc.} \circ \text{defl} \circ \text{Tr}(\text{im}(h)),$$

which yields a reduction in space complexity of a factor equal to

$$\frac{d}{|\rho(T^{-1})_{\text{triv}} \backslash \{e_i\}_{i=1}^{d}|},$$

since we can perform all computations in the space

$$(\frac{\mathbb{Q}^{\text{ab}}[\![q^{1/N}]\!]}{q^s \mathbb{Q}^{\text{ab}}[\![q^{1/N}]\!]})^{|\rho(T^{-1})_{\text{triv}} \backslash \{e_i\}_{i=1}^{d}|}, \text{ instead of } (\frac{\mathbb{Q}^{\text{ab}}[\![q^{1/N}]\!]}{q^s \mathbb{Q}^{\text{ab}}[\![q^{1/N}]\!]})^{d}.$$

Chapter 5

Invariant spaces

5.1 Spaces of homomorphisms

From the previous chapter we see that one of the central computational challenges is to compute bases for spaces of the form

$$H^0((\rho_{N_1}^{\times})^{\vee} \otimes (\rho_{N_2}^{\times})^{\vee} \otimes \rho)$$

rapidly.

We note that $(\rho_N^{\times})^{\vee} \cong \rho_N^{\times}$ by the orthogonality of permutation matrices, so we could just as well compute

$$H^0(\rho_{N_1}^{\times} \otimes \rho_{N_2}^{\times} \otimes \rho).$$

For the same reason, we have that the dual of a twisted permutation type remains a twisted permutation type, though not necessarily with the same twists.

Spaces of the form $H^0(\rho)$ where ρ is a twisted permutation type, or the tensor product of a permutation type with an arbitrary arithmetic type, can be computed via the orbits of $\mathrm{SL}_2(\mathbb{Z})$ acting on a set that corresponds to a basis of $V(\rho)$.

For trivially twisted permutation types, the following example shows how.

Example 5.1. Let ρ be any trivially twisted permutation type, such as for example $\rho = \rho_N^{\times}$ for some positive integer N, and let $\{e_i\}_{i=1}^d$ be a basis for $V(\rho)$.

We have that $v \in H^0(\rho)$ if and only if

$$v = \rho(S)v \text{ and } v = \rho(T)v.$$

Let now $\mathrm{SL}_2(\mathbb{Z})$ act on $\{e_i\}_{i=1}^d$ by $(g, e_i) \mapsto \rho(g)e_i$ for $g \in \mathrm{SL}_2(\mathbb{Z})$.

Suppose that we have computed the orbits under this action. Then we have a partition

$$\{e_i\}_{i=1}^d = \bigsqcup_{j=1}^{m} \mathrm{SL}_2(\mathbb{Z})h_j,$$

for some representatives $h_j \in \{e_i\}_{i=1}^d$. Since the orbits $\mathrm{SL}_2(\mathbb{Z})h_j$ are disjoint, the vectors

$$v_j = \sum_{f \in \mathrm{SL}_2(\mathbb{Z})h_j} f,$$

are linearly independent. Now since all the elements f in the sum above lie in the same orbit, they are simply permuted in the orbit by $\rho(S)$ and $\rho(T)$, and hence

$$\rho(S)v_j = v_j, \text{ and } \rho(T)v_j = v_j,$$

for every j. Hence the span of the v_j is contained in $H^0(\rho)$. We expect the reverse to also hold true, and indeed it does. To see this, we take an arbitrary $v \in H^0(\rho)$, and write

$$\mathrm{SL}_2(\mathbb{Z})h_j = \{e_{\alpha_j(1)}, e_{\alpha_j(2)}, \ldots, e_{\alpha_j(n_j)}\},$$

for some subsequence α_j of $(i)_{i=1}^d$ of length $n_j > 0$, then

$$v = \sum_{j=1}^{m} \sum_{i=1}^{n_j} c_{ij} e_{\alpha_j(i)},$$

for some coefficients c_{ij}. Let now j be arbitrary. Since all the $e_{\alpha_j(i)}$ lie in the same orbit, we have that for each $1 \leq k \leq n_j$ there exists an element $g_k \in \mathrm{SL}_2(\mathbb{Z})$ satisfying

$$\rho(g_k) e_{\alpha_j(1)} = e_{\alpha_j(k)},$$

but we also have that $\rho(g_k)v = v$, and this gives us that $c_{1,j} = c_{k,j}$ for all k. Or in other words, we can write v as

$$v = \sum_{j=1}^{m} d_j \sum_{i=1}^{n_j} e_{\alpha_j(i)},$$

for some coefficients d_j. But then v is in the span of the v_j.

Thus we conclude that computing a basis for $H^0(\rho)$ is equivalent to computing the orbits of $\{e_i\}_{i=1}^d$ with respect to the action $(g, e_i) \mapsto \rho(g)e_i$.

A similar correspondence holds for twisted permutation types, and tensor products of permutation types with arbitrary arithmetic types, with the caveat that we need to keep track of twists.

5.1.1 Invariants for twisted permutation types

In the setting of twisted permutation types ρ, we have that $\mathrm{SL}_2(\mathbb{Z})$ acts on a set of the form

$$\{ce_i : c \in \mathbb{C}, 1 \leq i \leq d\},$$

for $\{e_i\}_{i=1}^d$ a basis of $V(\rho)$. As we saw in example 5.1, for trivially twisted permutation types $\mathrm{SL}_2(\mathbb{Z})$ instead acts just on the basis.

For the purposes of making upcoming duscission easier, we introduce some notation.

Notation 5. Let ρ be a twisted permutation type with basis $V(\rho) = \{e_1, \ldots, e_d\}$. Then for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we write

$$\rho(\gamma)e_i = \rho_{\mathrm{tw}}(\gamma)(e_i)\rho_{\mathrm{triv}}(\gamma)(e_i),$$

where $\rho_{\mathrm{tw}} \in \mathbb{C}$ and ρ_{triv} is a trivially twisted permutation type.

When computing $H^0(\rho)$ we make the identification $\rho = \mathbf{1} \otimes \rho$, where $\mathbf{1}$ is the 1-dimensional trivial arithmetic type. Then we can compute $H^0(\rho)$ in the same way as $H^0(\rho_1 \otimes \rho_2)$ where ρ_1 is an arbitrary arithmetic type and ρ_2 is twisted permutation type.

5.1.2 Invariants for arbitrary arithmetic types

When ρ is an arbitrary arithmetic type, we can apply Gaußian elimination to the system

$$\begin{pmatrix} \rho(S) \\ \rho(T) \end{pmatrix} v = \begin{pmatrix} v \\ v \end{pmatrix},$$

or equivalently, to the system

$$\begin{pmatrix} \rho(S) - I \\ \rho(T) - I \end{pmatrix} v = 0.$$

This is precisely what we did in example 2.7. If ρ is a Weil type, there are fast methods for computing $H^0(\rho)$, see for example [ES17].

5.1.3 Invariants for tensor products of twisted permutation types with arbitrary arithmetic types

As we have hinted at, computing $H^0(\rho)$ for ρ a tensor product of a permutation type and an arbitrary arithmetic type combines the approach of example 5.1 and the previous section.

Specifically, let ρ_1 be an arithmetic type and ρ_2 be a twisted permutation type, say with $V(\rho_2)$ having the basis $\{e_1, \dots, e_d\}$. Recall that for an arbitrary $v \in V(\rho_1)$ we have that

$$(\rho_1 \otimes \rho_2)(\gamma)(v \otimes e_i) = \rho_{2,\mathrm{tw}}(\gamma)(e_i)\rho_1(\gamma)v \otimes \rho_{2,\mathrm{triv}}(\gamma)e_i.$$

We recall that $\rho_{2,\mathrm{triv}}(\gamma)e_i = e_{\pi_\gamma(i)}$ where π_γ is a permutation depending on γ.

We see this as $\mathrm{SL}_2(\mathbb{Z})$ acting on the linear transformations $\varphi_i : V(\rho_1) \to V(\rho_1 \otimes \rho_2)$ defined by $\varphi_i(v) = v \otimes e_i$ by

$$\gamma.\varphi_i = (\rho_{2,\mathrm{tw}}(\gamma)(e_i)\rho_1(\gamma) \otimes \mathrm{id}) \circ \varphi_{\pi_\gamma(i)}.$$

We can then compute the orbits of this action in the obvious way. Details will follow in the next section.

Analogous to the case of scalar twists, this is an action on the set

$$\{A\varphi_i : 1 \le i \le d, A \in \mathrm{GL}_d(\mathbb{C})\}.$$

Hence we view $\gamma.\varphi_i$ as an action induced by a permutation type twisted by matrices, instead of complex numbers. To make the upcoming discussion less cumbersome, we introduce the following notation.

Notation 6. Let ρ_1 be any arithmetic type type and ρ_2 a trivially twisted permutation type. Let $\rho = \rho_1 \otimes \rho_2$. Let $\{e_i\}_{i=1}^d$ be a basis for $V(\rho_2)$. Then we write

$$\rho_{\mathrm{tw}}(\gamma)(\varphi_i) = \rho_{2,\mathrm{tw}}(\gamma)(e_i)\rho_1(\gamma),$$

and

$$\rho_{\mathrm{triv}}(\gamma)(\varphi_i) = \varphi_{\pi_\gamma(i)},$$

where the permutation π_γ is defined as above.

The following proposition tells us how to compute H^0 in terms of the orbits of the aforementioned action and a criterion on some of the stabilizer subgroups.

Proposition 5.1. Let $\rho = \rho_1 \otimes \rho_2$ be an arithmetic type, with ρ_1 any arithmetic type and ρ_2 a twisted permutation type. Let $\{e_1, \dots, e_d\}$ be a basis of $V(\rho_2)$ and let $\varphi_i(v) = v \otimes e_i$. Let $\mathrm{Stab}(\varphi_i)$ denote the stabilizer subgroup with respect to the action induced by ρ_{triv}.

Let $\{h_j\}_{j=1}^m$ be a system of representatives for the orbit space coming from the action of ρ_{triv}, so that

$$\{\varphi_i\}_{i=1}^d = \bigsqcup_{j=1}^m \mathrm{SL}_2(\mathbb{Z})h_j.$$

Select subsequences α_j of $(i)_{i=1}^d$ such that $\mathrm{SL}_2(\mathbb{Z})h_j = \{e_{\alpha_j(1)}, \dots, e_{\alpha_j(n_j)}\}$ for some integer n_j. Select also elements $g_{j,i} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\rho_{2,\mathrm{triv}}(g_{j,i})e_{\alpha_j(1)} = e_{\alpha_j(i)}$. Let N_j for $1 \le j \le m$ denote the following subspace of $V(\rho_1)$

$$\{v \in V(\rho_1) : \rho_{\mathrm{tw}}(\gamma)(\varphi_{\alpha_j(1)})v = v \text{ for all } \gamma \in \mathrm{Stab}(\varphi_{\alpha_j(1)})\}.$$

Let the subset $J \subseteq \{j\}_{j=1}^m$ be defined by

$$J = \{j : \dim(N_j) > 0\}.$$

Finally, for $j \in J$, let $\{n_{1,j}, \dots, n_{\dim(N_j),j}\}$ be a basis for N_j.

Then

$$\mathcal{B}_J = \{\sum_{i=1}^{n_j} \rho(g_{j,i})\varphi_{\alpha_j(1)}(n_{k,j})\}_{\substack{j \in J \\ 1 \le k \le \dim(N_j)}},$$

forms a basis for $H^0(\rho)$.

Proof. Since $\rho(g_{j,i})$ is injective, and the elements $\varphi_{\alpha_j(1)}(n_{k,j})$ are linearly independent, it is clear that the elements in $\mathcal{B}_J$ are linearly independent, so we are done if we can show that

$$\operatorname{span}\mathcal{B}_J = H^0(\rho).$$

Let first $v \in H^0(\rho)$, and write

$$v = \sum_{j=1}^{m}\sum_{i=1}^{n_j} \varphi_{\alpha_j(i)}(w_{ij}),$$

for some vectors $w_{ij} \in V(\rho_1)$. Notice that

$$\varphi_{\alpha_j(i)}(w_{ij}) = \rho(g_{j,i})\varphi_{\alpha_j(1)}(\rho_{\mathrm{tw}}(g_{j,i}^{-1})(\varphi_{\alpha_j(i)})w_{ij}).$$

Hence, after letting $u_{ij} = \rho_{\mathrm{tw}}(g_{j,i}^{-1})(\varphi_{\alpha_j(i)})w_{ij}$, we obtain

$$v = \sum_{j=1}^{m}\sum_{i=1}^{n_j} \rho(g_{j,i})\varphi_{\alpha_j(1)}(u_{ij}).$$

Now we apply the invariance. Since $\rho(g_{j,k}^{-1})v = v$ for any k, injectivity gives that

$$u_{kj} = u_{1,j},$$

for any k. So we put $u_j = u_{1,j} = u_{2,j} = \cdots = u_{n_j,j}$ and write

$$v = \sum_{j=1}^{m}\sum_{i=1}^{n_j} \rho(g_{j,i})\varphi_{\alpha_j(1)}(u_j).$$

To obtain the inclusion of $H^0(\rho)$ into the span, we only need to show that $u_j \in N_j$. To this end, select an index $1 \leq j' \leq m$ and let $\sigma \in \mathrm{Stab}(\varphi_{\alpha_{j'}(1)})$ be arbitrary. Then $\rho(\sigma)v = v$ but also

$$\rho(\sigma)v = \rho(\sigma)\varphi_{\alpha_{j'}(1)}(u_{j'}) + \rho(\sigma)\sum_{i=2}^{n_{j'}} \rho(g_{j',i})\varphi_{\alpha_{j'}(1)}(u_{j'})$$

$$+ \rho(\sigma)\sum_{j\neq j'}\sum_{i=1}^{n_j} \rho(g_{j,i})\varphi_{\alpha_j(1)}(u_j).$$

However, it holds that

$$\rho(\sigma)\varphi_{\alpha_{j'}(1)}(u_{j'}) = \rho_{\mathrm{tw}}(\sigma)(\varphi_{\alpha_{j'}(1)})u_{j'} \otimes e_{\alpha_{j'}(1)},$$

whence by injectivity we obtain that

$$\rho_{\mathrm{tw}}(\sigma)(\varphi_{\alpha_{j'}(1)})u_{j'} = u_{j'},$$

and therefore $u_{j'} \in N_{j'}$, since σ was arbitrary. If in addition $j' \notin J$, then since $N_{j'} = \{0\}$, we have that $u_{j'} = 0$. Since j' was arbitrary, it follows that $v \in \operatorname{span}\mathcal{B}_J$.

Let now $v \in \operatorname{span}\mathcal{B}_J$. By linearity, we can assume that

$$v = \sum_{i=1}^{n_j} \rho(g_{j,i})\varphi_{\alpha_j(1)}(n_{kj}),$$

for some j and k. Select an arbitrary element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and a permutation π_γ such that

$$\rho_{\mathrm{triv}}(\gamma g_{j,i})\varphi_{\alpha_j(1)} = \rho_{\mathrm{triv}}(g_{j,\pi_\gamma(i)})\varphi_{\alpha_j(1)}.$$

Then we have that $g_{j,\pi_\gamma(i)}^{-1}\gamma g_{j,i} \in \mathrm{Stab}(\varphi_{\alpha_j(1)})$. This implies

$$\rho_{\mathrm{tw}}(g_{j,\pi_\gamma(i)}^{-1}\gamma g_{j,i})(\varphi_{\alpha_j(1)})n_{kj} = n_{kj},$$

which is equivalent to

$$\rho_{\mathrm{tw}}(\gamma g_{j,i})(\varphi_{\alpha_j(1)})n_{kj} = \rho_{\mathrm{tw}}(g_{j,\pi_\gamma(i)})(\varphi_{\alpha_j(1)})n_{kj}.$$

From this, we obtain

$$\rho(\gamma)v = \sum_{i=1}^{n_j} \rho(\gamma g_{j,i})\varphi_{\alpha_j(1)}(n_{kj}) = \sum_{i=1}^{n_j} \rho(g_{j,\pi_\gamma(i)})\varphi_{\alpha_j(1)}(n_{kj}) = v.$$

Hence $v \in H^0(\rho)$ and we are done. □

5.2 Computing a basis of invariants through an orbit-stabilizer algorithm

In this section we will retain the notation from the previous section.

Now that we understand the relation between orbits of the relevant action of $\mathrm{SL}_2(\mathbb{Z})$ on $\{\varphi_i\}_{i=1}^d$ and $H^0(\rho)$, we can compute a basis for $H^0(\rho)$ by extending the standard orbit-stabilizer algorithm. For details on algorithms of this sort and of related computational methods, the reader is advised to consult for example [HEO05, p. 79].

For completeness, we state the complete algorithm in pseudocode below.

Algorithm 2: Computing a basis through an orbit-stabilizer algorithm.

```
function BasisOrbStab (ρ, B);
  Input  : ρ = ρ_1 ⊗ ρ_2 – a tensor product of an arithmetic type ρ_1 with a trivially twisted
           permutation type ρ_2
           B – an ordered basis {e_i}_{i=1}^d for V(ρ_2)
  Output: B – basis for H^0(ρ)
U ← [φ_1, ..., φ_d];
B ← ∅;
A ← [];
while U ≠ [] do
    φ ←pop(U);
    Stab ← [];
    O_elt ← [φ];
    O_gen(φ) ← I;
    O_u ← [φ];
    while O_u ≠ [] do
        φ ←pop(O_u);
        for γ ∈ {S, T} do
            ψ ← ρ_triv(γ)φ;
            if ψ ∉ O_elt then
                push(O_elt, ψ);
                O_gen(ψ) ← γ · O_gen(φ);
                push(O_u, ψ);
                delete(U, ψ);
            else
                push(Stab, O_gen(ψ)^{-1} · γ · O_gen(φ));
            end
        end
    end
    N ← {v ∈ V(ρ_1) : ρ_tw(γ)v = v for all γ ∈ Stab};
    if dim(N) > 0 then
        N_B ← any basis of N;
        B ← B ∪ {∑_{g∈O_elt} ρ(O_gen(g))g(n) : n ∈ N_B};
    end
end
return B;
```

We now declare this thesis ended.

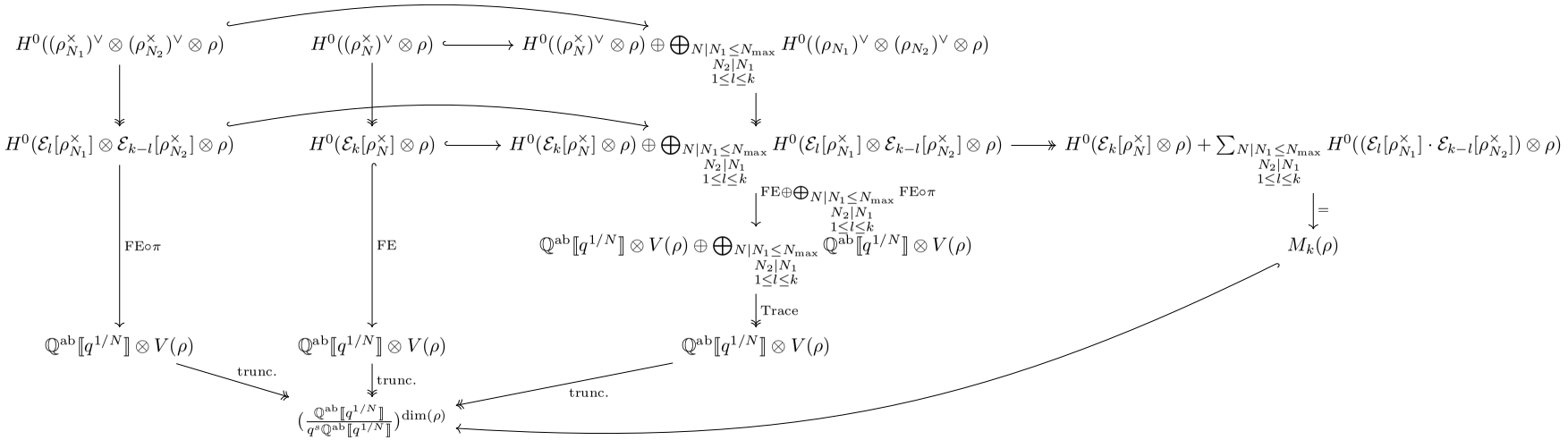

Figure 5: Commutative diagram describing connections between invariant subspaces of arithmetic types and the spaces $M_k(\rho)$.

Appendix A

The metaplectic group

A.1 Half-integer weights

Recall that $f \in M_k(\rho)$ for some representation $\rho : \Gamma \to V(\rho)$ where $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$ if and only if

$$f(\gamma.\tau) = (c\tau + d)^k \rho(\gamma) f(\tau),$$

for every $\tau \in \mathbb{H}$ and $\gamma \in \Gamma$. Or, in terms of the slash action, that $f|_k\gamma = f$ for every $\gamma \in \Gamma$. Obviously this definition is fine if $k \in \mathbb{Z}$ but if say $k \in \frac{1}{2} + \mathbb{Z}$, then we run into problems.

Clearly, we need the slash action to actually be an action, and therefore we need $(\gamma, \tau) \mapsto (c(\gamma)\tau + d(\gamma))^k$ to be a co-cycle. That is; we need

$$(c(\gamma_1\gamma_2)\tau + d(\gamma_1\gamma_2))^k = (c(\gamma_1)(\gamma_2.\tau) + d(\gamma_1))^k (c(\gamma_2)\tau + d(\gamma_2))^k,$$

for every $\gamma_1, \gamma_2 \in \Gamma$ and $\tau \in \mathbb{C}$. But this is not necessarily true if $k \in \frac{1}{2} + \mathbb{Z}$, even if we agree on a branch of the square root.

Indeed, consider the case $k = 1/2$ and recall that for example

$$1 = \sqrt{(-1)^2} \neq \sqrt{-1}^2 = -1,$$

and if $z = e(2/3)$ and $w = e(3/4)$ then also $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$, and so on. More to the point, we make the following trivial but important observation.

Observation A.1. Let $z \mapsto z^{1/2}$ be the prinicipal square root and let $z \mapsto \mathrm{Arg}(z)$ be the principal argument. Then if $z_1, z_2 \in \mathbb{C}$ satisfy $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \in (-\pi, \pi]$ we have that

$$(z_1 z_2)^{1/2} = z_1^{1/2} z_2^{1/2},$$

and otherwise

$$(z_1 z_2)^{1/2} = -z_1^{1/2} z_2^{1/2}.$$

Proof. Trivial. □

To circumvent this issue, we introduce the metaplectic group.

A.2 The metaplectic group

As a source for much of the basic theory of half-integer weight modular forms we use [Bor00], but also [CS17].

Definition A.1. We define the metaplectic group (also known as the double cover of $\mathrm{SL}_2(\mathbb{Z})$) as follows

$$\mathrm{Mp}_1(\mathbb{Z}) = \{(\gamma, \omega) : \gamma \in \mathrm{SL}_2(\mathbb{Z}), \omega : \mathbb{H} \to \mathbb{C}, \omega(\tau)^2 = c(\gamma)\tau + d(\gamma)\},$$

together with the following multiplication

$$(\gamma_1, \omega_1)(\gamma_2, \omega_2) = (\gamma_1\gamma_2, \tau \mapsto \omega_1(\gamma_2.\tau)\omega_2(\tau)).$$

There is a canonical projection from $\mathrm{Mp}_1(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\pi : \mathrm{Mp}_1(\mathbb{Z}) \ni (\gamma, \omega) \mapsto \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

From here and in the sequel we let $\sqrt{\cdot}$ and $(\cdot)^{1/2}$ denote the principal square root.

Let us state some basic facts about the metaplectic group before we continue.

Proposition A.1. It holds that

(i) $\mathrm{Mp}_1(\mathbb{Z})$ is generated by $\tilde{T} = (T, 1)$ and $\tilde{S} = (S, \tau \mapsto \sqrt{\tau})$, and

(ii) with $Z = \tilde{S}^2$ it holds that $\mathrm{Z}(\mathrm{Mp}_1(\mathbb{Z})) = \langle Z \rangle$, and $Z^4 = \mathrm{Id}$, and

(iii) $\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Mp}_1(\mathbb{Z})/\langle Z^2 \rangle$.

Notation 7. Concerning subgroups of $\mathrm{Mp}_1(\mathbb{Z})$, we write for a positive integer N

(i) $\tilde{\Gamma}_0(N) = \pi^{-1}(\Gamma_0(N))$

(ii) $\tilde{\Gamma}_1(N) = \pi^{-1}(\Gamma_1(N))$

(iii) $\tilde{\Gamma}(N) = \pi^{-1}(\Gamma(N))$

In analogy with the definitions above, we say that a subgroup $\Gamma \subseteq \mathrm{Mp}_1(\mathbb{Z})$ is a congruence subgroup if and only if $\pi(\Gamma)$ is a congruence subgroup.

We do the same for arithmetic types and congruence types. That is; an arithmetic type associated to a subgroup $\Gamma \subseteq \mathrm{Mp}_1(\mathbb{Z})$ is a complex finite-dimensional representation of Γ and a congruence type is an arithmetic type whose kernel contains $\tilde{\Gamma}(N)$ for some positive integer N.

Recall that $\omega^2(\tau) = c\tau + d$ implies that $\omega(\tau) = \pm\sqrt{c\tau + d}$. So in order to do computations in $\mathrm{Mp}_1(\mathbb{Z})$ we only need to store tuples on the form (A, r) where $r \in \{\pm 1\}$ and $A \in \mathrm{SL}_2(\mathbb{Z})$, and then figure out how the sign changes under multiplication. To help with this, we introduce a function $\sigma : \mathrm{SL}_2(\mathbb{Z}) \to \{\pm 1\}$ defined implicitly as follows.

Definition A.2. Let $(A, \sqrt{c(A)\tau + d(A)}), (B, \sqrt{c(B)\tau + d(B)}) \in \mathrm{Mp}_1(\mathbb{Z})$. We then let $\sigma(A, B) \in \{\pm 1\}$ be given by

$$(A, \sqrt{c(A)\tau + d(A)})(B, \sqrt{c(B)\tau + d(B)}) = (AB, \sigma(A, B)\sqrt{c(AB)\tau + d(AB)}).$$

Strömberg [Str13] shows that there exists an explicit formula for $\sigma(A, B)$, and more to the point; a formula that is computationally efficient.

First we need some simplifying notation. We use the same as Strömberg [Str13].

Notation 8. Let $M \in \mathrm{SL}_2(\mathbb{Z})$ and $a, b \in \mathbb{Z}$. Then we write

$$\sigma(M) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{otherwise,} \end{cases}$$

and

$$(a, b)_\infty = \begin{cases} -1 & \text{if } a < 0 \text{ and } b < 0, \\ 1 & \text{otherwise.} \end{cases}$$

The latter of these is referred to as the Hilbert symbol at infinity.

We now have the following proposition.

Proposition A.2. Let $A, B \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\sigma(A, B) = \begin{cases} (\sigma(AB)\sigma(A), \sigma(AB)\sigma(B))_\infty & \text{if } \sigma(A)\sigma(B)\sigma(AB) \neq 0 \\ (\sigma(A), \sigma(B))_\infty & \text{if } \sigma(A)\sigma(B) \neq 0 \text{ and } \sigma(AB) = 0, \text{ or } \sigma(A) = \sigma(B) = 0 \\ (-\sigma(A), \sigma(B))_\infty & \text{if } \sigma(A) \neq 0, \sigma(B) = 0 \\ (-\sigma(B), \sigma(A))_\infty & \text{if } \sigma(A) = 0, \sigma(B) \neq 0. \end{cases}$$

It is now easy to compute arbitrary products. Indeed, say $A = (\alpha, (-1)^r\sqrt{c(\alpha)\tau + d(\alpha)})$ and $B = (\beta, (-1)^s\sqrt{c(\beta)\tau + d(\beta)})$ are arbitrary elements in $\mathrm{Mp}_1(\mathbb{Z})$. Then

$$A = Z^{2r}(\alpha, \sqrt{c(\alpha)\tau + d(\alpha)}),$$

and analogously for B. Hence, since Z is in the center, we have that

$$AB = Z^{2(r+s)}(\alpha\beta, \sigma(\alpha, \beta)\sqrt{c(\alpha\beta)\tau + d(\alpha\beta)}) = (\alpha\beta, (-1)^{r+s}\sigma(\alpha, \beta)\sqrt{c(\alpha\beta)\tau + d(\alpha\beta)}),$$

and thus, using $\sigma(\alpha, \beta)$ we can compute the sign of the second component of AB, as desired.

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