



Causality in quantum field theory with classical sources

Downloaded from: <https://research.chalmers.se>, 2026-04-06 00:48 UTC

Citation for the original published paper (version of record):

Skagerstam, B., Eriksson, K., Rekdal, P. (2019). Causality in quantum field theory with classical sources. *Journal of Physics Communications*, 3(8). <http://dx.doi.org/10.1088/2399-6528/ab3c1c>

N.B. When citing this work, cite the original published paper.

TOPICAL REVIEW • OPEN ACCESS

Causality in quantum field theory with classical sources

To cite this article: Bo-Sture K Skagerstam *et al* 2019 *J. Phys. Commun.* **3** 082001

View the [article online](#) for updates and enhancements.

You may also like

- [Local Fisher- and skew-information coherence of two Nitrogen-Vacancy centers induced by two coupled dissipative nanocavities](#)
A-B A Mohamed and Abdel-Haleem Abdel-Aty
- [Thermal local Fisher information and quantum uncertainty in Heisenberg model](#)
A-B A Mohamed and H Eleuch
- [Dynamics of local quantum uncertainty for a two-qubit system under dephasing noise](#)
Cheng Yang, You-Neng Guo, Hu-Ping Peng et al.



TOPICAL REVIEW

Causality in quantum field theory with classical sources

OPEN ACCESS

RECEIVED
30 March 2019REVISED
15 July 2019ACCEPTED FOR PUBLICATION
16 August 2019PUBLISHED
30 August 2019

Original content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](#).

Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Bo-Sture K Skagerstam^{1,4} , Karl-Erik Eriksson² and Per K Rekdal³¹ Department of Physics, NTNU, Norwegian University of Science and Technology, N-7491 Trondheim, Norway² Department of Space, Earth and Environment, Chalmers University of Technology, SE-412 96 Göteborg, Sweden³ Molde University College, PO Box 2110, N-6402 Molde, Norway⁴ Author to whom any correspondence should be addressed.E-mail: bo-sture.skagerstam@ntnu.no, frtkee@chalmers.se and per.k.rekdal@himolde.no**Keywords:** causality, retardation, quantum field theory, gauge invariance, radiation processes, quantum uncertainty, classical and quantum interference**Abstract**

In an exact quantum-mechanical framework we show that space-time expectation values of the second-quantized electromagnetic fields in the Coulomb gauge in the presence of a classical conserved source automatically lead to *causal* and properly *retarded* \hbar -independent electromagnetic field strengths. The classical \hbar -independent and gauge invariant Maxwell's equations naturally emerge in terms of quantum-mechanical expectation values and are therefore also consistent with the classical special theory of relativity. The fundamental difference between interference phenomena due to the linear nature of the classical Maxwell theory as considered in, e.g., classical optics, and interference effects of quantum states is clarified. In addition to these issues, the framework outlined also provides for a simple approach to invariance under time-reversal, some spontaneous photon emission and/or absorption processes as well as an approach to Vavilov-Čerenkov radiation. The inherent and necessary quantum uncertainty, limiting a precise space-time knowledge of expectation values of the quantum fields considered, is, finally, recalled.

1. Introduction

The roles of causality and retardation in classical, \hbar -independent, and quantum-mechanical versions of electrodynamics are issues that one encounters in various contexts (for recent discussions see, e.g., [1–14]). In electrodynamics it is natural to introduce gauge-dependent scalar and vector potentials. These potentials do not have to be local in space-time. It can then be a rather delicate issue to verify that gauge-independent observables obey the physical constraint of causality and that they also are properly retarded. Attention to this and related issues are often discussed in a classical framework where one explicitly shows how various choices of gauge give rise to the same electromagnetic field strengths (see, e.g., the excellent discussion in [8]). Even though issues related to causality in physics have been discussed for many years they are still open for investigations and we are facing new insights regarding such fundamental concepts. In a recent investigation [12] the near-, intermediate-, and far-field causal properties of classical electromagnetic fields have, e.g., been discussed in great detail. In terms of experimental and theoretical considerations, locally backward velocities and apparent super-luminal features of electromagnetic fields were demonstrated. Such observations do not challenge our understanding of causality since they describe phenomena that occur behind the light front of electromagnetic signals (see, e.g., [7, 12, 14] and references cited therein).

In the present paper, we investigate the problems mentioned above, in a quantum-mechanical framework. Some aspects of this were, in fact, already considered a long time ago by Fermi [15]. Here we consider, in particular, the *finite time* and *exact* time-evolution as dictated by quantum mechanics with second-quantized electromagnetic fields in the presence of arbitrary classical conserved currents.

In terms of suitable and well-known optical quadratures (see, e.g., [16]), the corresponding \hbar -dependent dynamical equations can then be reduced to a system of decoupled harmonic oscillators with space-time dependent external forces. No pre-defined global causal order is assumed other than the deterministic

time-evolution as prescribed by the Schrödinger equation. The classical \hbar -independent theory of Maxwell then naturally emerges in terms of properly causal and retarded expectation values of the second-quantized electromagnetic field for any initial quantum state. This is in line with more general S -matrix arguments due to Weinberg [17]. Thus we demonstrate explicitly that the quantized theory of electro-magnetic fields is an adequate framework for solving a whole set of classical field-theory problems. We also clarify the fundamental role of quantum-mechanical interference in comparison to classical interference, as expressed by the linearity of Maxwell's equations.

In a straightforward manner, we have also extended this kind of presentation to a quantized theory of gravitational fluctuations around a flat Minkowski space-time, in the presence of a classical source in terms of a conserved energy-momentum tensor. From this, the weak-field limit of Einstein's general theory of relativity emerges. Our work on this was presented in a separate publication [18].

The paper is organized as follows. In section 2 we recall, for reasons of completeness, the classical version of electrodynamics in vacuum and the corresponding issues of causality and retardation in the presence of a space-time dependent source, and the extraction of a proper set of physical but non-local degrees of freedom. The exact quantum-mechanical framework approach is illustrated in terms of a second-quantized single-mode electromagnetic field in the presence of a time-dependent classical source in section 3, where emergence of the classical \hbar -independent physics is also made explicit. In section 4, the analysis of section 3 is extended to multi-modes and to a general space-time dependent classical source. The issues of causality, retardation, and time-reversal are then discussed in section 5. The framework also provides for a discussion of some radiative processes, and in section 6 we consider dipole radiation, and the famous classical Vavilov-Čerenkov radiation is reproduced in a straightforward and exact manner. In section 7, we briefly discuss the role of the intrinsic quantum uncertainty of expectation values considered. Finally, in section 8, we present conclusions and final remarks. Some multi-mode considerations as referred to in the main text are presented in an [appendix](#).

2. Maxwell's equations with a classical source

Unless stated explicitly, we often make use of the notation $\mathbf{E} \equiv \mathbf{E}(\mathbf{x}, t)$ for the electric field and similarly for other fields. The microscopic classical Maxwell's equations in vacuum are then (see, e.g., [19]):

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (2.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (2.4)$$

with the velocity of light in vacuum as given by $c = 1/\sqrt{\epsilon_0 \mu_0}$. Equations (2.1) and (2.4) imply current conservation, i.e.,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (2.5)$$

The classical Maxwell's equations can, of course, be written in a form that is explicitly covariant under Lorentz transformations but this will not be of importance here.

The general vector identity

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}, \quad (2.6)$$

applied to the electric field \mathbf{E} , making use of Maxwell's equations (2.3) and (2.4), implies that

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial \mathbf{j}}{\partial t} - \nabla \left(\frac{\rho}{\epsilon_0} \right), \quad (2.7)$$

with retarded as well as advanced solutions. By physical arguments one selects the retarded solution, even though Maxwell's equations are invariant under time-reversal as, e.g., discussed by Rohrlich [6].

We now write the electric field \mathbf{E} and the magnetic field \mathbf{B} in terms of the vector potential \mathbf{A} and the scalar potential ϕ , i.e.,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \quad (2.8)$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} . \quad (2.9)$$

The Coulomb (or radiation) gauge, which, of course, is not Lorentz covariant, is defined by the requirement

$$\nabla \cdot \mathbf{A} = 0 , \quad (2.10)$$

and therefore leads to at most two physical degrees of freedom of the electromagnetic field. \mathbf{A} defined with this gauge-choice restriction is denoted by \mathbf{A}_T . By making use of the vector identity equation (2.6) with $\mathbf{F} = \mathbf{A}_T$, Ampère's law, i.e., equation (2.3), may then be written in the form

$$\frac{\partial^2 \mathbf{A}_T}{\partial t^2} - c^2 \nabla^2 \mathbf{A}_T = \frac{\mathbf{j}_T}{\epsilon_0} , \quad (2.11)$$

where we have introduced a transverse current \mathbf{j}_T according to

$$\mathbf{j}_T \equiv \mathbf{j} - \epsilon_0 \frac{\partial}{\partial t} \nabla \phi . \quad (2.12)$$

Equation (2.11) is, of course, the well-known wave-equation for the vector potential \mathbf{A}_T in the Coulomb gauge. The transversality condition $\nabla \cdot \mathbf{j}_T = 0$ follows from charge conservation and

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi = \frac{\rho}{\epsilon_0} , \quad (2.13)$$

in the Coulomb gauge. Equation (2.13) is, therefore, not dynamical but should rather be regarded as a constraint on the physical degrees of freedom in the Coulomb gauge enforcing current conservation. The instantaneous scalar potential ϕ degree of freedom can therefore be eliminated entirely in terms of the physical charge density ρ (in this context see, e.g., [20, 21]).

In passing we also recall that in the Coulomb gauge, the scalar potential ϕ is, according to equation (2.13), given by

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} . \quad (2.14)$$

Due to the conservation of the current, i.e., equation (2.5), the time derivative of ϕ may be written in the form

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = -\frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\nabla' \cdot \mathbf{j}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} . \quad (2.15)$$

According to the well-known Helmholtz decomposition theorem $\mathbf{F} = \mathbf{F}_L + \mathbf{F}_T$ for a vector field (see, e.g., [19]), formally written in the form

$$\mathbf{F} = \frac{1}{\nabla^2} (\nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})) , \quad (2.16)$$

using equation (2.6), we can identify the corresponding longitudinal current \mathbf{j}_L , i.e.,

$$\mathbf{j}_L(\mathbf{x}, t) \equiv -\frac{1}{4\pi} \nabla \int d^3x' \frac{\nabla' \cdot \mathbf{j}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} . \quad (2.17)$$

It is now evident that the right-hand side of the wave-equation equation (2.11) for the vector potential can be expressed in terms of the current $\mathbf{j}(\mathbf{x}, t)$, i.e.,

$$\mathbf{j}_T(\mathbf{x}, t) \equiv \frac{1}{4\pi} \nabla \times (\nabla \times \int d^3x' \frac{\mathbf{j}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}) . \quad (2.18)$$

The important point here is that \mathbf{j}_T is an instantaneous and non-local function in space of the physical current $\mathbf{j}(\mathbf{x}, t)$. When the Helmholtz decomposition theorem is applied to the vector potential $\mathbf{A} = \mathbf{A}_L + \mathbf{A}_T$, it follows that the transverse part \mathbf{A}_T is gauge-invariant but, again, a non-local function in space of the vector potential \mathbf{A} .

At the classical level, we now make a normal-mode Ansatz for the real-valued vector field \mathbf{A} confined in, e.g., a cubic box with volume $V = L^3$ and with periodic boundary conditions. With $\mathbf{k} = 2\pi(n_x, n_y, n_z)/L$, where n_x, n_y, n_z are integers, we therefore write

$$\mathbf{A}_T(\mathbf{x}, t) = \sum_{\mathbf{k}\lambda} \sqrt{\frac{1}{V\epsilon_0}} (q_{\mathbf{k}\lambda}(t) \boldsymbol{\epsilon}(\mathbf{k}; \lambda) e^{i\mathbf{k}\cdot\mathbf{x}} + q_{\mathbf{k}\lambda}^*(t) \boldsymbol{\epsilon}^*(\mathbf{k}; \lambda) e^{-i\mathbf{k}\cdot\mathbf{x}}) , \quad (2.19)$$

with time-dependent Fourier components $q_{\mathbf{k}\lambda}(t)$. The, in general, complex-valued polarization vectors $\boldsymbol{\epsilon}(\mathbf{k}; \lambda)$ obey the transversality condition $\mathbf{k} \cdot \boldsymbol{\epsilon}(\mathbf{k}; \lambda) = 0$. They are normalized in such a way that

$$P_{ij} \equiv P_{ij}(\hat{\mathbf{k}}) \equiv \sum_{\lambda} \epsilon_i^*(\mathbf{k}; \lambda) \epsilon_j(\mathbf{k}; \lambda) = \delta_{ij} - \hat{k}_i \hat{k}_j , \quad (2.20)$$

where we have defined the unit vector $\hat{\mathbf{k}} \equiv \mathbf{k}/|\mathbf{k}|$. In the case of linear polarization the real-valued, orthonormal, and linear polarization unit vectors $\epsilon(\mathbf{k}; \lambda)$, with $\lambda = 1, 2$, are such that $\epsilon_i(-\mathbf{k}; \lambda) = (-1)^{\lambda+1} \epsilon_i(\mathbf{k}; \lambda)$. Since \mathbf{A}_T itself is independent of the actual realization of the polarization degrees of freedom $\epsilon(\mathbf{k}; \lambda)$, it is without any difficulty to express equation (2.19) in terms of, e.g., the complex circular polarization vectors with $\lambda = \pm$, i.e.,

$$\epsilon(\mathbf{k}; \pm) = \frac{1}{\sqrt{2}}(\epsilon(\mathbf{k}; 1) \pm i\epsilon(\mathbf{k}; 2)), \quad (2.21)$$

such that $\epsilon(-\mathbf{k}; \pm) = \epsilon^*(\mathbf{k}; \pm)$.

The Ansatz equation (2.19) for \mathbf{A}_T is, of course, consistent with transversality of the current \mathbf{j}_T in equation (2.11). Due to the transversality of \mathbf{j}_T , we can then also write that

$$\mathbf{j}_T(\mathbf{x}, t) = \sum_{\mathbf{k}\lambda} \sqrt{\frac{\epsilon_0}{V}} (j_{\mathbf{k}\lambda}(t) \epsilon(\mathbf{k}; \lambda) e^{i\mathbf{k}\cdot\mathbf{x}} + j_{\mathbf{k}\lambda}^*(t) \epsilon^*(\mathbf{k}; \lambda) e^{-i\mathbf{k}\cdot\mathbf{x}}). \quad (2.22)$$

The time-dependence of $q_{\mathbf{k}\lambda}(t)$ is now determined by the dynamical equation (2.11) for \mathbf{A}_T , i.e.,

$$\ddot{q}_{\mathbf{k}\lambda}(t) + \omega_k^2 q_{\mathbf{k}\lambda}(t) = j_{\mathbf{k}\lambda}(t), \quad (2.23)$$

with $\omega_k = c|\mathbf{k}|$. If we define classical real-valued quadratures

$$Q_{\mathbf{k}\lambda}(t) \equiv q_{\mathbf{k}\lambda}(t) + q_{\mathbf{k}\lambda}^*(t), \quad (2.24)$$

then

$$\ddot{Q}_{\mathbf{k}\lambda}(t) + \omega_k^2 Q_{\mathbf{k}\lambda}(t) = j_{\mathbf{k}\lambda}(t) + j_{\mathbf{k}\lambda}^*(t) \equiv f_{\mathbf{k}\lambda}(t). \quad (2.25)$$

This equation has the same form as the dynamical equation for a time-dependent forced harmonic oscillator. The corresponding quantum dynamics will be treated in the next session.

3. Single mode considerations

As seen in the previous section, a single mode of the electromagnetic field reduces to a dynamical system equivalent to a forced harmonic oscillator with a time-dependent external force. The quantization of such a system is well-known (see, e.g., [22–27]) and is presented here in a form suitable for illustrating a calculational procedure to be used in later sections for *finite* time intervals.

With only one mode present, we write $Q \equiv Q_{\mathbf{k}\lambda}(t)$, $\omega \equiv \omega_k$, as well as $f(t) \equiv f_{\mathbf{k}\lambda}(t)$. Equation (2.25) then takes the form

$$\ddot{Q} + \omega^2 Q = f(t). \quad (3.1)$$

This classical equation of motion can, of course, be obtained from the classical time-dependent Hamiltonian $H_{cl}(t)$ for a forced harmonic oscillator with unit mass, i.e.,

$$H_{cl}(t) = \frac{P^2}{2} + \frac{1}{2}\omega^2 Q^2 - f(t)Q. \quad (3.2)$$

We quantize this classical system by making use of the canonical commutation relation

$$[Q, P] = i\hbar. \quad (3.3)$$

We express Q and P in terms of the quantum-mechanical quadratures

$$Q = \sqrt{\frac{\hbar}{2\omega}}(a + a^*), \quad (3.4)$$

as well as

$$P = i\sqrt{\frac{\hbar\omega}{2}}(a^* - a), \quad (3.5)$$

where $[a, a^*] = 1$. The classical Hamiltonian $H_{cl}(t)$ is then promoted to the explicitly time-dependent quantum-mechanical Hamiltonian $H(t)$ according to

$$H_{cl}(t) \rightarrow H(t) = \hbar\omega\left(a^*a + \frac{1}{2}\right) + g(t)(a + a^*), \quad (3.6)$$

where we have defined

$$g(t) \equiv -f(t)\sqrt{\frac{\hbar}{2\omega}}. \quad (3.7)$$

In general, it is notoriously difficult to solve the Schrödinger equation with an explicitly time-dependent Hamiltonian. Due to the at most quadratic dependence of a and a^* in equation (3.6) it is, however, easy to solve exactly for the unitary quantum dynamics. Indeed, if one considers the dynamical evolution of the system in the interaction picture with $|\psi(t)\rangle_I \equiv \exp(itH_0/\hbar)|\psi(t)\rangle$, where we for convenience make the choice $t_0 = 0$ of initial time, then

$$i\hbar \frac{d|\psi(t)\rangle_I}{dt} = H_I(t)|\psi(t)\rangle_I. \quad (3.8)$$

For observables \mathcal{O} in the interaction picture we also have that

$$\mathcal{O}_I(t) \equiv \exp(itH_0/\hbar)\mathcal{O}\exp(-itH_0/\hbar). \quad (3.9)$$

In our case $H_0 = \hbar\omega(a^*a + 1/2)$ and therefore

$$H_I(t) = g(t)(ae^{-i\omega t} + a^*e^{i\omega t}). \quad (3.10)$$

The explicit solution for $|\psi(t)\rangle_I$ is then given by

$$|\psi(t)\rangle_I = \exp\left(\frac{i}{\hbar}\phi(t)\right)\exp\left(-\frac{i}{\hbar}\int_0^t dt' H_I(t')\right)|\psi(0)\rangle, \quad (3.11)$$

for any initial pure state $|\psi(0)\rangle$. Equation (3.11) can easily be verified by, e.g., considering the limit $\Delta t \rightarrow 0$ of $(|\psi(t + \Delta t)\rangle_I - |\psi(t)\rangle_I)/\Delta t$ using that $\exp(A + B) = \exp(A)\exp(B)\exp(-[A, B]/2)$ if $[A, B]$ is a c -number. The c -number phase $\phi(t)$ can then be computed according to

$$i\phi(t) = \frac{1}{2\hbar}\int_0^t dt' [N(t'), H_I(t')] = \frac{1}{2\hbar}\int_0^t dt' \int_0^{t'} dt'' [H_I(t''), H_I(t')], \quad (3.12)$$

with

$$N(t) \equiv \int_0^t dt' H_I(t'), \quad (3.13)$$

since, in our case, $[N(t'), H_I(t')]$ is a c -number. We therefore see that, apart from a phase, the time-evolution in the interaction picture is controlled by a conventional displacement operator as used in various studies of coherent states (see, e.g., [28, 29] and references cited therein).

The expectation value of the quantum-mechanical quadrature Q in equation (3.4) at time t , i.e., $\langle Q \rangle(t) \equiv \langle \psi(t) | Q | \psi(t) \rangle = {}_I \langle \psi(t) | Q_I(t) | \psi(t) \rangle_I$, can now easily be evaluated for an arbitrary initial pure state $|\psi(0)\rangle$ with the result

$$\langle Q \rangle(t) = \langle Q \rangle \cos(\omega t) + \frac{1}{\omega} \langle P \rangle \sin(\omega t) + \frac{1}{\omega} \int_0^t dt' f(t') \sin(\omega(t - t')), \quad (3.14)$$

where $\langle \mathcal{O} \rangle \equiv \langle \mathcal{O} \rangle(0)$ for the initial expectation value of an observable \mathcal{O} . In equation (3.14) we, of course, recognize the general classical solution of the forced harmonic oscillator equations of motion equation (3.1), i.e.,

$$\frac{d^2}{dt^2} \langle Q \rangle(t) + \omega^2 \langle Q \rangle(t) = f(t), \quad (3.15)$$

in terms of its properly retarded Green's function (see, e.g., [30]). The last term in equation (3.14) is classical in the sense that it does not depend on \hbar . Possible quantum-interference effects are hidden in the homogeneous solution of equation (3.15). Similarly, we find for the P -quadrature in equation (3.5) that $\langle P \rangle(t) = d \langle Q \rangle(t) / dt$ or more explicitly:

$$\begin{aligned} \langle P \rangle(t) &= \langle P \rangle \cos(\omega t) - \langle Q \rangle \omega \sin(\omega t) \\ &\quad + \int_0^t dt' f(t') \cos(\omega(t - t')). \end{aligned} \quad (3.16)$$

Even though the classical equation of motion emerges in terms of quantum-mechanical expectation values, intrinsic quantum uncertainty for any observable \mathcal{O} as defined by

$$(\Delta \mathcal{O})^2(t) \equiv \langle \psi(t) | (\mathcal{O} - \langle \mathcal{O} \rangle)^2 | \psi(t) \rangle, \quad (3.17)$$

is in general present. For $\mathcal{O} = Q$ one finds that

$$(\Delta Q)^2(t) = (\Delta Q)^2 \cos^2 \omega t + (\Delta P)^2 \frac{\sin^2 \omega t}{\omega^2} + \frac{\cos \omega t \sin \omega t}{\omega} \langle PQ + QP - 2 \langle Q \rangle \langle P \rangle \rangle, \quad (3.18)$$

independent of the external force $f(t)$. For minimal dispersion states, i.e., states for which $\Delta Q \Delta P = \hbar/2$, the last term is zero. For coherent states one then finds the intrinsic and time-independent quantum-mechanical uncertainty $(\Delta Q)^2(t) = \hbar/2\omega$ and $(\Delta P)^2(t) = \hbar\omega/2$.

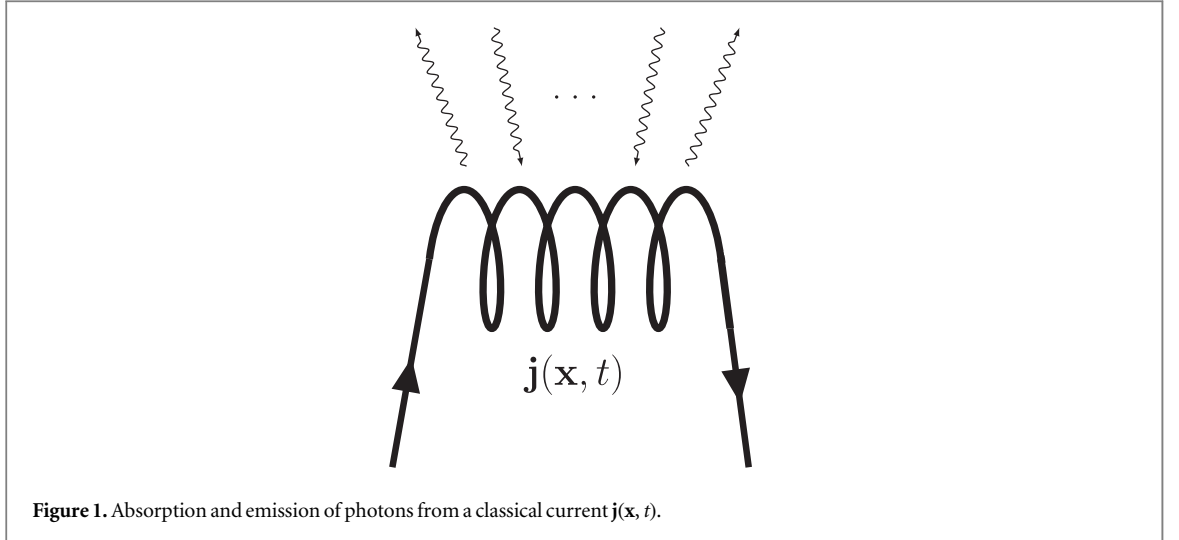


Figure 1. Absorption and emission of photons from a classical current $\mathbf{j}(\mathbf{x}, t)$.

The classical equation of motion equation (3.15) allows for linear superpositions of solutions. Such linear superposition are, however, not directly related to quantum-mechanical superpositions of the initial quantum states since expectation values are non-linear functions of quantum states. For number states $|n\rangle \equiv (a^*)^n|0\rangle/\sqrt{n!}$, with $a|0\rangle = 0$, which in terms of, e.g., a Wigner function have no classical interpretation except for the vacuum state $|0\rangle$ (see, e.g., [16]), we have that $\langle Q \rangle = \langle P \rangle = 0$ but $(\Delta Q)^2(t) = (\Delta P)^2(t)/\omega^2 = (n + 1/2)\hbar/\omega$. For an initial state of the form $|\psi(0)\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ we find that $\langle Q \rangle = \sqrt{\hbar/2\omega}$ and $\langle P \rangle = 0$ with an intrinsic time-dependent quantum uncertainty, e.g., $(\Delta Q)^2(t) = \hbar(2 - \cos^2 \omega t)/2\omega$. This initial state therefore leads to expectation values that do not correspond to a superposition of the classical solutions obtained from the initial states $|0\rangle$ or $|1\rangle$. This simple example demonstrates the fundamental difference between the role of the superposition principle in classical and in quantum physics. It is a remarkable achievement of experimental quantum optics that such quantum-mechanical interference effects between the vacuum state and a single-photon state have been observed [31, 32] (for a related discussions also see [33–37]). In the next section we extend this simple single-mode case to the general multi-mode space-time dependent situation.

4. Multi-mode considerations

We will now consider emission as well as absorption processes of photons in the presence of a general space-time dependent classical source as illustrated in figure 1. In the multi-mode case the interaction Hamiltonian $H_I(t)$ for a classical current \mathbf{j} is now an extension of the single-mode version equation (3.10). In the Coulomb gauge and in the interaction picture, we therefore consider

$$\begin{aligned} H_I(t) &= - \int_V d^3x \mathbf{j}(\mathbf{x}, t) \cdot \mathbf{A}_T(\mathbf{x}, t) \\ &= - \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_k}} (\mathbf{j}(\mathbf{k}, t) \cdot \boldsymbol{\epsilon}(\mathbf{k}; \lambda) a_{\mathbf{k}\lambda} e^{-i\omega_k t} + \mathbf{j}^*(\mathbf{k}, t) \cdot \boldsymbol{\epsilon}^*(\mathbf{k}; \lambda) a_{\mathbf{k}\lambda}^* e^{i\omega_k t}), \end{aligned} \quad (4.1)$$

with the ‘free’ field Hamiltonian

$$H_0 = \sum_{\mathbf{k}\lambda} \hbar\omega_k \left(a_{\mathbf{k}\lambda}^* a_{\mathbf{k}\lambda} + \frac{1}{2} \right). \quad (4.2)$$

Here we have introduced the Fourier transformed current

$$\mathbf{j}(\mathbf{k}, t) \equiv \int_V d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{j}(\mathbf{x}, t). \quad (4.3)$$

Since $\mathbf{j}_L(\mathbf{k}, t) = \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{j}(\mathbf{k}, t))$ and $\mathbf{j}_T(\mathbf{k}, t) = \mathbf{j}(\mathbf{k}, t) - \mathbf{j}_L(\mathbf{k}, t)$, it is clear due to the transversality condition $\mathbf{k} \cdot \boldsymbol{\epsilon}(\mathbf{k}; \lambda) = 0$ that only the transverse part of the current contributes in equation (4.1).

The interaction equation (4.1) therefore corresponds to a system of independent forced harmonic oscillators of the one-mode form as discussed in the previous section. The second-quantized version of the vector-potential in equation (2.19) then has the form of a free quantum field, i.e.,

$$\mathbf{A}_T(\mathbf{x}, t) = \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_k}} (a_{\mathbf{k}\lambda} \boldsymbol{\epsilon}(\mathbf{k}; \lambda) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_k t)} + a_{\mathbf{k}\lambda}^* \boldsymbol{\epsilon}^*(\mathbf{k}; \lambda) e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_k t)}), \quad (4.4)$$

with the basic canonical commutation relation

$$[a_{\mathbf{k}\lambda}, a_{\mathbf{k}'\lambda'}^*] = \delta_{\lambda\lambda'} \delta_{\mathbf{k}\mathbf{k}'}, \quad (4.5)$$

and where we recall that $\omega_k = c|\mathbf{k}|$. The vacuum state $|0\rangle$ is then such that $a_{\mathbf{k}\lambda}|0\rangle = 0$ for all quantum numbers $\mathbf{k}\lambda$. The quantum field \mathbf{A}_T is then normalized in such a way that

$$H_0 = \int_V d^3x \frac{1}{2} \left(\epsilon_0 \mathbf{E}_T^2(\mathbf{x}, t) + \frac{1}{\mu_0} \mathbf{B}^2(\mathbf{x}, t) \right), \quad (4.6)$$

where, for the free field in equation (4.4), we make use of $\mathbf{E}_T = -\partial\mathbf{A}_T/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}_T$.

If we consider the circular polarization vectors $\boldsymbol{\epsilon}(\mathbf{k}; \pm)$ according to equation (2.21), we have to replace the annihilation operators $a_{\mathbf{k}\lambda}$ with

$$a_{\mathbf{k}\pm} = \frac{1}{\sqrt{2}} (a_{\mathbf{k}1} \mp ia_{\mathbf{k}2}). \quad (4.7)$$

We then observe that

$$\sum_{\lambda=1,2} \boldsymbol{\epsilon}(\mathbf{k}; \lambda) a_{\mathbf{k}\lambda} = \sum_{\lambda=\pm} \boldsymbol{\epsilon}(\mathbf{k}; \lambda) a_{\mathbf{k}\lambda}, \quad (4.8)$$

which means that the quantum field \mathbf{A}_T does not depend on the actual realization of the choice of polarization degrees of freedom.

The single photon quantum states $|\mathbf{k}\lambda\rangle \equiv a_{\mathbf{k}\lambda}^*|0\rangle$, with $\lambda = \pm$, will then carry the energy $\hbar\omega_k$, momentum $\hbar\mathbf{k}$ as well as the intrinsic spin angular momentum $\pm\hbar$ along the direction $\hat{\mathbf{k}}$, i.e., the helicity quantum number of a massless spin-one particle. In passing, we remark that the latter property can be inferred from a consideration of a rotation with an angle θ around the wave-vector \mathbf{k} in terms of a rotation matrix $R_{ij}(\theta)$, which implies that $a_{\mathbf{k}\pm} \rightarrow a_{\mathbf{k}\pm}(\theta) = \exp(\pm i\theta) a_{\mathbf{k}\pm}$. In terms of the corresponding rotated polarization vectors $\boldsymbol{\epsilon}_i(\mathbf{k}; \lambda|\theta) = R_{ij}(\theta) \boldsymbol{\epsilon}_j(\mathbf{k}; \lambda)$ we then have, in accordance with equation (4.8), that

$$\sum_{\lambda=1,2} \boldsymbol{\epsilon}(\mathbf{k}; \lambda) a_{\mathbf{k}\lambda} = \sum_{\lambda=\pm} \boldsymbol{\epsilon}(\mathbf{k}; \lambda|\theta) a_{\mathbf{k}\lambda}(\theta). \quad (4.9)$$

In addition to the intrinsic spin angular momentum, photon states can also carry conventional orbital angular momentum which plays an important role in many current contexts (see, e.g., [38] and references cited therein) but will not be of concern in the present work. A complete set of physical and well-defined Fock-states can then be generated in a conventional manner. By construction, these states have positive norm avoiding the presence of indefinite norm states in manifestly covariant formulations (for some considerations see, e.g., [39–41]).

Since, obviously,

$$\int_0^t dt' H_I(t') = - \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_k}} \int_0^t dt' (a_{\mathbf{k}\lambda} e^{-i\omega_k t'} \boldsymbol{\epsilon}(\mathbf{k}; \lambda) \cdot \mathbf{j}(\mathbf{k}, t') + a_{\mathbf{k}\lambda}^* e^{i\omega_k t'} \boldsymbol{\epsilon}^*(\mathbf{k}; \lambda) \cdot \mathbf{j}^*(\mathbf{k}, t')), \quad (4.10)$$

we conclude that the time-evolution for $|\psi(t)\rangle_I$ in equation (3.11) is, apart from a phase factor, given by a multi-mode displacement operator

$$D(\boldsymbol{\alpha}) \equiv \exp\left(-\frac{i}{\hbar} \int_0^t dt' H_I(t')\right) = \prod_{\mathbf{k}\lambda} \exp(\alpha_{\mathbf{k}\lambda}(t) a_{\mathbf{k}\lambda}^* - \alpha_{\mathbf{k}\lambda}^*(t) a_{\mathbf{k}\lambda}). \quad (4.11)$$

Here $\alpha_{\mathbf{k}\lambda}(t)$ is, as inferred from equation (4.10), explicitly given by

$$\alpha_{\mathbf{k}\lambda}(t) \equiv \frac{i}{\hbar} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_k}} \int_0^t dt' e^{i\omega_k t'} \mathbf{j}^*(\mathbf{k}, t') \cdot \boldsymbol{\epsilon}^*(\mathbf{k}; \lambda), \quad (4.12)$$

with $\mathbf{j}^*(\mathbf{k}, t) = \mathbf{j}(-\mathbf{k}, t)$. The displacement operator $D(\boldsymbol{\alpha})$ has the form of a product of independent single-mode displacement operators. By making use of equation (3.11), and by considering the action on the vacuum state, the quantum-mechanical time-evolution generates a multi-mode coherent state $D(\boldsymbol{\alpha})|0\rangle$, apart from the \hbar -dependent phase $\phi(t)$ in equation (3.11). As in the single-mode case, the time-dependent expectation value of the transverse quantum field $\mathbf{A}_T(\mathbf{x}, t)$ will then obey a classical equation of motion similar to equation (3.15), i.e., (see appendix)

$$\frac{\partial^2 \langle \mathbf{A}_T(\mathbf{x}, t) \rangle}{\partial t^2} - c^2 \nabla^2 \langle \mathbf{A}_T(\mathbf{x}, t) \rangle = \frac{\mathbf{j}_T(\mathbf{x}, t)}{\epsilon_0}, \quad (4.13)$$

to be investigated in more detail in section 5. In other words, there are particular quantum states of the radiation field, namely multi-mode coherent states, which naturally lead to the classical electromagnetic fields obeying Maxwell's equations equations (2.1)–(2.4) in terms of quantum-mechanical expectation values.

5. The causality issue

The expectation value of the transverse second-quantized vector field \mathbf{A}_T is now given by equation (A.4), i.e.,

$$\langle \mathbf{A}_T(\mathbf{x}, t) \rangle = \frac{1}{\epsilon_0} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \int_0^t dt' \frac{\sin(\omega_k(t-t'))}{\omega_k} \int d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \mathbf{j}_T(\mathbf{x}', t'), \quad (5.1)$$

where we have carried out a sum over polarizations according to equation (2.20) as in equation (A.5), and where the sum over \mathbf{k} in the large volume V limit is replaced by

$$\sum_{\mathbf{k}} = \frac{V}{(2\pi)^3} \int d^3k. \quad (5.2)$$

The Fourier transform of the transverse current vector in equation (5.1) is as above given by

$$\mathbf{j}_T(\mathbf{k}, t) \equiv \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{j}_T(\mathbf{x}, t). \quad (5.3)$$

The time-derivative of equation (5.1) can now be written in the form

$$\frac{\partial \langle \mathbf{A}_T(\mathbf{x}, t) \rangle}{\partial t} = \frac{1}{\epsilon_0} \int d^3x' \int_0^t dt' \frac{\partial}{\partial t} G(\mathbf{x} - \mathbf{x}', t - t') (\mathbf{j}(\mathbf{x}', t') - \mathbf{j}_L(\mathbf{x}', t')), \quad (5.4)$$

by making use of the Helmholtz decomposition of the current vector $\mathbf{j}(\mathbf{x}, t)$, and where we identified the Green's function $G(\mathbf{x}, t)$

$$\begin{aligned} G(\mathbf{x}, t) &\equiv \lim_{V \rightarrow \infty} \frac{1}{V} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin(\omega_k t)}{\omega_k} = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin(\omega_k t)}{\omega_k} \\ &= \frac{1}{4\pi c^2 |\mathbf{x}|} (\delta(t - |\mathbf{x}|/c) - \delta(t + |\mathbf{x}|/c)). \end{aligned} \quad (5.5)$$

This Green's function is a solution to the homogeneous wave-equation

$$\frac{\partial^2 G(\mathbf{x}, t)}{\partial t^2} = c^2 \nabla^2 G(\mathbf{x}, t), \quad (5.6)$$

such that $G(\mathbf{x}, t) = 0$ and $\partial G(\mathbf{x}, t)/\partial t = 0$ at $t = 0$. For the second term in equation (5.4) we need to consider the integral

$$\mathbf{I} \equiv \int d^3x' \int_0^t dt' \frac{\partial}{\partial t} G(\mathbf{x} - \mathbf{x}', t - t') \nabla' \frac{\partial}{\partial t'} \phi(\mathbf{x}', t'). \quad (5.7)$$

This is so since the longitudinal vector current $\mathbf{j}_L(\mathbf{x}, t)$ may be written in the form

$$\mathbf{j}_L(\mathbf{x}, t) = \frac{\partial}{\partial t} \frac{1}{4\pi} \nabla \int d^3x' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} = \epsilon_0 \frac{\partial}{\partial t} \nabla \phi(\mathbf{x}, t), \quad (5.8)$$

where we make use of the Helmholtz decomposition equation (2.17) and current conservation. After a partial integration in the time variable t' and by making use of equation (5.6), the integral \mathbf{I} can therefore be written in the following form

$$\mathbf{I} = \nabla \phi(\mathbf{x}, t) + c^2 \int d^3x' \int_0^t dt' \nabla' \cdot \nabla' G(\mathbf{x} - \mathbf{x}', t - t') \nabla' \phi(\mathbf{x}', t'), \quad (5.9)$$

where we have used the fact that $\partial G(\mathbf{x}, t)/\partial t = \delta^3(\mathbf{x})$ at $t = 0$ as well as the initial condition $\mathbf{j}_L(\mathbf{x}, 0) = 0$ for all \mathbf{x} . We now perform two partial integrations over the spatial variable and by using equation (2.13), we finally see that

$$\mathbf{I} = \nabla \phi(\mathbf{x}, t) - \frac{c^2}{\epsilon_0} \int d^3x' \int_0^t dt' G(\mathbf{x} - \mathbf{x}', t - t') \nabla' \rho(\mathbf{x}', t'), \quad (5.10)$$

neglecting spatial boundary terms and using the initial condition $\rho(\mathbf{x}, 0) = 0$ for all \mathbf{x} . The first term in equation (5.10) exactly cancels the instantaneous Coulomb potential contribution in the expectation value of the quantized electric field observable

$$\langle \mathbf{E}(\mathbf{x}, t) \rangle = -\frac{\partial \langle \mathbf{A}_T(\mathbf{x}, t) \rangle}{\partial t} - \nabla \phi(\mathbf{x}, t). \quad (5.11)$$

For $t > t'$, we therefore obtain the desired result

$$\langle \mathbf{E}(\mathbf{x}, t) \rangle = -\frac{\partial}{\partial t} \left(\frac{1}{4\pi\epsilon_0 c^2} \int d^3x' \frac{\mathbf{j}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} \right) - \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\nabla' \rho(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|}, \quad (5.12)$$

where $\nabla' \rho(\mathbf{x}', t')$ in equation (5.12) has to be evaluated for a fixed value of $t' = t - |\mathbf{x} - \mathbf{x}'|/c$. In a similar manner we also see that

$$\begin{aligned} \langle \mathbf{B}(\mathbf{x}, t) \rangle &= \nabla \times \langle \mathbf{A}_T(\mathbf{x}, t) \rangle = \frac{1}{\epsilon_0} \int d^3x' \int_0^t dt' G(\mathbf{x} - \mathbf{x}', t - t') \nabla' \times \mathbf{j}_T(\mathbf{x}', t') \\ &= \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \nabla' \times \mathbf{j}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c), \end{aligned} \quad (5.13)$$

since $\nabla' \times \mathbf{j}_T(\mathbf{x}', t') = \nabla' \times \mathbf{j}(\mathbf{x}', t')$. In equation (5.13), we remark again that $\nabla' \times \mathbf{j}(\mathbf{x}', t')$ has to be evaluated for a fixed value of $t' = t - |\mathbf{x} - \mathbf{x}'|/c$. The causal and properly retarded form of the electric and magnetic quantum field expectation values in terms of the physical and local sources given have therefore been obtained (see in this context, e.g., [19], section 6.5).

The expectation values as given by equations (5.12) and (5.13) obey Maxwell's equations in terms of the classical charge density ρ and current \mathbf{j} . The quantization procedure above of the electromagnetic field explicitly breaks Lorentz covariance. Since, however, Maxwell's equations transform covariantly under Lorentz transformations we can, nevertheless, now argue that the special theory of relativity emerges in terms of expectation values of gauge-invariant second-quantized electromagnetic fields.

Maxwell's equations of motion according to equations (2.1)–(2.4) are invariant under the discrete time-reversal transformation $t \rightarrow t' = -t$ with $\mathbf{E}(\mathbf{x}, t) \xrightarrow{\mathcal{T}} \mathbf{E}'(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}, -t)$ and $\mathbf{B}(\mathbf{x}, t) \xrightarrow{\mathcal{T}} \mathbf{B}'(\mathbf{x}, t) = -\mathbf{B}(\mathbf{x}, -t)$ provided $\mathbf{j}(\mathbf{x}, t) \xrightarrow{\mathcal{T}} \mathbf{j}'(\mathbf{x}, t) = -\mathbf{j}(\mathbf{x}, -t)$ and $\rho(\mathbf{x}, t) \xrightarrow{\mathcal{T}} \rho'(\mathbf{x}, t) = \rho(\mathbf{x}, -t)$. At the classical level, the corresponding transverse vector potential transforms according to $\mathbf{A}_T(\mathbf{x}, t) \xrightarrow{\mathcal{T}} \mathbf{A}'_T(\mathbf{x}, t) = -\mathbf{A}_T(\mathbf{x}, -t)$. The anti-unitary time-reversal transformation \mathcal{T} is implemented on second-quantized fields in the interaction picture according to the rule (see, e.g., [20, 42–44])

$${}_I \langle \psi(t) | \mathbf{A}_T(\mathbf{x}, t) | \psi(t) \rangle_I \xrightarrow{\mathcal{T}} {}_I \langle \psi(t) | \mathbf{A}'_T(\mathbf{x}, t) | \psi(t) \rangle_I = {}_I \langle \psi(-t) | \mathcal{T} \mathbf{A}_T(\mathbf{x}, t) \mathcal{T}^{-1} | \psi(-t) \rangle_I. \quad (5.14)$$

It then follows that ${}_I \langle \psi(t) | \mathbf{A}'_T(\mathbf{x}, t) | \psi(t) \rangle_I = -{}_I \langle \psi(-t) | \mathbf{A}_T(\mathbf{x}, -t) | \psi(-t) \rangle_I$ if $\mathcal{T} a_{\mathbf{k}\lambda} \mathcal{T}^{-1} = a_{-\mathbf{k}\lambda} (-1)^\lambda$ and provided the vacuum state $|0\rangle$ is invariant under time-reversal. We therefore find that $\langle \mathbf{A}'_T(\mathbf{x}, t) \rangle = -\langle \mathbf{A}_T(\mathbf{x}, -t) \rangle$. We therefore obtain $\langle \mathbf{E}(\mathbf{x}, t) \rangle \xrightarrow{\mathcal{T}} \langle \mathbf{E}(\mathbf{x}, -t) \rangle$ and $\langle \mathbf{B}(\mathbf{x}, t) \rangle \xrightarrow{\mathcal{T}} -\langle \mathbf{B}(\mathbf{x}, -t) \rangle$ as it should. Due to the form of the Green's function $G(\mathbf{x}, t)$ in equation (5.5) it can be verified that equation (5.1) also leads to expectation values $\langle \mathbf{E}(\mathbf{x}, t) \rangle$ and $\langle \mathbf{B}(\mathbf{x}, t) \rangle$ that transform correctly under time-reversal.

The arrow of time can therefore, as expected, not be explained by our approach but as soon as the direction of time is defined the observable quantities $\langle \mathbf{E}(\mathbf{x}, t) \rangle$ and $\langle \mathbf{B}(\mathbf{x}, t) \rangle$ are causal and properly retarded. In the presence of external sources we could have an apparent breakdown of time-reversal invariance unless one also time-reverses the external sources.

6. Electromagnetic radiation processes

The rate for spontaneous emission of a photon from, e.g., an excited hydrogen atom can now be obtained in a straightforward manner in terms of a slight extension of the interaction equation (4.10) as to be made use of in first-order time-dependent perturbation theory. We then make use of the long wave-length approximation

$$\mathbf{j}(\mathbf{k}, t) \approx \int d^3x \mathbf{j}(\mathbf{x}, t) = \frac{\partial}{\partial t} \int d^3x \mathbf{x} \rho(\mathbf{x}, t) = q \frac{d}{dt} \mathbf{x}(t), \quad (6.1)$$

taking current conservation equation (2.5) into account, where $\rho(\mathbf{x}, t) = q\delta^{(3)}(\mathbf{x} - \mathbf{x}(t))$ in terms of the position $\mathbf{x}(t)$ of the charged electron in the interaction picture. For the spontaneous single photon transition $|i\rangle \rightarrow |f\rangle$ with $|i\rangle = |a_i\rangle \otimes |0\rangle$ and $|f\rangle = |a_f\rangle \otimes |\mathbf{k}\lambda\rangle$, we then arrive at the standard dipole radiation first-order matrix element

$$\langle f | H_I(0) | i \rangle = \text{i.e.} \sqrt{\frac{\hbar}{2V\epsilon_0\omega_k}} \omega_{if} \boldsymbol{\epsilon}^*(\mathbf{k}; \lambda) \cdot \langle a_f | \mathbf{x} | a_i \rangle, \quad (6.2)$$

using equation (4.10) with $q = -e$ in the interaction picture. The relevant matrix element $\langle a_f | \mathbf{x}(t) | a_i \rangle$ is then given by $\exp(-i\omega_{if}t) \langle a_f | \mathbf{x} | a_i \rangle$. For the atomic transition from $|a_i\rangle = |nlm\rangle = |2pm\rangle$ to the final atomic ground state $|a_f\rangle = |1s\rangle$, the corresponding rate is then given by

$$\Gamma \equiv \frac{2\pi}{\hbar^2} \sum_{\mathbf{k}\lambda} \delta(\omega_k - \omega_{if}) |\langle i|H_I(0)|f\rangle|^2 = \left(\frac{2}{3}\right)^8 \alpha^4 \frac{c}{a_B}, \quad (6.3)$$

in terms of the fine-structure constant $\alpha \equiv e^2/4\pi\epsilon_0\hbar c$ and the Bohr radius a_B . The rate Γ is independent of the quantum number m . Stimulated emission gives rise to a multiplicative factor $(1 + n_{\mathbf{k}\lambda})$. Equation (6.3) is, of course, a well-known text-book result in agreement with the experimental value (see, e.g., [45]). The considerations above can be extended to graviton quadrupole radiation processes in an analogous manner [18].

The power of electromagnetic emission from a classical conserved electric current in, e.g., a non-dissipative dielectric medium and the famous Vavilov-Čerenkov [46] radiation can, furthermore, now also be derived in terms of the quantum-mechanical framework above. This form of radiation was first explained by Frank and Tamm [47] using the framework of Maxwell's classical theory of electromagnetism. The exact classical \hbar -independent expression for the power of Vavilov-Čerenkov radiation (see, e.g., Section 13.4 in [19]), neglecting possible spin effects to be discussed elsewhere [48], can now be obtained as follows. For a particle with electric charge q , mass m , and an initial velocity \mathbf{v} , moving in dielectric medium such that $\epsilon_0 \rightarrow \epsilon\epsilon_0$, with $\epsilon > 1$, the interaction $H_I(t)$ in equation (4.10) leads to a displacement operator $D(\alpha)$ (see equation (4.11)) with $\alpha_{\mathbf{k}\lambda}(t)$ now replaced by

$$\begin{aligned} \alpha_{\mathbf{k}\lambda}(t) &= \frac{i}{\hbar} \sqrt{\frac{\hbar}{2V\epsilon\epsilon_0\omega_k}} \int_{t_0}^t dt' e^{i\omega_k t'} \int d^3x \mathbf{j}(\mathbf{x}, t') \cdot \boldsymbol{\epsilon}^*(\mathbf{k}; \lambda) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= q \frac{i}{\hbar} \sqrt{\frac{\hbar}{2V\epsilon\epsilon_0\omega_k}} \mathbf{v} \cdot \boldsymbol{\epsilon}^*(\mathbf{k}; \lambda) \int_{t_0}^t dt' e^{i(\omega_k - \mathbf{k}\cdot\mathbf{v})t'}, \end{aligned} \quad (6.4)$$

where the relativistic current $\mathbf{j}(\mathbf{x}, t)$ in an inertial frame is given by

$$\mathbf{j}(\mathbf{x}, t) = q\mathbf{v}\delta^{(3)}(\mathbf{x} - \mathbf{v}t). \quad (6.5)$$

The power $P(\omega)d\omega$ of emitted radiation in the range ω to $\omega + d\omega$ is then obtained by evaluating the exact expression $d\langle H_0\rangle(t)/dt$, using equation (6.4), where

$$\frac{d\langle H_0\rangle(t)}{dt} = \frac{d}{dt} \sum_{\mathbf{k}\lambda} |\alpha_{\mathbf{k}\lambda}(t)|^2 \equiv \int_0^{\omega_c} d\omega P(\omega), \quad (6.6)$$

in the large volume V limit, and by considering the large $T \equiv t - t_0$ limit. The cut-off angular frequency $\omega_c \ll mc^2/\hbar$ is to be determined in a standard manner taken the ω -dependence of ϵ into account (see, e.g., [19]). Here we can, of course, disregard the additive divergent zero-point fluctuations in $\langle H_0\rangle(t)$. The large T limit leads to a phase-matching condition $\omega_k = \mathbf{k} \cdot \mathbf{v} = vk \cos \theta_C$, using $v \equiv |\mathbf{v}|$ and $k \equiv |\mathbf{k}|$, expressed in terms of the well-known Čerenkov angle $\cos \theta_C \equiv c/nv$, where $\omega_k = ck/n$ with the refractive index $n \equiv \sqrt{\epsilon}$. The λ -sum over the polarization degrees of freedom in equation (6.6) leads to

$$\sum_{\lambda=1,2} |(\mathbf{v} \cdot \boldsymbol{\epsilon}(\mathbf{k}; \lambda))|^2 = |\mathbf{v}|^2 (1 - \cos^2 \theta), \quad (6.7)$$

using equation (2.20), where, in general, $\cos \theta \equiv \hat{\mathbf{v}} \cdot \hat{\mathbf{k}}$ in terms of the unit vectors. In summing over the angular distribution of the radiation emitted in equation (6.6), the large T phase-matching condition is taken into account. We then easily find the well-known \hbar -independent power spectrum

$$P(\omega) = \frac{e^2}{4\pi\epsilon_0 c} \frac{v}{c} \omega (1 - \cos^2 \theta_C). \quad (6.8)$$

Alternatively, but in a less rigorous manner, one may consider $\langle H_0\rangle(t)/T$ and make use of equation (6.4) in the large T -limit, i.e.,

$$\alpha_{\mathbf{k}\lambda}(t) = 2\pi q \frac{i}{\hbar} \sqrt{\frac{\hbar}{2V\epsilon\epsilon_0\omega_k}} \delta(\omega_k - \mathbf{k} \cdot \mathbf{v}) \mathbf{v} \cdot \boldsymbol{\epsilon}^*(\mathbf{k}; \lambda). \quad (6.9)$$

By inspection we then observe that $\alpha_{\mathbf{k}\lambda}(t)$ in equation (6.9) exactly corresponds the quantum-mechanical amplitude for the emission of one photon from the source to first-order in time-dependent perturbation theory even though our expression for $\alpha_{\mathbf{k}\lambda}(t)$ is exact.

We have therefore derived a power spectrum that exactly corresponds to the 1937 Frank-Tamm expression [47] in terms of the Čerenkov angle $\cos \theta_C$ as obtained from the δ -function constraint in equation (6.9). In the quantum-mechanical perturbation theory language this constraint corresponds to an energy-conservation δ -function as a well as to conservation of momentum taking the refractive index $n \equiv \sqrt{\epsilon}$ into account. The corresponding energy of the emitted photon is then given by $E_\gamma = \hbar\omega$ and the Minkowski canonical momentum by $\mathbf{p}_\gamma = \hbar\mathbf{k}$ (see, e.g., [49]), with $\omega = c|\mathbf{k}|/n$. The expression for the Čerenkov angle $\cos \theta_C$ is then modified according to [50]

$$\cos \theta_C = \frac{c}{nv} \left(1 + \hbar \omega_k (n^2 - 1) \frac{\sqrt{1 - v^2/c^2}}{2mc^2} \right). \quad (6.10)$$

As was first noted by Ginzburg ([50] and references cited therein), and also presented in various text-books accounts (see, e.g., [51, 52]), first-order perturbation theory in quantum mechanics actually leads to the same exact power spectrum for Vavilov-Čerenkov radiation. The explanation of this curious circumstance can be traced back to the fact that all higher order corrections are taken into account by the presence of the phase $\phi(t)$ in equation (3.11).

7. Quantum uncertainty

The displacement of quantum states as induced by $D(\alpha)$, as defined in equation (4.11), acting on an arbitrary pure initial state again leads to Maxwell's equations for the expectation value of the quantum field changing, at most, the homogeneous solution of the expectation value of the wave-equation (4.13). The corresponding quantum uncertainty of $\mathbf{E}(\mathbf{x}, t)$, however, depends on the choice of the initial state along the same reasoning as in the single-mode case in section 3. An essential and additional ingredient with regard to the approach to the classical limit is to consider the variance of, e.g., the second-quantized $\mathbf{E}(\mathbf{x}, t)$ -field suitably defined. We consider the scalar quantity

$$(\Delta \mathbf{E}(\mathbf{x}, t))^2 \equiv \langle \mathbf{E}^2(\mathbf{x}, t) \rangle - \langle \mathbf{E}(\mathbf{x}, t) \rangle^2. \quad (7.1)$$

We observe that the uncertainty in equation (7.1) in general does not depend on the complex parameters α when evaluated for the displaced state $D(\alpha)|\psi(0)\rangle$ and is therefore determined by the uncertainty as determined by the initial state $|\psi(0)\rangle$.

In order to be specific, we will evaluate the uncertainty $\Delta \mathbf{E}(\mathbf{x}, t)$ for a displaced Fock state with $|\psi(0)\rangle = |..., n_{\mathbf{k}\lambda}, \dots, n_{\mathbf{k}'\lambda'}, \dots\rangle$. We then obtain

$$(\Delta \mathbf{E}(\mathbf{x}, t))^2 = \frac{\hbar}{V\varepsilon_0} \sum_{\mathbf{k}\lambda} \omega_k \left(n_{\mathbf{k}\lambda} + \frac{1}{2} \right). \quad (7.2)$$

Physical requirements now demand that the uncertainty $\Delta \mathbf{E}(\mathbf{x}, t)$ must be smaller than expectation values of the components of the second-quantized electromagnetic field $\mathbf{E}(\mathbf{x}, t)$. If the sum in equation (7.2) had been convergent, the variance would have vanished in the naive limit $\hbar \rightarrow 0$. Since the natural constant \hbar is non-zero, the sum in equation (7.2) is, however, divergent.

Even though the expectation value of the quantum field at a space-time point (\mathbf{x}, t) in our case is well-defined, the corresponding uncertainty is therefore actually divergent. This means that the observable value of the quantum field in a space-time point (\mathbf{x}, t) is physically ill-defined. In the early days of quantum field theory, this fact was actually noticed already in 1933 by Bohr and Rosenfeld [53] and later proved in a rigorous manner by Wightman [54]. Bohr and Rosenfeld also provided a solution of this apparent physical contradiction. The basic idea is to introduce quantum field observables averaged over some finite space-time volume. Bohr and Rosenfeld made use of a cube centered at the space-point \mathbf{x} at a fixed time t which, however, makes some of the expressions obtained rather complicated. We will follow another approach which makes the expressions more tractable (see, e.g., problem 2.3 in [20]), i.e., we consider

$$\mathbf{E}_\sigma(\mathbf{x}, t) \equiv \int_{-\infty}^{\infty} dt' \int_V d^3x' f_s(\mathbf{x} - \mathbf{x}') f_t(t - t') \mathbf{E}(\mathbf{x}', t'), \quad (7.3)$$

where

$$f_s(\mathbf{x}) = \frac{1}{(2\pi\sigma_s^2)^{3/2}} \exp\left(-\frac{\mathbf{x}^2}{2\sigma_s^2}\right), \quad (7.4)$$

and

$$f_t(t) = \frac{1}{(2\pi\sigma_t^2)^{1/2}} \exp\left(-\frac{t^2}{2\sigma_t^2}\right). \quad (7.5)$$

The parameter σ_s gives a characteristic scale for the space-volume around the point \mathbf{x} where we perform the space average. Correspondingly, the parameter σ_t gives a characteristic time-scale for the time average procedure.

The linear classical Maxwell's equations can then again be obtained as in the previous sections in terms of the quantum-mechanical average of fields like $\mathbf{E}_\sigma(\mathbf{x}, t)$ provided that the classical sources are space and/or time averaged in the same manner. It now follows that

$$\mathbf{E}_\sigma(\mathbf{x}, t) = \sum_{\mathbf{k}\lambda} i \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2V\epsilon_0}} \exp\left(-\frac{\sigma_s^2 \mathbf{k}^2}{2} - \frac{\sigma_t^2 \omega_{\mathbf{k}}^2}{2}\right) (a_{\mathbf{k}\lambda} \epsilon(\mathbf{k}; \lambda) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}}t)} - a_{\mathbf{k}\lambda}^* \epsilon^*(\mathbf{k}; \lambda) e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}}t)}). \quad (7.6)$$

The variance $(\Delta\mathbf{E}_\sigma(\mathbf{x}, t))^2$ of the space and time averaged electric quantum field $\mathbf{E}_\sigma(\mathbf{x}, t)$, in a sufficiently large quantization volume V , will then be finite and corresponds to an energy

$$\epsilon_0(\Delta\mathbf{E}_\sigma(\mathbf{x}, t))^2 \sigma^3 \geq E_\sigma \equiv \frac{1}{(2\pi)^3} \frac{2\pi\hbar c}{\sigma}, \quad (7.7)$$

localized in a volume $V_\sigma \equiv \sigma^3 \ll V$, where

$$\sigma^2 \equiv \sigma_s^2 + c^2 \sigma_t^2. \quad (7.8)$$

It is now clear that E_σ will be finite in the cases $\sigma_t = 0$, $\sigma_s \neq 0$ as well as $\sigma_s = 0$, $\sigma_t \neq 0$.

Physically, the expression in equation (7.7) corresponds, apart from an irrelevant numerical factor, to the energy of a photon with a wave-length $\lambda \simeq \sigma$ and, hence, a wave-number $k \simeq 2\pi/\sigma$ and therefore to an energy $E_\sigma \simeq \hbar ck \simeq 2\pi\hbar c/\sigma$, in a typical localization volume V_σ . It is now clear that E_σ will tend to infinity as $\sigma \rightarrow 0$, i.e., we would then obtain an arbitrarily large energy and/or energy density if we try to localize the quantum field in the sense above in an arbitrary small V_σ . A macroscopic field, however, corresponds to a localization volume much larger than V_σ , and therefore these quantum uncertainties can be disregarded in the classical regime.

This latter feature can be illustrated by evaluating $(\Delta\mathbf{E}_\sigma(\mathbf{x}, t))^2$ for a thermal Planck distribution of $n_{\mathbf{k}\lambda}$ at a temperature T with a typical coherence length scale $\sigma_T \equiv \hbar c/k_B T$. For localization scales $\sigma \ll \sigma_T$, i.e., at sufficiently small temperatures, one then finds that $\epsilon_0(\Delta\mathbf{E}_\sigma(\mathbf{x}, t))^2 \sigma^3 = E_\sigma(1 + 4\pi^4(\sigma/\sigma_T)^4/15)$. The thermal induced uncertainty can therefore be neglected in comparison with E_σ for large thermal coherence lengths σ_T as compared to σ . If, on the other hand, $\sigma_T \leq \sigma$, i.e., at sufficiently high temperatures, it follows that $\epsilon_0(\Delta\mathbf{E}_\sigma(\mathbf{x}, t))^2 \sigma^3 = k_B T(1 + (\sigma_T/\sigma)^2/8)/4\pi^{3/2}$ and, as expected, the thermal uncertainty will then be dominating at sufficiently high temperatures.

As was predicted a long time ago for single-mode quantum fields [55], it is possible to reduce the uncertainty $(\Delta\mathbf{E}(\mathbf{x}, t))^2$ below the vacuum value by making use of initial squeezed quantum states $|\psi(0)\rangle$. This feature has recently been confirmed experimentally ([56] references cited therein). For multi-mode considerations, relevant for the framework of the present work, this may also be possible for $(\Delta\mathbf{E}_\sigma(\mathbf{x}, t))^2$ but this will not be a topic in the present paper.

8. Final remarks

We have shown how a quantum-mechanical framework offers a good platform to analyze causality and retardation issues in the classical theory of Maxwell. As we have shown elsewhere, our quantum-mechanical framework can rather easily be extended to a derivation of the weak-field limit of Einstein's general theory of relativity [18]. From second-quantization of the physical degrees of freedom under the condition of current conservation the well established classical theory for electromagnetism naturally emerges. The overwhelming experimental support for Maxwell's classical theory does not necessarily imply the existence of photons and doubts on the existence of such quantum states are sometimes put forward (see, e.g., [57]). However, the quantum-mechanical derivation of the classical theory necessarily implies the existence of single particle quantum states corresponding to a photon.

We have also observed that various radiation processes including the classical, i.e., \hbar -independent, Vavilov-Čerenkov radiation can be obtained in a straightforward manner. It may come as a surprise that a first-order quantum-mechanical perturbation theory calculation can give an exact \hbar -independent answer. This, as it seems, remarkable fact is explained by the factorization of the time-evolution operator in terms of a displacement operator for quantum states in the interaction picture according to equation (3.11) making use of equation (4.10). The phase $\phi(t)$ then contains the non-perturbative effects of all higher-order corrections to the first-order result.

As a matter of fact, similar features are known to occur also in some other situations. As is well-known, the famous differential cross-section for Rutherford scattering can be obtained exactly in terms of the first-order Born approximation. All higher order corrections will then contribute with an overall phase for probability amplitudes which follows from the exact solution (see, e.g., the excellent discussion in [58]). The classical Thomson cross-section for low-energy light scattering on a charged particle is also exactly obtained from a Born approximation due to the existence of an exact low-energy theorem in quantum electrodynamics (see, e.g., the discussions in [2, 3, 59]).

Acknowledgments

The research by B.-S. S. was supported in part by Molde University College and NTNU. K.-E. E. gratefully acknowledges hospitality at Chalmers University of Technology. The research of P. K. R. was supported by Molde University College. The authors are grateful for the hospitality provided for at Department of Space, Earth and Environment, Chalmers University of Technology, Sweden, and Molde University College, Norway, during the finalization of the present work. The authors also appreciate helpful comments from anonymous referees.

Appendix. expectation value of the quantum field $A_T(\mathbf{x}, t)$

Apart from a phase-factor, the time-evolution in the interaction picture is controlled by the operator

$$\exp\left(-\frac{i}{\hbar} \int_0^t dt' H_I(t')\right) = \prod_{\mathbf{k}\lambda} \exp(\alpha_{\mathbf{k}\lambda}(t) a_{\mathbf{k}\lambda}^* - \alpha_{\mathbf{k}\lambda}^*(t) a_{\mathbf{k}\lambda}), \quad (\text{A.1})$$

where $\alpha_{\mathbf{k}\lambda}(t)$ is given by equation (4.12) in the main text. Since expectation values are independent of the picture used, i.e.,

$${}_S\langle\psi(t)|\mathcal{O}|\psi(t)\rangle_S = {}_I\langle\psi(t)|\mathcal{O}_I(t)|\psi(t)\rangle_I \equiv \langle\mathcal{O}_I(t)\rangle, \quad (\text{A.2})$$

we see that

$$\langle A_T(\mathbf{x}, t) \rangle = \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2V\omega_k \epsilon_0}} (\epsilon(\mathbf{k}; \lambda) \alpha_{\mathbf{k}\lambda}(t) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_k t} + \epsilon^*(\mathbf{k}; \lambda) \alpha_{\mathbf{k}\lambda}^*(t) e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega_k t}). \quad (\text{A.3})$$

If we, in particular, consider initial states $|\psi(0)\rangle$ such that $\langle\psi(0)|a_{\mathbf{k}\lambda}|\psi(0)\rangle = 0$ for all $\mathbf{k}\lambda$, like quantum states with a fixed number of photons, we find that

$$\begin{aligned} \langle A_T(\mathbf{x}, t) \rangle &= \sum_{\mathbf{k}\lambda} \frac{i}{2V\omega_k \epsilon_0} \left(\epsilon(\mathbf{k}; \lambda) \int_0^t dt' e^{i\omega_k(t'-t) + i\mathbf{k}\cdot\mathbf{x}} \epsilon^*(\mathbf{k}; \lambda) \cdot \mathbf{j}^*(\mathbf{k}, t') \right. \\ &\quad \left. - \epsilon^*(\mathbf{k}; \lambda) \int_0^t dt' e^{-i\omega_k(t'-t) - i\mathbf{k}\cdot\mathbf{x}} \epsilon(\mathbf{k}; \lambda) \cdot \mathbf{j}(\mathbf{k}, t') \right) \\ &= -\sum_{\mathbf{k}\lambda} \frac{1}{V\epsilon_0} e^{i\mathbf{k}\cdot\mathbf{x}} \int_0^t dt' \frac{\sin(\omega_k(t'-t))}{\omega_k} \epsilon(\mathbf{k}; \lambda) (\epsilon^*(\mathbf{k}; \lambda) \cdot \mathbf{j}^*(\mathbf{k}, t')), \end{aligned} \quad (\text{A.4})$$

after a change $\mathbf{k} \rightarrow -\mathbf{k}$ in the last term above, using $\mathbf{j}(-\mathbf{k}, t) = \mathbf{j}^*(\mathbf{k}, t)$ as well as equation (4.12). The second time-derivative of this expression will then contain the following factor:

$$\sum_{\mathbf{k}\lambda} \frac{1}{V\epsilon_0} e^{i\mathbf{k}\cdot\mathbf{x}} \epsilon(\mathbf{k}; \lambda) (\epsilon^*(\mathbf{k}; \lambda) \cdot \mathbf{j}^*(\mathbf{k}, t)) = \sum_{\mathbf{k}} \frac{1}{V\epsilon_0} e^{i\mathbf{k}\cdot\mathbf{x}} (\mathbf{j}^*(\mathbf{k}, t) - \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{j}^*(\mathbf{k}, t))), \quad (\text{A.5})$$

where $\mathbf{j}_T^*(\mathbf{k}, t) = \mathbf{j}_T(-\mathbf{k}, t) \equiv \mathbf{j}(-\mathbf{k}, t) - \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{j}(-\mathbf{k}, t))$ corresponds to the Fourier-components of a transverse current $\mathbf{j}_T(\mathbf{x}, t)$ such that $\nabla \cdot \mathbf{j}_T(\mathbf{x}, t) = 0$, and where use have been made of equation (2.20). The transverse term obtained using equation (A.5) can therefore be written in the form:

$$\frac{1}{V\epsilon_0} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{j}_T(-\mathbf{k}, t) = \frac{1}{V\epsilon_0} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \int d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \mathbf{j}_T(\mathbf{x}', t) = \frac{1}{\epsilon_0} \mathbf{j}_T(\mathbf{x}, t), \quad (\text{A.6})$$

where we make use of the fact that

$$\frac{1}{V} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \delta^{(3)}(\mathbf{x} - \mathbf{x}'). \quad (\text{A.7})$$

We have therefore reproduced the source-term in the wave-equation equation (4.13) in the main text and terms we have left out in the evaluation of $\partial^2 \langle A_T(\mathbf{x}, t) \rangle / \partial t^2$ for $\langle\psi(0)|a_{\mathbf{k}\lambda}|\psi(0)\rangle \neq 0$ will satisfy the homogeneous wave-equation.

ORCID iDs

Bo-Sture K Skagerstam  <https://orcid.org/0000-0001-8124-1540>

References

- [1] Brill O L and Goodman B 1967 Causality in the coulomb gauge *Am. J. Phys.* **35** 832–7
- [2] Itzykson C 1972 Soft quanta *Ann. Phys.* **7** 59–83 Various classical aspects are also discussed [3].

- [3] Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
- [4] Gardiner C W and Drummond P D 1988 Causality in the coulomb gauge: a direct proof *Phys. Rev. A* **38** 4897–8
- [5] Carlip S 2000 Aberration and the speed of gravity *Phys. Lett. A* **267** 81–7
- [6] Rohrlich F 2002 Causality, the coulomb field, and Newton's law of gravitation *Am. J. Phys.* **70** 411–4
- [7] Jackson A D, Lande A and Lautrup B 2001 Apparent superluminal behavior in wave propagation *Phys. Rev. A* **64** 044101–1–4
- [8] Jackson J D 2002 From lorenz to coulomb and other explicit gauge transformations *Am. J. Phys.* **70** 917–28
As cited in this reference, a sign error in equation (9) in Skagerstam B-S K 1983 A note on the Poincaré gauge *Am. J. Phys.* **51** 1148–9 was correctly observed by J D Jackson.
- [9] Stewart A M 2003 Vector potential of the coulomb gauge *Eur. J. Phys.* **24** 519–24
- [10] Hnizdo V 2004 Comment on 'Vector potential of the coulomb gauge' *Eur. J. Phys.* **25** L21–2
- [11] Heras J A 2007 How potentials in different gauges yield the same retarded electric and magnetic fields *Am. J. Phys.* **75** 176–83
- [12] Budko N V 2009 Observation of locally negative velocity of the electromagnetic field in free space *Phys. Rev. Lett.* **102** 020401–1
020401–1–020401-4
Budko N V 2009 Electromagnetic radiation in a time-varying background medium *Phys. Rev. A* **80** 053817–1053817–1–8
Budko N V 2011 Superluminal, subluminal, and negative velocities in free-space electromagnetic propagation *Advances in Imaging and Electron Physics* vol 165 ed P W Hawkes (New York: Academic) pp 47–72
- [13] Dickinson R, Forshaw J and Millington P 2016 Probabilities and signaling in quantum field theory *Phys. Rev. D* **93** 065054–1
065054–1–12
Dickinson R, Forshaw J and Millington P 2017 Working directly with probabilities in quantum field theory *VIII International Workshop DICE2016: Spacetime—Matter - Quantum Mechanics. IOP Conf. Series: Journal of Physics: Conf. Series* **880** 1–8
- [14] Skagerstam B-S K 2019 Comment on: 'Observation of locally negative velocity of electromagnetic field in free space' arXiv:1903.03966 [quant-ph]
- [15] Fermi E 1932 Quantum theory of radiation *Rev. Mod. Phys.* **4** 88–132
- [16] Schleich W 2001 *Quantum Optics in Phase Space* (Berlin: WILEY-VCH)
- [17] Weinberg S 1965 Photons and gravitons in perturbation theory: derivation of Maxwell's and Einstein's equations *Phys. Rev.* **138** B988–1002
- [18] Skagerstam B-S K, Eriksson K-E and Rekdal P K 2019 Quantum field theory with classical sources—linearized quantum gravity *Class. Quantum Grav.* **36** 015011–1015011–1–12
- [19] Jackson J D 1998 *Classical Electrodynamics* 3rd edn (New York: Wiley)
- [20] Sakurai J J 1967 *Advanced Quantum Mechanics* (Reading: Addison-Wesley Publ. Comp.)
- [21] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1989 *Photons and Atoms—Introduction to Quantum Electrodynamics* (New York: Wiley)
- [22] Feynman R P 1951 An operator calculus having applications in quantum electrodynamics *Phys. Rev.* **84** 108–28
- [23] Glauber R J 1951 Some notes on multiple-boson processes *Phys. Rev.* **84** 395–400
Glauber R J 1963 Coherent and incoherent states of the radiation field *Phys. Rev.* **131** 2766–88
Glauber R J 1965 Optical coherence and photon statistics *Quantum Optics and Electronics (Les Houches)* ed A Blandin, C DeWitt-Morette and C Cohen-Tannoudji (New York: Gordon and Breach) pp 65–185
Glauber R J 2007 *Quantum Theory of Optical Coherence* (Weinheim: Wiley-VCH)
- [24] Carruthers P and Nieto M M 1965 Coherent states and the forced harmonic oscillator *Am. J. Phys.* **33** 537–44 (reprinted in [28])
- [25] Eriksson K-E 1967 Summation methods for radiative corrections (*Cargèse Lectures in Physics* vol 2) ed M Lévy (New York: Gordon and Breach) pp 245–74
- [26] Ferrell R A 1977 Forced harmonic oscillator in the interaction picture *Am. J. Phys.* **45** 468–9
- [27] Gerry C C and Knight P L 1977 Quantum superpositions and Schrödinger cat states in quantum optics *Am. J. Phys.* **65** 964–73
- [28] Klauder J R and Skagerstam B-S K 1988 *Coherent States—Applications in Physics and Mathematical Physics* (Singapore: World Scientific) 1985 and Beijing
- [29] Skagerstam B-S K 1994 Coherent states—some applications in quantum field theory and particle physics *Coherent States: Past, Present, and the Future* ed D H Feng, J R Klauder and M R Strayer (Singapore: World Scientific) pp 469–506
Klauder J R 2001 The current state of coherent states *contribution to the VII ICSSUR Conf. (2001) (Boston)* arXiv:quant-ph/0110108
- [30] Byron F W and Fuller R W 1992 *Mathematics of Classical and Quantum Physics* (New York: Dover Pub. Inc.)
Stone M and Goldbart P 2010 *Mathematics for Physics—A Guided Tour for Graduate Students* (Cambridge: Cambridge University Press)
- [31] Ou Z Y, Wang L J, Zou X Y and Mandel L 1990 Evidence for phase memory in two-photon down conversion through entanglement with the vacuum *Phys. Rev. A* **41** 556–68
- [32] Mandel L 1999 Quantum effects in one-photon and two-photon interference *Rev. Mod. Phys.* **71** S274–82
- [33] Resch K J, Lundeen J S and Steinberg A M 2002 Quantum state preparation and conditional coherence *Phys. Rev. Lett.* **88** 113601–1
113601–1–4
- [34] Lvovsky A I and Mlynek J 2002 Quantum-optical catalysis: generating nonclassical states of light by means of linear optics *Phys. Rev. Lett.* **88** 250401–1250401–1–4
- [35] Hessmo B, Usachev P, Heydari H and Björk G 2004 Experimental demonstration of single photon nonlocality *Phys. Rev. Lett.* **92** 180401–1180401–1–4
- [36] Pozza N D, Wiseman H M and Huntington E H 2015 Deterministic preparation of superpositions of vacuum plus one photon by adaptive homodyne detection: experimental considerations *New J. Phys.* **17** 013047–1013047–1–16
- [37] Sathyamoorthy S R, Bengtsson A, Bens S, Simoen M, Delsing P and Johansson G 2016 Simple, robust, and on-demand generation of single and correlated photons *Phys. Rev. A* **93** 063823–1063823–1–9
- [38] Allen L, Barnett S M and Padgett M J (ed) 2003 *Optical Angular Momentum* (Bristol, England: Institute of Physics Publishing)
Franke-Arnold S, Allen L and Padgett M J 2008 Advances in optical angular momentum *Laser & Photon. Rev.* **2** 299–313
Yao A M and Padgett M J 2011 Orbital angular momentum: origins, behavior and applications *Advances in Optics and Photonics* **3** 161–204
Willner A E et al 2015 Optical communications using orbital angular momentum beams *Advances in Optics and Photonics* **7** 66–106
- [39] Lautrup B 1966 Canonical quantum electrodynamics in covariant gauges *Mat. Fys. Medd. Dan. Vid. Selsk.* **35** 1–29
- [40] Nakanishi N 1972 Indefinite-metric quantum field theory *Prog. Theor. Phys.* **51** 1–95
- [41] Futakuchi S and Usui K 2017 Eliminating unphysical photon components from dirac-maxwell hamiltonian quantized in the lorentz gauge *J. Math. Anal. Appl.* **446** 1060–104
- [42] Bjorken J D and Drell S D 1965 *Relativistic Quantum Fields* (New York: McGraw-Hill)

- [43] Chern B and Tubis A 1967 Invariance principles in classical and quantum mechanics *Am. J. Phys.* **35** 254–70
- [44] Lopes J L 1969 Lectures on symmetries *Documents on Modern Physics* ed E W Montroll, G H Vineyard and M Lévy (New York: Gordon and Breach)
- [45] Bickel W S and Goodman A S 1966 Mean lives of the 2p and 3p levels in atomic hydrogen *Phys. Rev.* **148** 1–4
- [46] Čerenkov P A 1934 The visible radiation of pure liquids caused by γ -rays *Dokl. Akad. Nauk SSSR* **2** 451–7
- [47] Frank I M and Tamm I E 1937 The coherent radiation of a fast electron in a medium *Dokl. Akad. Nauk SSSR* **14** 107–13
- [48] Skagerstam B-S K *Spin-Polarization Effects in Vavilov-Čerenkov Radiation* to appear
- [49] Griffiths J D 2012 Resource letter EM-1: electromagnetic momentum *Am. J. Phys.* **80** 7–18
- [50] Ginzburg V L 1940 Quantum theory of luminous radiation from an electron traveling uniformly in a medium *J. Phys. USSR* **2** 441–425
Recent discussions and reviews are given by Ginzburg V L 1993 V. Radiation by uniformly moving sources: Vavilov-Čerenkov effect, doppler effect in a medium, transition radiation and associated phenomena *Progress in Optics* **32** 267–312
Ginzburg V L 1996 Radiation by uniformly moving sources (Vavilov - Čerenkov effect, transition radiation, and other phenomena) *Physics—Uspekhi* **39** 973–82
Ginzburg V L 2002 Some remarks on the radiation of charges and multipoles uniformly moving in a medium *Physics—Uspekhi* **45** 341–4
ch. 7 Ginzburg V L 1989 *Applications of Electrodynamics in Theoretical Physics and Astrophysics* (New York: Gordon and Breach Sci. Pub.)
- [51] Harris E G and Pedestrian A 1972 *Approach to Quantum Field Theory* (N. Y: Wiley-InterScience)
- [52] Marcuse D 1980 *Principles of Quantum Electronics* (N. Y: Academic)
- [53] Bohr N and Rosenfeld L 1950 Field and charge measurements in quantum electrodynamics *Phys. Rev.* **78** 794–8 references therein
- [54] Wightman A S 1964 La Théorie Quantique Locale et la Théorie Quantiques des Champs *Ann. Inst. Poincaré* **1** 403–20
- [55] Walls D F and Zoller P 1981 Reduced quantum fluctuations in resonance fluorescence *Phys. Rev. Lett.* **47** 709–11
Gardiner C W 1986 Inhibition of atomic phase decays by squeezed light: a direct effect of squeezing *Phys. Rev. Lett.* **56** 1917–20
- [56] Schulte C H H, Hansom J, Jones A E, Matthiesen C, Le Gall C and Atatüre M 2015 Quadrature squeezed photons from a two-level system *Nature* **525** 222–5
- [57] Lamb W E Jr 1995 Anti-Photon *Appl. Phys. B* **60** 77–84
- [58] Gottfried K 1966 *Quantum Mechanics: Vol I: Fundamentals* (New York: W. A. Benjamin, Inc.)
- [59] All radiative corrections to Thomson scattering vanishes in the low-energy limit. This theorem goes back to Thirring W 1950 Radiative corrections in the non-relativistic limit *Phil. Mag.* **41** 1193–4 More details can be found in section 19.3 of [42] where the importance of soft photon emission in low energy scattering processes with a finite experimental resolution is clarified.